Palestine Journal of Mathematics

Vol. 3(1) (2014) , 61–69

On some properties of W-curvature tensor

Zafar Ahsan and Musavvir Ali

Communicated by Jose Luis Lopez-Bonilla

MSC 2010 Classifications: 53C25,53C50, 83C20, 83C50. 11R32.

Keywords and phrases: curvature tensors, collineation, electromagnetic fields.

Abstract. Relationship between W-curvature tensor and its divergence with that of other curvature tensors has been established. A symmetry of the spacetime named as W-collineation, has been introduced and conditions under which the spacetimes of general relativity may admit such collineations are obtained.

1 Introduction

The construction of gravitational potentials satisfying Einstein's field equations is the principal aim of all investigations in gravitational physics and this has been often been achieved by imposing symmetries on the geometry compatible with the dynamics of the chosen distribution of matter. The geometrical symmetries of the spacetime are expressible through the vanishing of the Lie derivative of certain tensors with respect to a vector. This vector may be time-like, space-like or null. The role of symmetries in general theory of relativity has been introduced by Katzin, Levine and Davis in a series of papers ([11] - [13]). These symmetries, also known as collineations, were further studied by Ahsan ([1] - [5]), Ahsan and Ali [7] and Ahsan and Husain [9].

In the differential geometry of certain F-structures, W-curvature tensor has been studied by a number of workers especially by Pokhriyal [16] for a Sasakian manifold; while for a P-Sasakian manifold Matsumoto et al [14] have studied this tensor. Shaikh et al [18] have introduced the notion of weekly W_2-symmetric manifolds in terms of W_2-tensor and studied their properties along with numerous non-trivial examples. The role of W_2-tensor in the study of Kenmotsu manifolds has been investigated by Yildiz and De [23] while N(k)-quasi Einstein manifolds satisfying the conditions $R(\xi, X).W_2 = 0$ have been considered by Taleshian and Hosseinzadeh [20]. Most recently, Venkatesha et al [21] have studied Lorentzian para-Sasakian manifolds satisfying certain conditions on W-curvature tensor. Motivated by the all important role of W-curvature tensor in the study of certain differential geometric structures, Ahsan et al. [8] have made a detailed study of this tensor on the spacetime of general relativity.

The purpose of this paper is to develop the relationships between the divergences of W, projective, conformal, conharmonic and concircular curvature tensors and to introduce a symmetry property of spacetime of general relativity, known as W-collineation, defined through the vanishing of Lie derivative of W-curvature tensor with respect to a vector field. The divergences are given in Section 3; while in Section 4, we have discussed W-collineation with some results and the cases of non-null and null electromagnetic fields are discussed in this context. Finally, in Section 5 summary of the work is given.

2 Preliminaries

So far more than twenty six different types of collineations have been studied and the literature on such collineations is very large and still expanding with results of elegance (cf., [5]). However, here we shall mention only those symmetry assumptions that are required for our investigations and we have

Definition 2.1. A spacetime is said to admit motion if there exists a vector field ξ^a such that

$$\pounds_\xi g_{ab} = \xi_{a;b} + \xi_{b;a} = 0 \tag{2.1}$$

Equation (2.1) is known as Killing equation and vector ξ^a is called a Killing vector field.

Definition 2.2. A spacetime admits curvature collineation if there is a vector field ξ^a such that

$$\pounds_\xi R^a_{bcd} = 0, \tag{2.2}$$

where R^a_{bcd} is the Riemann curvature tensor.

Definition 2.3. A spacetime is said to admit Ricci collineation if there is a vector field ξ^i such that

$$\pounds_\xi R_{ab} = 0, \tag{2.3}$$

where R_{ab} is the Ricci tensor.

