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ON A CLASS OF TWO-INDEX REAL HERMITE POLYNOMIALS†

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Abstract We discuss some basic properties of a class of doubly indexed real Hermite polynomials including recurrence formulae, Runge's addition formula, generating function and Nielsen's identity.

1 Introduction

The Burchnall's operational formula ([2])

$$\left(-\frac{d}{dx}+2x\right)^m(f)=m!\sum_{k=0}^{m}\frac{(-1)^k}{k!}\frac{H_{m-k}(x)}{(m-k)!}\frac{d^k}{dx^k}(f), \tag{1.1}$$

where $H_m(x)$ denotes the usual Hermite polynomial ([5, 10])

$$H_m(x)=(-1)^m e^{x^2}\frac{d^m}{dx^m}\left(e^{-x^2}\right), \tag{1.2}$$

enjoy a number of remarkable properties. It is used by Burchnall [2] to give a direct proof of Nielsen's identity ([8])

$$H_{m+n}(x)=m!n!\sum_{k=0}^{min(m,n)}\frac{(-2)^k}{k!}\frac{H_{m-k}(x)}{(m-n)!}\frac{H_{n-k}(x)}{(n-k)!}. \tag{1.3}$$

The special case of (1.1) where $f=1$, i.e.,

$$H_m(x)=\left(-\frac{d}{dx}+2x\right)^m\cdot(1). \tag{1.4}$$

can be employed to recover in a easier way the generating function

$$\sum_{m=0}^{+\infty}H_m(x)\frac{t^m}{m!}=\exp(2xt-t^2) \tag{1.5}$$

as well as the Runge addition formula ([9, 7])

$$H_m(x+y)=\left(\frac{1}{2}\right)^{m/2}m!\sum_{k=0}^{n}\frac{H_k(\sqrt{2}x)}{k!}\frac{H_{m-k}(\sqrt{2}y)}{(m-k)!}. \tag{1.6}$$

Many generalizations of such Hermite polynomials can be found in the literature including multi-index ones [11, 6, 1, 3]. In this paper, we consider the following class of two-index Hermite polynomials of single real variable:

$$H_{m,n}(x)=\left(-\frac{d}{dx}+2x\right)^m\cdot(x^n), \tag{1.7}$$

and we derive some of their useful properties. More essentially, we discuss the associated recurrence formulae, Runge's addition formula, generating function and Nielsen's identity.

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2 Doubly indexed real Hermite polynomials $H_{m,n}(x)$

By taking $f(x) = x^n$ in (1.1), we obtain

$$H_{m,n}(x) := \left(-\frac{d}{dx} + 2x\right)^m (x^n) \tag{2.1}$$

$$= m!n! \sum_{k=0}^{min(m,n)} \frac{(-1)^k}{k!} \frac{x^{n-k}}{(n-k)!} \frac{H_{m-k}(x)}{(m-k)!}. \tag{2.2}$$

It follows that $H_{m,n}(x)$ is a polynomial of degree $m+n$, since

$$Q(x) := H_{m,n}(x) - x^n H_m(x)$$

is a polynomial of degree $deg(Q) \leq n+m-2$. For the unity of the formulations, we shall define trivially

$$H_{m,n}(x) = 0$$

whenever $m < 0$ or $n < 0$. We call them doubly indexed real Hermite polynomials. Note that $H_{m,0}(x) = H_m(x)$, $H_{0,n}(x) = x^n$ and

$$H_{m,n}(0) = \begin{cases} 0 & m < n \\ (-1)^n \frac{m!}{(m-n)!} H_{m-n}(0) & m \geq n \end{cases}. \tag{2.3}$$

A direct computation using (2.1) gives rise to

$$H_{1,n}(x) = -nx^{n-1} + 2x^{n+1}$$

for every integer $n \geq 1$. Note also that, since $H_1(x) = 2x$, it follows

$$H_{m+1}(x) = \left(-\frac{d}{dx} + 2x\right)^m (H_1(x)) = \left(-\frac{d}{dx} + 2x\right)^m (2x) = 2H_{m,1}(x). \tag{2.4}$$

