Palestine Journal of Mathematics

Vol. 4 (Spec. 1) (2015) , 508–514

GAUSS LEGENDRE MULTIPLICATION FORMULA FOR p-ADIC BETA FUNCTION

Hamza Menken and Özge Çolakoğlu

Communicated by Ali Bülent Ekin

MSC 2010 Classifications: Primary 11S80; Secondary, 11E95.

Keywords and phrases: p-adic number, p-adic gamma function, p-adic beta function, Gauss Lengendre multiplication formula.

Abstract In the present work we study a p-adic analogue of the classical beta function by using Y. Morita's p-adic gamma function. We obtain Gauss Lengendre multiplication type formulas for the p-adic beta function.

1 Introduction

Let p be a odd prime number and let $\mathbb{Z}_p$ and $\mathbb{Q}_p$ denote the ring of p-adic integers and the field of p-adic numbers, respectively. The p-adic analogue of the classical gamma function Γ is defined by Y. Morita (1975) [10] as the continuous extension to $\mathbb{Z}_p$ of the function $n \to (-1)^n \prod\limits_{\substack{1\leq j<n \\ (p,,j)=1}} j$, i.e., $\Gamma_p(x)$ is defined by the formula

$$\Gamma_p\left(x\right) = \lim_{n\to x}(-1)^n \prod_{\substack{j<n \\ (p,,j)=1}} j$$

for $x \in \mathbb{Z}_p$, where n approaches x through positive integers. The p-adic gamma function Γ_p had been studied by J. Diamond (1977) [5], D. Barsky (1979) [2] , T. K. Kim (1997) [8] and others. The relationship between some special functions and the p-adic gamma function $\Gamma_p(x)$ were investigated by B. Gross and N. Koblitz (1979) [7], H. Cohen and E. Friedman (2008) [4] and I. Shapiro (2012) [12].

We use the following properties of the p-adic gamma function Γ_p (for details see [6] and [11])

(i) For all $x \in \mathbb{Z}_p$

$$\Gamma_p(x+1) = h_p(x)\Gamma_p(x) \tag{1.1}$$

where

$$h_p(x) := \begin{cases} -x, \text{ if } |x|_p = 1 \\ -1, \text{ if } |x|_p < 1 \end{cases}.$$

(ii) $\Gamma_p(0) = 1, \Gamma_p(1) = -1, \Gamma_p(2) = 1$ and $|\Gamma_p(x)|_p = 1$ for all $x \in \mathbb{Z}_p$.

(iii) For all $x \in \mathbb{Z}_p$

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{\ell(x)} \quad (x \in \mathbb{Z}_p) \tag{1.2}$$

where $\ell : \mathbb{Z}_p \to \{1, 2, ..., p\}$ assigns to $x \in \mathbb{Z}_p$ its residue $\in \{1, 2, ..., p\}$ modulo $p\mathbb{Z}_p$. Hence,

$$\Gamma_p(\frac{1}{2})^2 = (-1)^{\ell(\frac{1}{2})} \tag{1.3}$$

and

$$\Gamma_p(\frac{1}{2})^2 = \begin{cases} 1 \text{ if } p \equiv 3(mod\, 4) \\ -1 \text{ if } p \equiv 1(mod\, 4). \end{cases}$$

Assume that $m, n \in \mathbb{N}$ and $m \geq 2$ is not divisible by p. The first version of the p-adic Gauss Legendre multiplication formula ([11]) states that

$$\Gamma_p(n)\Gamma_p(n+\frac{1}{m})...\Gamma_p(n+\frac{m-1}{m}) = \left(\prod_{j=0}^{m-1}\Gamma_p(\frac{j}{m})\right)\Gamma_p(mn)m^{-\mu(mn)} \quad (n \in \mathbb{N}) \tag{1.4}$$

where $\mu(k)$ is the number of elements of $\{1, 2, ..., k-1\}$ that are not divisible by p, i.e.

