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Additively Regular Rings and Marot Rings

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Abstract. An ideal I of a commutative ring is regular if it contains a regular element (an element that is not a zero divisor). If each regular ideal of a ring is generated by its regular elements, then the ring is said to be a Marot ring. Also, a ring R is additively regular if for each pair of elements $a, b \in R$ with b regular, there is an element $r \in R$ such that $a + br$ is a regular element of R. Of primary interest here are rings with only finitely many regular maximal ideals (but perhaps infinitely many maximal ideals that are not regular). For example, if R is additively regular and has only finitely many regular maximal ideals, then each invertible ideal is principal. In contrast, an example is given of a Marot ring R with exactly two regular maximal ideals where each regular maximal ideal is invertible, but neither is principal.

1 Introduction

Throughout the paper R denotes a commutative ring with identity and $T(R)$ denotes its total quotient ring. Also we let $Z(R)$ denote the set of zero divisors of R and refer to an element that is not contained in $Z(R)$ as being regular. A regular ideal is an ideal that contains at least one regular element. We let $\text{Max}(R)$ denote the set of maximal ideals of R and let $\text{Max}(R, I)$ denote the set of maximal ideals (of R) that contain the ideal I.

A *Marot ring* is a ring in which each regular ideal can be generated by the regular elements it contains (see for example [7]). A related concept is that of an additively regular ring, R is *additively regular* if for each $t \in T(R)$ there is an element $r \in R$ such that $t + r$ is a regular element (of $T(R)$). An equivalent formulation is that for $a, b \in R$ with b regular, there is an element $s \in R$ such that $a + sb$ is regular (alternately, $a \in T(R)$ and $b \in R\backslash Z(R)$) [see [6, Lemma 7]]. It is clear that an additively regular ring is Marot. Examples exist to show the concepts are not equivalent (see for example [7, Example 12, Section 27]).

For a multiplicatively closed subset S of a ring R, the *regular ring of quotients* of R with respect to S is the ring $R_{(S)} = \{t \in T(R) \mid ts \in R \text{ for some regular } s \in S\}$. A second ring of quotients related to S is the *large ring of quotients* $R_{[S]} = \{t \in T(R) \mid ts \in R \text{ for some } s \in S\}$. It is clear that $R_{(S)} \subseteq R_{[S]}$, but the containment can be proper. In the case $S = R\backslash P$ for some prime P of R, one uses $R_{(P)}$ and $R_{[P]}$ in place of $R_{(S)}$ and $R_{[S]}$. If R is a Marot ring, then $R_{(P)} = R_{[P]}$ [13, Proposition 6]. However, as we will see below it is possible to have proper containment $R_{(S)} \subsetneq R_{[S]}$ even if S is the complement of a finite set of regular maximal ideals in a Marot ring [Example 2.5].

For an ideal I of R and multiplicative set S, the “simple" extension $IR_{(S)}$ is the same as the set $\{t \in R_{(S)} \mid ts \in I \text{ for some regular element } s \in S\}$. On the other hand the ideal $[I]R_{[S]} = \{t \in R_{[S]} \mid ts \in I \text{ for some } s \in S\}$ may be larger than the simple extension $IR_{[S]}$. For example, if $ts = 0$ for some $s \in S$, then $t \in [I]R_{[S]}$ for each ideal I of R. If I is a regular ideal of a Marot ring R and P is a prime ideal that does not contain I, then $IR_{(P)} = R_{(P)}$ as the Marot property guarantees the existence of a regular element in $I\backslash P$. Hence if Q is a regular prime of R and R is Marot, then $QR_{(Q)}$ is the unique regular maximal ideal of $R_{(Q)}$. A simple conductor argument shows that if $R \neq T(R)$, then $I = \bigcap\{IR_{(M)} \mid M \in \text{RMax}(R)\}$ for each ideal I of R where $\text{RMax}(R)$ denotes the (nonempty) set of regular maximal ideals of R. Note that $R = R_{(M)}$ in the event M is the unique regular maximal ideal of R. Also $R_{(P)} = T(R)$ if P is a prime ideal that is not regular.