The first few values of $H_{m,n}$ are given by

$H_{m,n}$	$n = 1$	$n = 2$	$n = 3$
$m = 1$	$-1 + 2x^2 = H_2(x)$	$-2x + 2x^3$	$-3x^2 + 2x^4$
$m = 2$	$-6x + 4x^3 = H_3(x)$	$2 - 10x^2 + 4x^4$	$6x - 14x^3 + 4x^5$
$m = 3$	$6 - 24x^2 + 8x^4 = H_4(x)$	$24x - 36x^3 + 8x^5$	$-6 + 54x^2 - 48x^4 + 8x^6$

From (2.2), one can deduce easily the symmetry formula

$$H_{m,n}(-x) = (-1)^{n+m} H_{m,n}(x), \tag{2.5}$$

so that the $H_{m,n}(x)$ is odd (rep. even) if and only if $n+m$ is odd (resp. even). Furthermore, the Rodrigues formula for $H_{m,n}(x)$ is

$$H_{m,n}(x) = (-1)^m e^{x^2} \frac{d^m}{dx^m} \left(x^n e^{-x^2}\right). \tag{2.6}$$

Indeed, this can be proved easily making use of

$$\left(-\frac{d}{dx} + 2x\right)^m \cdot (f) = (-1)^m e^{x^2} \frac{d^m}{dx^m} \left(e^{-x^2} f\right). \tag{2.7}$$

Therefore, these polynomials constitute a subclass of the generalized Hermite polynomials

$$H_m^{\gamma}(x, \alpha, p) := (-1)^m x^{-\alpha} e^{px^{\gamma}} \frac{d^m}{dx^m} \left(x^{\alpha} e^{-px^{\gamma}}\right). \tag{2.8}$$

considered by Gould and Hopper in [4]. In fact, we have $H_{m,n}(x) = x^n H_m^2(x, n, 1)$.

Proposition 2.1. *The polynomials $H_{m,n}$; $m, n \geq 1$, satisfy the following recurrence formulae*

$$H'_{m,n}(x) + H_{m+1,n}(x) - 2xH_{m,n}(x) = 0, \tag{2.9}$$

$$H_{m,n}(x) + nH_{m-1,n-1}(x) - 2H_{m-1,n+1}(x) = 0, \tag{2.10}$$

$$H_{m,n}(x) + mH_{m-1,n-1}(x) - xH_{m,n-1}(x) = 0, \tag{2.11}$$

$$(m-n)H_{m-1,n-1}(x) + 2H_{m-1,n+1}(x) + xH_{m,n-1}(x) = 0. \tag{2.12}$$

Proof. The first one follows by writing the derivation operator as

$$\frac{d}{dx} = -\left(-\frac{d}{dx} + 2x\right) + 2x.$$

Indeed, we get

$$\begin{aligned}\frac{d}{dx}\left(H_{m,n}(x)\right) &= -\left(-\frac{d}{dx} + 2x\right) H_{m,n}(x) + 2xH_{m,n}(x)\\ &= -H_{m+1,n}(x) + 2xH_{m,n}(x).\end{aligned}$$

For the second one, write $H_{m,n}(x)$ as

$$\begin{aligned}H_{m,n}(x) &= \left(-\frac{d}{dx} + 2x\right)^{m-1} (H_{1,n}(x))\\ &= \left(-\frac{d}{dx} + 2x\right)^{m-1} \left(-nx^{n-1} + 2x^{n+1}\right)\\ &= -nH_{m-1,n-1}(x) + 2H_{m-1,n+1}(x).\end{aligned}$$