Definition 2.4. Infinitesimal transformation

$$'\xi^x = \xi^x + v^x dt \tag{2.4}$$

is called WCC if and only if

$$\pounds_\xi C^a_{bcd} = 0, \ \ (n > 3) \tag{2.5}$$

Definition 2.5. A spacetime admits Weyl projective collineation if there is a vector field ξ^a such that

$$\pounds_\xi P^a_{bcd} = 0, \tag{2.6}$$

where P^a_{bcd} is the Projective curvature tensor.

Definition 2.6. The electromagnetic field inherits the symmetry property of spacetime defined by

$$\pounds_\xi F_{ab} = F_{ab;c}\xi^c + F_{ac}\xi^c_{;b} + F_{bc}\xi^c_{;a} = 0 \tag{2.7}$$

where F_{ab} is the electromagnetic field tensor then transformation (2.4) be called as Maxwell collineation.

3 Divergence of W-tensor and other curvature tensors

In this section, we shall express W-curvature tensor in terms of projective, conformal, conharmonic and concircular curvature tensor and obtain the relationships between the divergence of W-tensor and these curvature tensors.

In 1970 Pokhariyal and Mishra [17] have introduced a W-curvature tensor or W_2-curvature tensor and studied its properties and this tensor is defined as

$$W_2(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}[g(X,Z)\text{Ric}(Y,T) - g(Y,Z)\text{Ric}(X,T)] \tag{3.1}$$

Or, in local coordinates

$$W_{abcd} = R_{abcd} + \frac{1}{n-1}[g_{ac}R_{bd} - g_{bc}R_{ad}] \tag{3.1a}$$

This satisfies the following properties:

$$W_{abcd} = -W_{bacd},\ W_{abcd} \neq -W_{abdc},\ W_{abcd} \neq W_{cdab} \tag{3.2}$$

$$W_{abcd} + W_{bcad} + W_{cabd} = 0 \tag{3.3}$$

The W-curvature tensor for the spacetime of general relativity, taking $n = 4$ in equation (3.1a) and contracting with g^{ah} is given by

$$W^h_{bcd} = R^h_{bcd} + \frac{1}{3}[\delta^h_c R_{bd} - g_{bc}R^h_d] \tag{3.4}$$

The Bianchi identities are given by

$$R^h_{bcd;e} + R^h_{bde;c} + R^h_{bec;d} = 0 \tag{3.5}$$

Contracting Equation (3.5) over h and e, using the symmetry properties of the Riemann curvature tensor, we get

$$R^h_{bcd;h} = R_{bc;d} - R_{bd;c} \tag{3.6}$$

Now from Equation (3.4) we have

$$W^h_{bcd;e} = R^h_{bcd;e} + \frac{1}{3}(\delta^h_c R_{bd;e} - g_{bc}R^h_{d;e}) \tag{3.7}$$

so that the divergence of W-curvature tensor is given by

$$W^h_{bcd;h} = R^h_{bcd;h} + \frac{1}{3}(R_{bd;c} - g_{bc}R_{,d}) \tag{3.8}$$

3.1 Projective curvature tensor

For a Riemannian space V_4, the projective curvature tensor P^h_{bcd} is defined as

$$P^h_{bcd} = R^h_{bcd} - \frac{1}{3}(R_{bc}\delta^h_d - R_{bd}\delta^h_c) \tag{3.9}$$

It may be noted that the contraction of P^h_{bcd} over h and d vanishes. Also

$$P_{abcd} = R_{abcd} - \frac{1}{3}(g_{ad}R_{bc} - g_{ac}R_{bd}) \tag{3.10}$$

From Equation (3.9), the covariant derivative of projective curvature tensor is given by

$$P^h_{bcd;e} = R^h_{bcd;e} - \frac{1}{3}(R_{bc;e}\delta^h_d - R_{bd;e}\delta^h_c) \tag{3.11}$$

so that the divergence of projective curvature tensor can be expressed as

$$P^h_{bcd;h} = R^h_{bcd;h} - \frac{1}{3}(R_{bc;d} - R_{bd;c}) \tag{3.12}$$

which on using Equation (3.6) leads to

$$P^h_{bcd;h} = \frac{2}{3}(R_{bc;d} - R_{bd;c}) \tag{3.13}$$

Remark 3.1. From Equation (3.13) it is evident that
(a) The divergence of projective curvature tensor vanishes if and only if the Ricci tensor is of Codazzi type.
(b) The divergence of projective curvature tensor vanishes for Einstein spaces.