Recall that a ring R is a *Prüfer ring* if each finitely generated regular ideal is invertible. Also it is a *regular Bézout ring* if each finitely generated regular ideal is principal. The ring in [2, Example 3.6] is a Prüfer ring with a unique regular maximal ideal and this ideal is invertible but not principal. Thus this ring is not regular Bézout. However, if R is a Marot Prüfer ring with a unique regular maximal ideal, then it is a regular Bézout ring [Corollary 2.2]. In contrast, the ring R in Example 2.4 is a Marot Prüfer ring with two regular maximal ideals where each regular ideal is invertible, but R is not regular Bézout. In particular, both regular maximal ideals are invertible yet neither is principal. For additively regular rings, Theorem 3.5 shows that if R is an additively regular ring with only finitely many regular maximal ideals, then each invertible ideal is principal. Hence an additively regular Prüfer ring with only finitely many regular maximal ideals is a regular Bézout ring.

For a ring V with prime ideal P, (V,P) is said to be a *valuation pair* of $T(V)$ if for each $t \in T(V) \backslash V$, there is an element $p \in P$ such that $tp \in V \backslash P$. Corresponding to such a pair there is a totally ordered Abelian group G and a corresponding surjective “valuation" map v : $T(V) \to G \bigcup \{\infty\}$ such that for all $a, b \in T(V)$:

(1) $\mathrm{v}(ab) = \mathrm{v}(a) + \mathrm{v}(b)$ (with $g < \infty = \infty + g = \infty + \infty$ for each $g \in G$),

(2) $\mathrm{v}(a+b) \geq \min\{\mathrm{v}(a), \mathrm{v}(b)\}$, and

(3) $V = \{t \in T(V) \mid \mathrm{v}(t) \geq 0\}$ and $P = \{t \in T(V) \mid \mathrm{v}(t) > 0\}$.

As with domains, V is referred to as a rank one discrete valuation ring if $G = \mathbb{Z}$. For a ring R, R is a Prüfer ring if and only if $(R_{[M]}, [M]R_{[M]})$ is a valuation pair for each (regular) maximal ideal M (see, for example, [7, Theorem 6.2]).

2 Marot Rings

We start with the case of a Marot ring with a unique regular maximal ideal.

Theorem 2.1. *If R is a Marot ring with a unique regular maximal ideal M, then each invertible ideal is principal.*

Proof. Suppose M is the unique regular maximal ideal of the Marot ring R and that I is an invertible ideal. Then I is regular and thus M is the only maximal ideal of R that contains I. Moreover, IR_M is invertible and thus principal. Since R is Marot, I can be generated by a finite set of regular elements and thus there is a regular element $s \in I$ such that $IR_M = sR_M$.

Since M is the only regular maximal ideal, $\mathrm{Max}(R,I) = \{M\}$ and $s/1$ is a unit in R_N for each maximal ideal $N \neq M$. Thus $sR_N = R_N = IR_N$. Therefore $I = sR$. □

Corollary 2.2. *If R is Marot ring with a unique regular maximal ideal M, then R is a Prüfer ring if and only if it is a regular Bézout ring.*

One might hope that an invertible ideal in a Marot ring with only finitely many regular maximal ideals would be principal, but this need not be the case even when there are only two regular maximal ideals (and each of these is invertible).

For the examples in this section we make use of the $A+B$ construction. Let D be an integral domain and let $\mathcal{P}$ be a nonempty set of prime ideals with index set $\mathcal{A}$. Next let $\mathcal{I} = \mathcal{A} \times \mathbb{N}$ and for each $i = (\alpha, n) \in \mathcal{I}$ we let $K_i = K_\alpha$ be the quotient field of D/P_α. For $B = \sum K_i$, we form a ring $R = D + B$ from $D \times B$ by defining addition and multiplication as $(r,b) + (s,c) = (r+s, b+c)$ and $(r,b)(s,c) = (rs, rb+sc+bc)$. We refer to R as the *ring of the form $A+B$ corresponding to D and $\mathcal{P}$*. For each $r \in D$, $b \in B$ and $i \in \mathcal{I}$, we let r_i denote the image of r in K_i and let b_i be the ith component of b. In the next theorem we recall some basic properties of these rings. Note that statements (7) and (8) are new. We assume all of the notation above.

Theorem 2.3. [cf. [9, Theorems 8.3 & 8.4]] *Let $\mathcal{P}$ be a nonempty set of prime ideals of a domain D and let $R = D + B$ be the $A+B$ ring corresponding to D and $\mathcal{P}$.*

(1) For each $i \in \mathcal{I}$, the set $M_i = \{(r,b) \in R \mid r_i = -b_i\}$ is both a maximal ideal and a minimal prime ideal of R. All other prime ideals of R are of the form $P + B$ for some prime ideal P of D.