$$\mu(k) = k - 1 - \left[\frac{k-1}{p}\right].$$

In the case $m = 2$, the first version of the p-adic Gauss Legendre formula says

$$\Gamma_p(n)\Gamma_p(n + \frac{1}{2}) = \Gamma_p(\frac{1}{2})\Gamma_p(2n)2^{-(2n-1)}2^{\left[\frac{2n-1}{p}\right]} \tag{1.5}$$

so that

$$\Gamma_p(2n) = 2^{2n-1}\Gamma_p(\frac{1}{2})^{-1}\Gamma_p(n)\Gamma_p(n + \frac{1}{2})2^{-\left[\frac{2n-1}{p}\right]}. \tag{1.6}$$

The Gauss Legendre multiplication formula for p-adic gamma function as follows:

For each $x \in \mathbb{Z}_p$, let $\ell(x) \in \{1, 2, ..., p\}$ be such that $|x - \ell(x)|_p < 1$. Further, let $\ell_1(x) = p^{-1}(x - \ell(x))$ $(x \in \mathbb{Z}_p)$. Then for $m > 1$, m not divisible by p

$$\prod_{j=0}^{m-1} \Gamma_p(x + \frac{1}{m}) = \left(\prod_{j=0}^{m-1} \Gamma_p(\frac{j}{m})\right) m^{1-\ell(mx)} \left(m^{p-1}\right)^{-\ell_1(mx)} \Gamma_p(mx). \tag{1.7}$$

A p-adic analogue of classical beta function by using Morita's p-adic gamma function can be defined as follows

Definition 1.1. The p-adic beta function $B_p : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Q}_p$ is defined by the formula

$$B_p(x, y) := \frac{\Gamma_p(x)\,\Gamma_p(y)}{\Gamma_p(x + y)}, \quad x, y \in \mathbb{Z}_p$$

In 1980 M. Boyarsky [3] used to the p-adic beta function in Dwork cohomology and gave an cohomological interpretation of the p-adic beta function. In 2006 F. Baldassarri [1] considered two constructions of the p-adic beta functions as the p-adic etale and p-adic crystalline beta functions, and gave some comparisons between them with relations via Fontaine's periods. In [9] the basic properties of the p-adic beta function are given.

In the present work we consider the p-adic beta function and we derive Gauss Legendre multiplication type formulas for the p-adic beta function.

2 Main Results

For the p-adic beta function, as a first version of Gauss Legendre multiplication formula we obtain the following result.

Theorem 2.1. *Let $m, n \in \mathbb{N}$ and $m \geq 2$ is not divisible by p.*

(i) If m is odd, then

$$\prod_{j=0}^{m-1} B_p(n + \frac{j}{m}, n + \frac{j}{m}) = \frac{(-1)^{\frac{m-1}{2}} B_p(mn, mn) \left(\prod_{j=1}^{\frac{m-1}{2}} B_p(\frac{j}{m}, \frac{m-j}{m})\right)}{\prod_{j=0}^{\frac{m-1}{2}-1} h_p(2n + \frac{2j+1}{m}) m^{2\mu(mn)-\mu(2mn)}} \tag{2.1}$$

(ii) If m is even, then

$$\prod_{j=0}^{m-1} B_p(n + \frac{j}{m}, n + \frac{j}{m}) = \frac{(-1)^{\frac{m}{2}} \left(\prod_{j=0}^{\frac{m}{2}-1} B_p(\frac{2j+1}{m}, \frac{m-(2j+1)}{m})\right) 2^{-2\mu(mn)}}{\prod_{j=0}^{\frac{m}{2}-1} h_p(2n + \frac{2j}{m})} \tag{2.2}$$

where

$$\mu(k) = k - 1 - \left[\frac{k-1}{p}\right]$$

Proof. From the first version of Gauss Legendre multiplication formula for p-adic gamma function that

$$\Gamma_p(n)\Gamma_p(n+\frac{1}{m})...\Gamma_p(n+\frac{m-1}{m})=\left(\prod_{j=0}^{m-1}\Gamma_p(\frac{j}{m})\right)\Gamma_p(mn)m^{-\mu(mn)}\quad (n\in\mathbb{N}).$$

Taking $2n$ in place of n, the relation becomes

$$\prod_{j=0}^{m-1}\Gamma_p(2n+\frac{j}{m})=\prod_{j=0}^{m-1}\Gamma_p(\frac{j}{m})m^{-\mu(2mn)}\Gamma_p(2mn). \tag{2.3}$$