From equations (3.8) and (3.12), the divergence of W-tensor in terms of the divergence of projective curvature tensor can be expressed as

$$W^h_{bcd;h} = P^h_{bcd;h} + \frac{2}{3}(R_{bc;d} - g_{bc}R_{,d}) \tag{3.14}$$

we thus have the following

Theorem 3.1. *For a V_4 of constant curvature,*

$$W^h_{bcd;h} = P^h_{bcd;h} \tag{3.15}$$

3.2 Conformal curvature tensor

The conformal curvature tensor C_{abcd} (also known as Weyl conformal curvature tensor), for a V_4 is defined through the equation

$$\begin{aligned} R_{abcd} = \quad & C_{abcd} + \frac{1}{2}(g_{ac}R_{bd} + g_{bd}R_{ac} - g_{ad}R_{bc} - g_{bc}R_{ad}) \\ & + \frac{R}{6}(g_{ad}g_{bc} - g_{ac}g_{bd}) \end{aligned} \tag{3.16}$$

so that

$$R^h_{bcd} = C^h_{bcd} + \frac{1}{2}(\delta^h_c R_{bd} + g_{bd}R^h_c - \delta^h_d R_{bc} - g_{bc}R^h_d) + \frac{R}{6}(\delta^h_d g_{bc} - g^h_c g_{bd}) \tag{3.17}$$

Taking the covariant derivative of Equation (3.17) so that the divergence of conformal curvature tensor can be expressed as

$$R^h_{bcd;h} = C^h_{bcd;h} + \frac{1}{2}(R_{bd;c} - R_{bc;d}) + \frac{2}{3}(g_{bd}R_{,c} - g_{bc}R_{,d}) \tag{3.18}$$

From equations (3.8) and (3.18), the divergence of W-tensor and conformal curvature tensor are related through the equation

$$W^h_{bcd;h} = C^h_{bcd;h} + \frac{5}{6}R_{bd;c} - \frac{1}{2}R_{bc;d} + \frac{2}{3}g_{bd}R_{,c} - g_{bc}R_{,d} \tag{3.19}$$

and we have

Theorem 3.2 For a V_4 of constant curvature, the divergences of W-tensor and Weyl conformal tensor are identical.

3.3 Conharmonic curvature tensor

For a V_4, the conharmonic curvature tensor L^h_{bcd} is defined as ([19])

$$L^h_{bcd} = R^h_{bcd} - \frac{1}{2}(g_{bc}R^h_d + \delta^h_d R_{bc} - \delta^h_c R_{bd} - g_{bd}R^h_c) \tag{3.20}$$

so that the divergence of L^h_{bcd} is given by

$$L^h_{bcd;h} = R^h_{bcd;h} - \frac{1}{2}(g_{bc}R^h_{d;h} + \delta^h_d R_{bc;h} - \delta^h_c R_{bd;h} - g_{bd}R^h_{c;h}) \tag{3.21}$$

Now from equations (3.8) and (3.21), we have

$$W^h_{bcd;h} = L^h_{bcd;h} + \frac{1}{6}(g_{bc}R_{,d} - R_{bd;c}) + \frac{1}{2}(R_{bc;d} - g_{bd}R_{,c}) \tag{3.22}$$