(2) *The total quotient ring of R can be identified with the ring $D_S + B$ where $S = D\backslash\bigcup\{P_\alpha \mid P_\alpha \in \mathcal{P}\}$.*

(3) *If I is a finitely generated ideal of D such that $I\bigcap S \neq \emptyset$, then $IR = I + B$ is a finitely generated regular ideal of R. Conversely, if J is a finitely generated regular ideal of R, then $J = I + B = IR$ for some finitely generated ideal I of D such that $I\bigcap S \neq \emptyset$.*

(4) *If I is a finitely generated ideal of D, then IR has a nonzero annihilator if and only if $I \subseteq P_\alpha$ for some $P_\alpha \in \mathcal{P}$.*

(5) *If I is a finitely generated ideal of D, then IR is an invertible of R if and only if I is an invertible ideal of D and $I\bigcap S \neq \emptyset$.*

(6) *If D is a Prüfer domain, then R is a Prüfer ring.*

(7) *Let P be a nonzero prime ideal P of D. If $P\bigcap S = \emptyset$, then $P + B \subseteq Z(R)$ and so $R_{[P+B]} = T(R) = R_{(P+B)}$. If $P\bigcap S \neq \emptyset$, then $PR = P + B$, $R_{(PR)} = D_X + B$ where $X = S\backslash P$ and $R_{[PR]} = (D_P\bigcap D_S) + B$.*

(8) *If P is a prime ideal of D such that D_P is a valuation domain and $P\bigcap S \neq \emptyset$, then $(R_{[PR]}, [P]R_{[PR]})$ is a valuation pair of $T(R)$.*

Proof. For statement (7), if $P\bigcap S = \emptyset$, then it is clear that each element of $P + B$ is a zero divisor. Hence $R_{(P+B)} = T(R) = R_{[P+B]}$. In the case $P\bigcap S \neq \emptyset$, $P + B = PR$ is a regular prime ideal of R. For $R_{(PR)}$, the regular elements of R that are not contained in PR have the form (s, b) where $s \in X = S\backslash P$ (where $s_i + b_i \neq 0$ for all i). Since B is a common ideal of R and $T(R)$, the "b" doesn't matter. If $(f, a) \in R_{(PR)}$, then there is an element $s \in X$ such that $(x, 0)(f, a) = (xf, xa) \in R$. Thus $fx^{-1} \in D_X$. Conversely, if $g \in D_X$, then there are elements $c, w \in D$ with $w \in X$ such that $g = cw^{-1}$. It follows that $(g, d)(w, 0) = (c, wd) \in R$ and thus $(g, d) \in R_{(PR)}$ for each $d \in B$.

Next consider the rings $R_{[PR]}$ and $(D_P\bigcap D_S) + B$. As above, start with an element $(f, a) \in R_{[PR]} \subseteq D_S + B$. Then there is an element $(x, c) \in R\backslash PR$ such that $(f, a)(x, c) \in R$. Since $B \subseteq P$, it must be that x is not in P. As $fx \in D$, we have $f \in D_P\bigcap D_S$ and hence $(f, a) \in (D_P\bigcap D_S) + B$. For the reverse containment, suppose $g \in D_P\bigcap D_S$. Then, as above, there is an element $w \in D\backslash P$ such that $gw = c \in D$. It follows that $(g, d)(w, 0) = (c, wd) \in R$. Hence $(g, d) \in R_{[PR]}$.

For (8), if $P\bigcap S \neq \emptyset$, then we have $PR = P + B$ and $R_{[PR]} = (D_P\bigcap D_S) + B$. To see that $(R_{[PR]}, [P]R_{[PR]})$ when D_P is a valuation domain it suffices to show that if $(f, a) \in T(R)\backslash R_{[PR]}$, then there is an element $(p, c) \in [P]R_{[PR]}$ such that $(f, a)(p, c) \in R_{[PR]}\backslash[P]R_{[PR]}$. Since (f, a) is not in $R_{[PR]}$, there is no element $(r, d) \in R\backslash PR$ such that $(f, a)(r, d) \in R$. Also f is not in D_P. Since D_P is a valuation domain, there is an element $x \in PD_P$ such that $fx \in D_P\backslash PD_P$. Thus there is an element $y \in D\backslash P$ such that both fxy and xy are in D, necessarily with $fxy \in D\backslash P$ and $xy \in P$. The element $(xy, 0) \in PR$ is such that $(f, a)(xy, 0) \in R_{[PR]}\backslash[P]R_{[PR]}$. Hence $(R_{[PR]}, [P]R_{[PR]})$ is a valuation pair of $T(R)$. □

Example 2.4. Let $D = \mathbb{Z}[\sqrt{10}]$. This is a Dedekind domain that is not a PID. Both $M = 2D + \sqrt{10}D$ and $N = 5D + \sqrt{10}D$ are maximal ideals and neither is principal as all three of 2, 5 and $\sqrt{10}$ are irreducible but not prime. Moreover, M is the only maximal ideal that contains 2, N is the only maximal ideal that contains 5, and each maximal ideal that contains $\sqrt{10}$ contains exactly one of 2 and 5. Let $\mathcal{P} = \text{Max}(D)\backslash\{M, N\}$ and let $R = D + B$ be the ring of the form $A + B$ corresponding to D and $\mathcal{P}$.