From Definition 1.1 we get

$$\begin{aligned} &B_p(n,n)B_p(n+\frac{1}{m},n+\frac{1}{m})...B_p(n+\frac{m-1}{m},n+\frac{m-1}{m})=\\ &= \frac{\Gamma_p(n)\Gamma_p(n)}{\Gamma_p(2n)}\frac{\Gamma_p(n+\frac{1}{m})\Gamma_p(n+\frac{1}{m})}{\Gamma_p(2n+\frac{2}{m})}...\frac{\Gamma_p(n+\frac{m-1}{m})\Gamma_p(n+\frac{m-1}{m})}{\Gamma_p(2n+\frac{2m-2}{m})}\\ &= \frac{\left(\prod_{j=0}^{m-1}\Gamma_p(n+\frac{j}{m})\right)^2}{\prod_{j=0}^{m-1}\Gamma_p(2n+\frac{2j}{m})}. \end{aligned}\tag{2.4}$$

Let

$$f(n):=\prod_{j=0}^{m-1}\Gamma_p(2n+\frac{2j}{m}). \tag{2.5}$$

Assume that m is odd. Thus, we can write

$$\begin{aligned} f(n) &= \Gamma_p(2n)\Gamma_p(2n+\frac{2}{m})...\Gamma_p(2n+\frac{m-1}{m})\Gamma_p(2n+\frac{m+1}{m})...\Gamma_p(2n+\frac{2m-2}{m})\\ &= \Gamma_p(2n)...\Gamma_p(2n+\frac{m-1}{m})\Gamma_p(2n+\frac{1}{m}+1)...\Gamma_p(2n+\frac{m-2}{m}+1). \end{aligned}$$

By the equality (1.1) we obtain

$$f(n)=\Gamma_p(2n)\Gamma_p(2n+\frac{1}{m})\Gamma_p(2n+\frac{2}{m})...\Gamma_p(2n+\frac{m-1}{m})h_p(2n+\frac{1}{m})...h_p(2n+\frac{m-2}{m})$$

such that

$$\begin{aligned} \Gamma_p(2n+\frac{m+1}{m}) &= \Gamma_p(2n+\frac{1}{m}+1)=\Gamma_p(2n+\frac{1}{m})h_p(2n+\frac{1}{m})\\ \Gamma_p(2n+\frac{m+3}{m}) &= \Gamma_p(2n+\frac{3}{m}+1)=\Gamma_p(2n+\frac{3}{m})h_p(2n+\frac{3}{m})\\ &.\\ &.\\ &.\\ \Gamma_p(2n+\frac{2m-2}{m}) &= \Gamma_p(2n+\frac{m-2}{m}+1)=\Gamma_p(2n+\frac{m-2}{m})h_p(2n+\frac{m-2}{m}). \end{aligned}$$

Then, we can write

$$f(n)=\prod_{j=0}^{m-1}\Gamma_p(2n+\frac{j}{m})\prod_{j=0}^{\frac{m-1}{2}-1}h_p(2n+\frac{2j+1}{m}).$$

Hence, using $f(n)$ in (2.4) we get the formula

$$\prod_{j=0}^{m-1}B_p(n+\frac{j}{m},n+\frac{j}{m})=\frac{\left(\prod_{j=0}^{m-1}\Gamma_p(n+\frac{j}{m})\right)^2}{\prod_{j=0}^{m-1}\Gamma_p(2n+\frac{j}{m})\prod_{j=0}^{\frac{m-1}{2}-1}h_p(2n+\frac{2j+1}{m})}$$

By using (1.4) and (2.3) we have

$$\begin{aligned}\prod_{j=0}^{m-1} B_p(n+\frac{j}{m}, n+\frac{j}{m}) &= \frac{\left[\left(\prod\limits_{j=0}^{m-1}\Gamma_p(\frac{j}{m})\right)\Gamma_p(mn)m^{-\mu(mn)}\right]^2}{\prod\limits_{j=0}^{m-1}\Gamma_p(\frac{j}{m})m^{-\mu(2mn)}\Gamma_p(2mn)\left(\prod\limits_{j=0}^{\frac{m-1}{2}-1}h_p(2n+\frac{2j+1}{m})\right)} \\ &= \frac{\left(\prod\limits_{j=0}^{m-1}\Gamma_p(\frac{j}{m})\right)\Gamma_p(mn)\Gamma_p(mn)m^{-2\mu(mn)}}{m^{-\mu(2mn)}\Gamma_p(2mn)\left(\prod\limits_{j=0}^{\frac{m-1}{2}-1}h_p(2n+\frac{2j+1}{m})\right)}.\end{aligned}$$