Thus we have the following

Theorem 3.3. *For a V_4 of constant curvature*

$$W^h_{bcd;h} = L^h_{bcd;h} \tag{3.23}$$

3.4 Concircular curvature tensor

The Concircular curvature tensor M_{abcd}, for a V_4 is defined as ([10])

$$M_{abcd} = R_{abcd} - \frac{R}{12}(g_{bc}g_{ad} - g_{bd}g_{ac}) \tag{3.24}$$

Also

$$M^h_{bcd} = R^h_{bcd} - \frac{R}{12}(g_{bc}\delta^h_d - g_{bd}\delta^h_c) \tag{3.25}$$

so that the divergence of concircular curvature tensor is

$$M^h_{bcd;h} = R^h_{bcd} - \frac{1}{12}(g_{bc}R_{,d} - g_{bd}R_{,c}) \tag{3.26}$$

Using equations (3.8) and (3.26), we get

$$W^h_{bcd;h} = M^h_{bcd;h} - \frac{1}{3}R_{bd;c} - \frac{1}{4}(g_{bc}R_{,d} + \frac{1}{3}g_{bd}R_{,c}) \tag{3.27}$$

We thus have the following

Theorem 3.4. For a V_4 of constant curvature, the divergence of W-tensor and concircular curvature tensor are identical.

Remark 3.2. Since a space of constant curvature is an Einstein space, therefore from the above discussions (cf., Theorems 3.1 - 3.4) it is evident that for Einstein spaces, the divergence of W-curvature tensor is identical to the divergence of projective, conformal, conharmonic and concircular curvature tensors although all the five curvature tensors have different properties.

4 W-Collineation

In this section, we shall define a symmetry property in terms of W-curvature tensor and obtain the condition under which the spacetime of general relativity may admit such symmetry. We have

Definition 4.1. A spacetime is said to admit W-collineation if there is a vector field ξ^a such that

$$\pounds_\xi W^a_{bcd} = 0, \tag{4.1}$$

where W^a_{bcd} is W-curvature tensor defined through the equation (3.4) (For detailed study of Lie derivatives and collineations see [22]).

Taking Lie derivative of Equation (3.4) with respect to vector field ξ

$$\pounds_\xi W^h_{bcd} = \pounds_\xi R^h_{bcd} + \frac{1}{3}\pounds_\xi[\delta^h_c R_{bd} - g_{bc}R^h_d] \tag{4.2}$$

Using the definitions and properties of Lie derivative ([6]), equation (4.2) leads to

$$\begin{aligned} \pounds_\xi W^h_{bcd} &= (\xi^m R^h_{bcd;m} - R^m_{bcd}\xi^h_{;m} + R^h_{mcd}\xi^m_{;b} + R^h_{bmd}\xi^m_{;c} \\ &+R^h_{bcm}\xi^m_{;d}) + \frac{1}{3}[\delta^h_c(\xi^m R_{bd;m} + R_{bm}\xi^m_{;d} + R_{md}\xi^m_{;c}) \\ &-(\xi_{b;c}+\xi_{c;b})R^h_d - g_{bc}(\xi^m R^h_{d;m} - R^m_d\xi^h_{;m} + R^h_m\xi^m_{;d})] \end{aligned} \tag{4.3}$$

Katzin et al. [11] have given the relationship chart of different symmetry properties of a spacetime manifold and from that chart we can have following

Lemma 4.1. Every motion in V_n imply curvature collineation (CC), Weyl projective collineation (WPC) and Weyl conformal collineation (W conf C).

In terms of Weyl projective curvature tensor P^h_{bcd}, the Riemann tensor is

$$R^h_{bcd} = P^h_{bcd} + \frac{1}{3}(R_{bc}\delta^h_d - R_{bd}\delta^h_c) \tag{4.4}$$

Using equations (3.4) and (4.4), the expression for W-curvature tensor come out to be

$$W^h_{bcd} = P^h_{bcd} + \frac{1}{3}(R_{bc}\delta^h_d - g_{bc}R^h_d) \tag{4.5}$$

Similarly for Weyl conformal tensor, we can write

$$\begin{aligned} W^h_{bcd} &= C^h_{bcd} + \frac{1}{2}(g_{bd}R^h_c - \delta^h_d R_{bc}) \\ &+\frac{5}{6}(\delta^h_c R_{bd} - g_{bc}R^h_d) + \frac{R}{6}(g_{ad}g_{bc} - g_{ac}g_{bd}) \end{aligned} \tag{4.6}$$

where C^h_{bcd} Weyl conformal curvature tensor and $R = g^{ab}R_{ab}$ is scalar curvature.