To prove (2.11), we use (2.6) combined with Leibnitz formula. Indeed,

$$\begin{aligned}H_{m,n}(x) &= (-1)^m e^{x^2} \frac{d^m}{dx^m}\left(x \cdot x^{n-1}e^{-x^2}\right)\\ &= (-1)^m e^{x^2}\left[x\frac{d^m}{dx^m}\left(x^{n-1}e^{-x^2}\right) + m\frac{d^{m-1}}{dx^{m-1}}\left(x^{n-1}e^{-x^2}\right)\right]\\ &= xH_{m,n-1}(x) - mH_{m-1,n-1}(x).\end{aligned}$$

Finally, (2.12) follows from (2.10) and (2.11) by substraction. □

Remark 2.2. According to (2.4), the (2.11) (corresponding to $n = 1$) leads to the well known recurrence formula $H_{m+1}(x) = 2xH_m(x) - 2mH_{m-1}(x)$ for $H_m(x)$. Note also that (2.9) reduces further to $H'_m(x) + H_{m+1}(x) - 2xH_m(x) = 0$ by taking $n = 0$, so that we recover the known result that $H'_m(x) = 2mH_{m-1}(x)$.

Proposition 2.3. *We have the following addition formula*

$$H_{m,n}(x+y) = m!n!\left(\frac{1}{\sqrt{2}}\right)^{m+n} \sum_{k=0}^{m}\sum_{j=0}^{n} \frac{H_{k,j}(\sqrt{2}x)}{k!j!}\frac{H_{m-k,n-j}(\sqrt{2}y)}{(m-k)!(n-j)!}. \tag{2.13}$$

Proof. We begin by writing have $H_{m,n}(x+y)$ as

$$\begin{aligned}H_{m,n}(x+y) &= \left(-\frac{d}{d(x+y)} + 2(x+y)\right)^m .((x+y)^n)\\ &= \left(-\frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) + 2(x+y)\right)^m .((x+y)^n)\\ &= \left(\frac{1}{\sqrt{2}}\right)^m (A_x + A_y)^m .((x+y)^n)\\ &= \left(\frac{1}{\sqrt{2}}\right)^m \sum_{j=0}^{n}\binom{n}{j}(A_x + A_y)^m .(x^j y^{n-j}),\end{aligned}$$

where A_t stands for $A_t = -\partial/(\partial\sqrt{2}t) + 2\sqrt{2}t$. Thus, since A_x and A_y commute, we can make use of the binomial formula to get

$$H_{m,n}(x+y) = \left(\frac{1}{\sqrt{2}}\right)^m \sum_{k=0}^{m}\sum_{j=0}^{n}\binom{m}{k}\binom{n}{j} A_x^k.(x^j)A_y^{m-k}.(y^{n-j}),$$

whence, we obtain the asserted result according to the fact that

$$A_t^r(t^s) = 2^{-s/2}H_{r,s}(\sqrt{2}t).$$

□

Remark 2.4. We recover the Runge addition formula (1.6) for the classical real Hermite polynomials $H_m(x) = H_{m,0}(x)$ by taking $n = 0$ in (2.13).

The following identities are immediate consequence of the previous proposition.

Corollary 2.5. *The identity*

$$H_{m,n}(t) = m!n!\left(\frac{1}{\sqrt{2}}\right)^{m+n}\sum_{j=0}^{n}\sum_{k=j}^{m}\frac{(-1)^j}{j!(k-j)!}H_{k-j}(0)\frac{H_{m-k,n-j}(\sqrt{2}t)}{(m-k)!(n-j)!}$$

holds by taking $x = 0$ and setting $t = y$ in (2.13)*, keeping in mind* (2.3)*. We get also*

$$H_{m,n}(t) = m!n!\left(\frac{1}{\sqrt{2}}\right)^{m+n}\sum_{k=0}^{m}\sum_{j=0}^{n}\frac{H_{k,j}(t/\sqrt{2})}{k!j!}\frac{H_{m-k,n-j}(t/\sqrt{2})}{(m-k)!(n-j)!}$$

by setting $x = y = t/2$ in (2.13)*. While for $t = -\sqrt{2}x = \sqrt{2}y$, we obtain*

$$\sum_{k=0}^{m}\sum_{j=0}^{n}(-1)^{k+j}\frac{H_{k,j}(t)}{k!j!}\frac{H_{m-k,n-j}(t)}{(m-k)!(n-j)!} = 0$$

whenever $m + n$ is odd or $m > n$.