According to Definition 1.1 we obtain the formula

$$\prod_{j=0}^{m-1} B_p(n+\frac{j}{m}, n+\frac{j}{m}) = \frac{B_p(mn,mn)m^{-2\mu(mn)+\mu(2mn)}\left(\prod\limits_{j=0}^{m-1}\Gamma_p(\frac{j}{m})\right)}{\prod\limits_{j=0}^{\frac{m-1}{2}-1}h_p(2n+\frac{2j+1}{m})}. \tag{2.6}$$

Now we must arrange $\left(\prod\limits_{j=0}^{m-1}\Gamma_p(\frac{j}{m})\right)$. First, notice that $\Gamma_p(1) = -1$ and $\Gamma_p(0) = 1$. By using Definition 1.1 it follows that

$$\begin{aligned}\prod_{j=0}^{m-1}\Gamma_p(\frac{j}{m}) &= \Gamma_p(0)\Gamma_p(\frac{1}{m})\Gamma_p(\frac{2}{m})...\Gamma_p(\frac{m-2}{m})\Gamma_p(\frac{m-1}{m}) \\ &= \frac{\Gamma_p(\frac{1}{m})\Gamma_p(\frac{m-1}{m})}{\Gamma_p(1)}\Gamma_p(1)...\frac{\Gamma_p(\frac{\left(\frac{m-1}{2}\right)}{m})\Gamma_p(\frac{m-\left(\frac{m-1}{2}\right)}{m})}{\Gamma_p(1)}\Gamma_p(1) \\ &= (-1)^{\frac{m-1}{2}}B_p(\frac{1}{m},\frac{m-1}{m})...B_p(\frac{\left(\frac{m-1}{2}\right)}{m},\frac{m-\left(\frac{m-1}{2}\right)}{m}) \\ &= (-1)^{\frac{m-1}{2}}\prod_{j=1}^{\frac{m-1}{2}}B_p(\frac{j}{m},\frac{m-j}{m}).\end{aligned}$$

Using this formula in (2.6) we prove the formula (2.1) for odd number m.

Assume that m be even. By (2.5) we have

$$f(n) = \Gamma_p(2n)...\Gamma_p(2n+\frac{\frac{m}{2}-1+\frac{m}{2}-1}{m})\Gamma_p(2n+\frac{m}{m})...\Gamma_p(2n+\frac{m-1+m-1}{m}).$$

Now, we arrange $f(n)$. By using the equality (1.1) we get

$$\begin{aligned}f(n) &= \Gamma_p(2n)\Gamma_p(2n+\frac{2}{m})...\Gamma_p(2n+\frac{m-2}{m})\Gamma_p(2n+1)...\Gamma_p(2n+\frac{m-2}{m}+1) \\ &= \Gamma_p(2n)...\Gamma_p(2n)h_p(2n)...\Gamma_p(2n+\frac{m-2}{m})h_p(2n+\frac{m-2}{m})\end{aligned}$$

or

$$f(n) = \left(\prod_{j=0}^{\frac{m-2}{2}}\Gamma_p(2n+\frac{2j}{m})\right)^2\left(\prod_{j=0}^{\frac{m-2}{2}}h_p(2n+\frac{2j}{m})\right). \tag{2.7}$$