Thus it is clear from equations (4.5) and (4.6) that motion and RC equate the Lie derivative of W-curvature tensor with that of Weyl projective and conformal curvature tensor. By using

Lemma 4.1, Equation (4.3) or (4.5) or (4.6) gives the following

Theorem 4.1. A spacetime admits W-collineation if it admits motion and Ricci collineation (RC).

Corollary 4.1. An empty spacetime ($R_{ij} = 0$) admits W-collineation if it admits motion.

For non-null electromagnetic field, the energy momentum tensor T_{ab} is expressed as

$$T_{ab} = F_{am}F_b^m - \frac{1}{4}g_{ab}F_{ij}F^{ij} \tag{4.7}$$

and Einstein equation for purely electromagnetic distribution are

$$R_{ij} = kT_{ij} \tag{4.8}$$

Using equations (4.7) and (4.8) in Equation (4.6), we get

$$\begin{aligned} W^h_{bcd} &= C^h_{bcd} + (\frac{k}{3} + \frac{R}{6})(\delta^h_d g_{bc} - \delta^h_c g_{bd})F_{ij}F^{ij} \\ &+\frac{k}{2}[(g_{bd}F^h_m F^m_c - \delta^h_d F_{bm}F^m_c) + \frac{5}{3}(\delta^h_c F_{bm}F^m_d - g_{bc}F^h_m F^m_d)] \end{aligned} \tag{4.9}$$

Lemma 4.2 [15] In a non-null electromagnetic field, the Lie derivative of electromagnetic field tensor F_{ab} with respect to a vector field ξ, vanishes if ξ is Killing vector.

The use of Lemmas 4.1 and 4.2 in equation (4.7), leads the following

Theorem 4.2. If a non-null electromagnetic field admits motion then it does admit W-collineation.

Remark 4.1. Similar result can also be obtained by using Equations (4.5), (4.6) and (4.7) and Lemmas 4.1 & 4.2.

The energy-momentum tensor for a null electromagnetic is given by

$$T_{ab} = F_{an}F^n_b \tag{4.10}$$

where $F_{ij} = s_i t_j - t_i s_j$ and $s_i s^i = s_i t^i = 0,\ t_i t^i = 1$, vectors s and t are the propagation and polarization vectors, respectively.

Now Using equations (4.8) and (4.10) in equation (4.6) we get

$$\begin{aligned} W^h_{bcd} &= C^h_{bcd} + \frac{k}{2}(g_{bd}F^h_n F^n_c - \delta^h_d F_{bn}F^n_c) + \frac{5}{3}(\delta^h_c F_{bn}F^n_b - g_{bc}F^h_n F^n_d) \\ &+\frac{R}{6}(\delta^h_d g_{bc} - \delta^h_c g_{bd}) \end{aligned} \tag{4.11}$$

It is known that

Lemma 4.3. [1] A null electromagnetic field admits Maxwell collineation along the propagation (polarization) vector if the propagation (polarization) vector is Killing and expansion-free.

From Equation (4.11) and Lemmas 4.1 and 4.3, the Lie derivative of W-curvature tensor with respect to propagation (polarization) vector, vanishes. Thus we can state

Theorem 4.3. A null electromagnetic field admits W-collineation along a propagation (polarization) vector if propagation (polarization) vector is Killing and expansion-free.