Next, we state the following

Proposition 2.6. *The generating function of $H_{m,n}$ is given by*

$$\sum_{m,n=0}^{+\infty} H_{m,n}(x)\frac{u^m}{m!}\frac{v^n}{n!} = \exp\left(-u^2 + (2u+v)x - uv\right). \tag{2.14}$$

Proof. According to the definition of $H_{m,n}$, we can write

$$\begin{aligned}\sum_{m,n=0}^{+\infty} H_{m,n}(x)\frac{u^m}{m!}\frac{v^n}{n!} &= \left[\sum_{m=0}^{+\infty}\frac{1}{m!}\left(-u\frac{d}{dx} + 2ux\right)^m\right]\cdot\left(\sum_{n=0}^{+\infty}\frac{v^n}{n!}x^n\right)\\ &= \exp\left(-u\frac{d}{dx} + 2ux\right)(e^{vx}).\end{aligned}$$

Making use of the Weyl identity which reads for the operators $A = 2xId$ et $B = -d/dx$ as

$$\exp(uA + uB) = \exp(uA)\exp(uB)\exp\left(-u^2 Id\right); \quad u \in \mathbb{R},$$

we get

$$\sum_{m,n=0}^{+\infty} H_{m,n}(x)\frac{u^m}{m!}\frac{v^n}{n!} = e^{2ux-u^2}\exp\left(-u\frac{d}{dx}\right)(e^{vx}).$$

Therefore, the desired result follows since

$$\exp\left(-u\frac{d}{dx}\right)(e^{vx}) = \sum_{k=0}^{\infty}\frac{(-u)^k}{k!}\left(\frac{d}{dx}\right)^k(e^{vx}) = e^{-uv}e^{vx}.$$

□

Remark 2.7. The special case of $v = 0$ (in (2.14)) infers the generating function (1.5) of the standard real Hermite polynomials H_m. Furthermore, for $y = u = -v$, we get

$$e^{xy} = \sum_{m,n=0}^{+\infty}(-1)^n H_{m,n}(x)\frac{y^{m+n}}{m!n!}. \tag{2.15}$$

Proposition 2.8. *We have the recurrence formula*

$$H'_{m,n}(x) = 2mH_{m-1,n}(x) + nH_{m,n-1}(x). \tag{2.16}$$

Proof. Differentiating the both sides of (2.14) and making appropriate changes of indices yield (2.16). □

Corollary 2.9. *We have*

$$\frac{d^\nu}{dx^\nu}(H_{r,n}(x)) = r!n!\sum_{j=0}^{\nu}\alpha_{j,\nu}\frac{H_{r-\nu+j,n-j}(x)}{(r-\nu+j)!(n-j)!}, \tag{2.17}$$

where

$$\alpha_{j,\nu} = \begin{cases} 2^\nu & \text{for} \quad j=0 \\ 2\alpha_{j,\nu-1}+\alpha_{j-1,\nu-1} & \text{for} \quad 1\le j<\nu \\ 1 & \text{for} \quad j=\nu \end{cases}.$$

Proof. This can be handled by mathematical induction using (2.16). □

Remark 2.10. The $\alpha_{j,\nu}$ are even positive numbers and their first values are

$\alpha_{j,\nu}$	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$\nu=0$	1					
$\nu=1$	$\boxed{2}$	1				
$\nu=2$	2^2	$\boxed{4}$	1			
$\nu=3$	2^3	12	$\boxed{6}$	1		
$\nu=4$	2^4	32	24	$\boxed{8}$	1	
$\nu=5$	2^5	80	80	40	$\boxed{10}$	1

.