Let $g(n) := \prod\limits_{j=0}^{\frac{m-2}{2}}\Gamma_p(2n+\frac{2j}{m})$. Taking $m = 2k$, we have

$$g(n) = \prod_{j=0}^{k-1}\Gamma_p(2n+\frac{j}{k})$$

Then, we can arrange $g(n)$. From the equality (2.3) we have

$$g(n)=\prod_{j=0}^{k-1}\Gamma_p(\frac{j}{k})k^{-\mu(2kn)}\Gamma_p(2kn).$$

Taking $k=\frac{m}{2}$ we get

$$g(n)=\prod_{j=0}^{\frac{m}{2}-1}\Gamma_p(\frac{2j}{m})m^{-\mu(mn)}2^{\mu(mn)}\Gamma_p(mn)$$

and using $g(n)$ in (2.7) we obtain

$$f(n)=\left(\prod_{j=0}^{\frac{m}{2}-1}\Gamma_p(\frac{2j}{m})m^{-\mu(mn)}2^{\mu(mn)}\Gamma_p(mn)\right)^2\left(\prod_{j=0}^{\frac{m-2}{2}}h_p(2n+\frac{2j}{m})\right).$$

Thus, using $f(n)$ in (2.4) we get

$$\prod_{j=0}^{m-1}B_p(n+\frac{j}{m},n+\frac{j}{m})=\frac{\left(\prod\limits_{j=0}^{m-1}\Gamma_p(n+\frac{j}{m})\right)^2}{\left(\prod\limits_{j=0}^{\frac{m}{2}-1}\Gamma_p(\frac{2j}{m})m^{-\mu(mn)}2^{\mu(mn)}\Gamma_p(mn)\right)^2\left(\prod\limits_{j=0}^{\frac{m-2}{2}}h_p(2n+\frac{2j}{m})\right)}.$$

Using the equality (1.4) we have that

$$\begin{aligned}\prod_{j=0}^{m-1}B_p(n+\frac{j}{m},n+\frac{j}{m}) &= \frac{\left(\prod\limits_{j=0}^{m-1}\Gamma_p(\frac{j}{m})\right)^2 m^{-2\mu(mn)}\left(\Gamma_p(mn)\right)^2}{\left(\prod\limits_{j=0}^{\frac{m}{2}-1}\Gamma_p(\frac{2j}{m})\right)^2 m^{-2\mu(mn)}2^{2\mu(mn)}\left(\Gamma_p(mn)\right)^2\prod\limits_{j=0}^{\frac{m-2}{2}}h_p(2n+\frac{2j}{m})}\\ &= \frac{\left(\prod\limits_{j=0}^{\frac{m-2}{2}}\Gamma_p(\frac{2j}{m})\right)^2\left(\prod\limits_{j=0}^{\frac{m-2}{2}}\Gamma_p(\frac{2j+1}{m})\right)^2}{\left(\prod\limits_{j=0}^{\frac{m}{2}-1}\Gamma_p(\frac{2j}{m})\right)^2 2^{2\mu(mn)}\left(\prod\limits_{j=0}^{\frac{m-2}{2}}h_p(2n+\frac{2j}{m})\right)}\end{aligned}$$

or

$$\prod_{j=0}^{m-1}B_p(n+\frac{j}{m},n+\frac{j}{m})=\left(\prod_{j=0}^{\frac{m-2}{2}}\Gamma_p(\frac{2j+1}{m})\right)^2 2^{-2\mu(mn)}\left(\prod_{j=0}^{\frac{m-2}{2}}h_p(2n+\frac{2j}{m})\right)^{-1}\tag{2.8}$$

Now we must arrange $\left(\prod\limits_{j=0}^{\frac{m-2}{2}}\Gamma_p(\frac{2j+1}{m})\right)^2$. Let $s(n):=\left(\prod\limits_{j=0}^{\frac{m-2}{2}}\Gamma_p(\frac{2j+1}{m})\right)^2$. Using $\Gamma_p(1)=-1$ and Definition 1.1 we get that

$$\begin{aligned}s(n) &= \left(\prod_{j=0}^{\frac{m}{2}-1}\Gamma_p(\frac{2j+1}{m})\right)^2\\ &= \Gamma_p(\frac{1}{m})^2\Gamma_p(\frac{3}{m})^2...\Gamma_p(\frac{m-3}{m})^2\Gamma_p(\frac{m-1}{m})^2\\ &= \Gamma_p(1)\frac{\Gamma_p(\frac{1}{m})\Gamma_p(\frac{m-1}{m})}{\Gamma_p(1)}...\Gamma_p(1)\frac{\Gamma_p(\frac{3}{m})\Gamma_p(\frac{m-3}{m})}{\Gamma_p(1)}\Gamma_p(1)\frac{\Gamma_p(\frac{1}{m})\Gamma_p(\frac{m-1}{m})}{\Gamma_p(1)}\\ &= (-1)^{\frac{m}{2}}\prod_{j=0}^{\frac{m}{2}-1}B_p(\frac{2j+1}{m},\frac{m-(2j+1)}{m}).\end{aligned}$$