Remark 4.2. A number similar results can be obtained for W-collineation as W-curvature tensor can be expressed in terms of other curvature tensors like concircular curvature tensor and conharmonic curvature tensor. (cf., [8])

5 Summary

In this paper an attempt has been made to investigate the relationship between the divergence of W-curvature tensor and other curvature tensors especially projective, conformal, concircular and conharmonic curvature tensor. Also we have introduce the notion of W-collineation and have obtained theconditions under which the spacetime of general relativity may the W-collineation. The cases of non-null and null electromagnetic field has also been discussed.

References

[1] Ahsan, Z.:*Tamkang Journal of Mathematics,* 9 No. 2 (1978) 237.

[2] Ahsan, Z.: *Journal of Math. Phys. Sci.*, 21 No.5 (1987).

[3] Ahsan, Z.: *Acta Phys. Sinica* 4, (1995) 337.

[4] Ahsan, Z.: *Braz. J. Phys.* 26 No.3 (1996) 572-576.

[5] Ahsan, Z.: *Bull. Call. Math. Soc.*, 97 No.3 (2005) 191.

[6] Ahsan, Z. : *"Tensor Analysis with Application"* Anshan Ltd, Tunbridge Wells, United Kingdom, 2008.

[7] Ahsan, Z. and Ali, M.: *Int. Jour. of Theo. Phys.*, 51 (2012) 2044-2055.

[8] Ahsan, Z., Ali, M. and Sirajuddin *"On a Curvature Tensor for the Spacetime of General Relativity"* (2012) Bull. Malay. Math. Soc. (accepted).

[9] Ahsan, Z. and Husain, S.I.: *Annali di Mathematical Pura ed Applicata*, 26(1980)339.

[10] Ahsan, Z. and Siddiqui, S.A.: *Int. Jour. of Theo. Phys.*, 48 (2009) 3202-3212.

[11] Katzin, G.H., Livine, J. and Davis, W.R.: *Journal of Mathematical Physics* 10 (1969) 617-620.

[12] Katzin, G.H., Livine, J. and Davis, W.R.: *Journal of Mathematical Physics* 11 (1970) 1875.

[13] Katzin, G.H. and Livine, J.: *Colloquium Mathematicum* XXVI 26 (1972) 21, *Tensor (N.S.)*, 22 (1971) 64.

[14] Matsumoto, K., Ianus, S. and Milrai, I. : *Publicationes Mathematicae Debrecen,* 33(3-4)199-204, 1986.

[15] Michalaski, H. and Wainwright, J.: *GRG* 6 (1975)289.

[16] Pokhriyal, G.P. : *Tensor,* 36(2) 222-226, 1982.

[17] Pokhriyal, G.P. and Mishra, R.S. :*Yokohama Math. Journal* 18,105-108, 1970.

[18] Shaikh, A.A., Jana, S.K. and Eyasmin, S. : *Sarajevo Journal of Mathematics,* 3(15)73-91, 2007.

[19] Siddiqui, S.A. and Ahsan, Z.: *Diff. Geom. and Dynamical System.*, 12 (2010) 213-220.

[20] Teleshian, A. and Hosseinzadeh, A.A. : *Journal of Mathematics and Computer Science,* 1(1), 28-32, 2010.

[21] Venkatesha, Bagewadi, C.S., and Pradeep Kumar, K.T. : *ISRN Geometry 2011, Article ID 161523, 9 pages, doi: 10.5402/2011/161523.*

[22] Yano, K.: *"The theory of Lie derivatives and its Application"* Vol. III, North Holand publishing co. Amsterdom p. Noordhoff L.T.D. Groningen(1957).

[23] Yildiz, A. and De, U.C. : *Diff. Geom-Dynamical Systems,* 12, 289-298, 2010.

Author information

Zafar Ahsan and Musavvir Ali, DEPARTMENT OF MATHEMATICS, ALIGARH MUSLIM UNIVERSITY, ALIGARH-202002, INDIA.
E-mail: zafar.ahsan@rediffmail.com musavvir.alig@gmail.com

Received: January 28, 2013

Accepted: April 21, 2013