We conclude this paper by giving a formula for the two-index Hermite polynomial $H_{m,n}(x)$ expressing it as a weighted sum of a product of the same polynomials. Namely, we state the following

Proposition 2.11. *Keep notation as above. Then the Nielsen identity for $H_{m,n}$; $n\ge 1$, reads*

$$H_{m+r,n}(x) = m!r!nn!\sum_{k,\nu,j=0}^{m,k,\nu}\alpha_{j,\nu}\frac{\Gamma(n+k-\nu)}{(k-\nu)!\nu!}\frac{(-x)^\nu}{x^{n+k}}\frac{H_{m-k,n}(x)}{(m-k)!n!}\frac{H_{r-\nu+j,n-j}(x)}{(r-\nu+j)!(n-j)!}.$$

Proof. Recall first that $H_m^\gamma(x,\alpha,p)$, the polynomials given through (2.8), can be rewritten in the following equivalent form ([4])

$$H_m^\gamma(x,\alpha,p) := \left(-\frac{d}{dx}+p\gamma x^{\gamma-1}-\frac{\alpha}{x}\right)^m(1).$$

Now, since for the special values $p=1$, $\gamma=2$ and $\alpha=n$, we have

$$\begin{aligned} H_{m+r,n}(x) &= x^n H^2_{m+r}(x,n,1) \\ &= x^n\left(-\frac{d}{dx}+2x-\frac{n}{x}\right)^m\left(H_r^2(x,n,1)\right) \\ &= x^n\left(-\frac{d}{dx}+2x-\frac{n}{x}\right)^m\left(x^{-n}H_{r,n}(x)\right), \end{aligned}$$

we can make use of the Burchnall's formula extension proved by Gould and Hopper [4], to wit

$$\left(-\frac{d}{dx}+p\gamma x^{\gamma-1}-\frac{\alpha}{x}\right)^m(f) = m!\sum_{k=0}^{m}\frac{(-1)^k}{k!}\frac{H^\gamma_{m-k}(x,\alpha,p)}{(m-k)!}\frac{d^k}{dx^k}(f).$$

Thus, for $f=x^{-n}H_{r,n}$, we obtain

$$H_{m+r,n}(x) = m!\sum_{k=0}^{m}\frac{(-1)^k}{k!}\frac{H_{m-k,n}(x)}{(m-k)!}\frac{d^k}{dx^k}(x^{-n}H_{r,n}(x)). \tag{2.18}$$

Therefore, by applying the Leibnitz formula and appealing the result of Corollary 2.9, we get

$$H_{m+r,n}(x) = m! \sum_{k=0}^{m} \frac{(-1)^k}{k!} \frac{H_{m-k,n}(x)}{(m-k)!} \sum_{\nu=0}^{k} \binom{k}{\nu} \frac{d^{k-\nu}}{dx^{k-\nu}}(x^{-n}) \frac{d^{\nu}}{dx^{\nu}}(H_{r,n}(x))$$

$$= m!r!nn! \sum_{k,\nu,j=0}^{m,k,\nu} \alpha_{j,\nu} \frac{\Gamma(n+k-\nu)}{(k-\nu)!\nu!} \frac{(-x)^{\nu}}{x^{n+k}} \frac{H_{m-k,n}(x)}{(m-k)!n!} \frac{H_{r-\nu+j,n-j}(x)}{(r-\nu+j)!(n-j)!}$$

for every integer $n \geq 1$. Note that for $n = 0$, (2.18) reads simply

$$H_{m+r}(x) = m! \sum_{k=0}^{m} \frac{(-1)^k}{k!} \frac{H_{m-k}(x)}{(m-k)!} \frac{d^k}{dx^k}(H_r(x)).$$

In this case, we recover the usual Nielsen formula (1.3) for the real Hermite polynomials H_m. □

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