Using $s(n)$ in (2.8) we prove the formula (2.2). □

In similar way, by using the equality (1.1) and the property (1.7) we can prove the Gauss Legendre multiplication formula for p-adic beta function:

Theorem 2.2. *For each $x \in \mathbb{Z}_p$, let $\ell(x) \in \{1, 2, ..., p\}$ be such that $|x - \ell(x)|_p < 1$. Further, let $\ell_1(x) = p^{-1}(x - \ell(x))$ $(x \in \mathbb{Z}_p)$. Then for $m > 1$, m not divisible by p*

(i) If m is odd, then

$$\prod_{j=0}^{m-1} B_p(x + \frac{j}{m}, x + \frac{j}{m}) = \frac{B_p(mx, mx)\left(\prod_{j=1}^{\frac{m-1}{2}} B_p(\frac{j}{m}, \frac{m-j}{m})\right)(m^{p-1})^{\ell_1(2mx)-2\ell_1(mx)}}{(-1)^{\frac{m-1}{2}}\left[\prod_{j=0}^{\frac{m-1}{2}-1} h_p(2x + \frac{2j+1}{m})\right] m^{-1+\ell(mx)-\ell(2mx)}}$$

(ii) If m is even, then

$$\prod_{j=0}^{m-1} B_p(x + \frac{j}{m}, x + \frac{j}{m}) = \frac{(-1)^{\frac{m}{2}}\left(\prod_{j=0}^{\frac{m}{2}-1} B_p(\frac{2j+1}{m}, \frac{m-(2j+1)}{m})\right)}{\left[\prod_{j=0}^{\frac{m}{2}-1} h_p(2x + \frac{2j}{m})\right] 2^{2\ell(mx)-2}(2^{p-1})^{2\ell_1(mx)}}$$

Corollary 2.3. *The equality*

$$B_p(n, n) = 2^{\left[\frac{2n-1}{p}\right]-(2n-1)} B_p(n, \frac{1}{2})$$

holds for all $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$. From Definition 1.1 we know that

$$B_p(n, n) = \frac{\Gamma_p(n)\Gamma_p(n)}{\Gamma_p(2n)}.$$

Then, by using (1.6) we get

$$\begin{aligned} B_p(n, n) &= \frac{\Gamma_p(n)\Gamma_p(n)}{2^{2n-1}\Gamma_p(\frac{1}{2})^{-1}\Gamma_p(n)\Gamma_p(n + \frac{1}{2})2^{-\left[\frac{2n-1}{p}\right]}} \\ &= \frac{\Gamma_p(n)\Gamma_p(\frac{1}{2})}{\Gamma_p(n + \frac{1}{2})} 2^{\left[\frac{2n-1}{p}\right]-(2n-1)}. \end{aligned}$$

Using Definition 1.1 we obtain the formula

$$B_p(n, n) = 2^{\left[\frac{2n-1}{p}\right]-(2n-1)} B_p(n, \frac{1}{2}).$$

□

Corollary 2.4. *For all $n \in \mathbb{N}$, the equality*

$$B_p(n, n)B_p(n + \frac{1}{2}, n + \frac{1}{2}) = (-1)^{\ell(\frac{1}{2})} 2^{-2(2n-1)+2\left[\frac{2n-1}{p}\right]} (h_p(2n))^{-1}$$

holds, where $\ell : \mathbb{Z}_p \to \{1, 2, ..., p\}$ assigns to $x \in \mathbb{Z}_p$ its residue $\in \{1, 2, ..., p\}$ modulo $p\mathbb{Z}_p$

Proof. Let $n \in \mathbb{N}$. Using Definition 1.1 we have

$$\begin{aligned} B_p(n, n)B_p(n + \frac{1}{2}, n + \frac{1}{2}) &= \frac{\Gamma_p(n)\Gamma_p(n)}{\Gamma_p(2n)} \frac{\Gamma_p(n + \frac{1}{2})\Gamma_p(n + \frac{1}{2})}{\Gamma_p(2n + 1)} \\ &= \frac{(\Gamma_p(n)\Gamma_p(n + \frac{1}{2}))^2}{\Gamma_p(2n)\Gamma_p(2n + 1)}. \end{aligned}$$

From the equalities (1.1) and (1.5) we get that

$$\begin{aligned} B_p(n, n)B_p(n + \frac{1}{2}, n + \frac{1}{2}) &= \frac{(\Gamma_p(\frac{1}{2})\Gamma_p(2n)2^{-(2n-1)}2^{\left[\frac{2n-1}{p}\right]})^2}{\Gamma_p(2n)\Gamma_p(2n)h_p(2n)} \\ &= \frac{(\Gamma_p(\frac{1}{2}))^2(\Gamma_p(2n))^2 2^{-2(2n-1)+2\left[\frac{2n-1}{p}\right]})}{(\Gamma_p(2n))^2 h_p(2n)} \\ &= \frac{(\Gamma_p(\frac{1}{2}))^2 2^{-2(2n-1)+2\left[\frac{2n-1}{p}\right]}}{h_p(2n)}. \end{aligned}$$

Finally, by using the equality (1.3) we obtain

$$B_p(n,n)B_p(n+\frac{1}{2},n+\frac{1}{2}) = (-1)^{\ell(\frac{1}{2})} 2^{-2(2n-1)+2\left[\frac{2n-1}{p}\right]} (h_p(2n))^{-1}.$$

□

References

[1] F. Baldassarri, Etale and crystalline beta and gamma functions via Fontaine's periods, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* **17**, no. 2, 175-198 (2006).

[2] D. Barsky, On Morita's p-adic gamma function, *Math. Proc. Cambridge Philos. Soc.* **89**, no. 1, 23-27 (1981).

[3] M. Boyarsky, p-adic gamma functions and Dwork cohomology, *Trans. Amer. Math. Soc.* **257**, no. 2, 359-369 (1980).

[4] H. Cohen and E. Friedman, Raabe's formula for p-adic gamma and zeta functions, *Ann. Inst. Fourier (Grenoble)* **58**, no. 1, 363-376 (2008).

[5] J. Diamond, The p-adic log gamma function and p-adic Euler constants, *Trans. Amer. Math. Soc.* **233**, 321-337 (1977).

[6] B. Dwork, A note on p-adic gamma function, *Study group on ultrametric analysis, 9th year: 1981/82*, no. 3 (Marseille, 1982), Exp. No. J5, 10 pp., Inst. Henri Poincaré, Paris, 1983.

[7] B. H. Gross and N. Koblitz, Gauss Sums and the p-adic Γ-function, *Ann. of Math. (2)*, vol. **109**, no. 3, 569-581 (1979).

[8] T. K. Kim, A note on analogue of gamma functions, in *Proc. Confer. on 5th Trancendental Number Theory* **5**, no.1, 111-118, Gakushin Univ., Tokyo, Japan, 1997.

[9] H. Menken and Ö. Çolakoğlu, Some properties of the p-adic beta function, *Eur. J. Pure Appl. Math.*, Vol. **8**, no.2, 214-231 (2015)

[10] Y. Morita, A p-adic analogue of the Γ-function, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **22**, no.2, 225- 266 (1975).

[11] W. H. Schikhof, *Ultrametric Calculus: An Introduction to p-adic Analysis*, Cambridge University Pres, 1984.

[12] I. Shapiro, Frobenius map and the p-adic gamma function, *J. Number Theory* **132**, no.8, 1770-1779 (2012).

Author information

Hamza Menken and Özge Çolakoğlu, Mersin University, Science and Arts Faculty, Department of Mathematics, Ciftlikkoy Campus, 33343, Mersin, Turkey.
E-mail: hmenken@mersin.edu.tr;ozgecolakoglu@mersin.edu.tr

Received: March 10, 2015.

Accepted: April 7, 2015.