(1) The only regular maximal ideals are $MR = M + B$ and $NR = N + B$.

(2) R is a Prüfer ring, but it is not a regular Bézout ring.

(3) Both $M + B$ and $N + B$ are invertible, but neither is principal.

(4) R is a Marot ring with exactly two regular maximal ideals where some invertible ideals are not principal.

(5) The regular elements of MR are contained in $M^2R\bigcup MNR$ but MR is not. Similarly, the regular elements of NR are contained in $N^2R\bigcup MNR$ but NR is not.

Proof. It is clear that neither 2 nor 5 divides $\sqrt{10}$, and $\sqrt{10}$ divides neither 2 nor 5. In addition, routine calculations show that 2, 5 and $\sqrt{10}$ are irreducible. On the other hand, $\sqrt{10}^{\,2} = 10 = 2 \cdot 5$. Thus each maximal ideal that contains 2, also contains $\sqrt{10}$. The only elements missing from $2D + \sqrt{10}D$ are the odd integers. Hence $M = 2D + \sqrt{10}D$ is the only maximal ideal that contains 2. Similarly, each maximal ideal that contains 5, also contains $\sqrt{10}$, and we have that $N = 5D + \sqrt{10}D$ is the only maximal ideal that contains 5. It is also the case that each maximal ideal that contains $\sqrt{10}$ must contain at least (exactly) one of 2 and 5. Hence $\mathrm{Max}(D, 2D) = \{M\}$, $\mathrm{Max}(D, 5D) = \{N\}$ and $\mathrm{Max}(D, \sqrt{10}D) = \{M, N\}$. It follows that $(\sqrt{10}, 0)$, $(2, 0)$ and $(5, 0)$ are regular elements of R and that $M + B = MR$ and $N + B = NR$ are regular maximal ideals of R.

Since $\mathcal{P} = \mathrm{Max}(D)\backslash\{M, N\}$, $M+B = MR$ and $N+B = NR$ are the only regular maximal ideals of R. Each of these is invertible, but neither is principal since neither M nor N is principal.

Let J be a regular proper ideal of R. Then $J = I + B = IR$ for some proper ideal I of D with $\mathrm{Max}(D, I) \subseteq \{M, N\}$. Since D is a Dedekind domain, there are nonnegative integers m and n (with $m + n > 0$) such that $I = M^m N^n$. It follows that I is generated by $2^m 5^n$, $2^m \sqrt{10}^{\,n}$, $5^n \sqrt{10}^{\,m}$ and $\sqrt{10}^{\,n+m}$. The corresponding regular elements $(2^m 5^n, 0)$, $(2^m \sqrt{10}^{\,n}, 0)$, $(5^n \sqrt{10}^{\,m}, 0)$ and $(\sqrt{10}^{\,n+m}, 0)$ generate J. Therefore R is a Marot ring, $MR = M + B$, $NR = N + B$ are the only regular maximal ideals and each of these is an invertible ideal that is not principal.

In D, $M^2 = 2D$, $MN = \sqrt{10}D$ and $N^2 = 5D$. If $p \in M$ is not in $M^2 \bigcup MN$, then at least one maximal ideal in the set $\mathcal{P}$ contains p, otherwise we have $MD_M = pD_M$ and $M = \sqrt{pD}$ which together imply $M = pD$, a contradiction. Translating into R, the regular elements of MR are contained in $M^2 R \bigcup MNR$. A similar proof holds for the regular elements of NR. Hence each regular element of NR is contained in $N^2 R \bigcup MNR$. □

Portelli and Spangher showed that if I is a regular ideal in an additively regular ring R and the regular elements of I are contained in a finite union of regular ideals, then I is also contained in this union [13, Proposition 8]. In [11], Matsuda gave an example of a nonreduced Marot ring that does not have this property.

While the ring in the previous example is a Marot Prüfer ring that is not a regular Bézout ring (even though it has only two regular maximal ideals), at least each of the regular maximal ideals contains a regular element that is comaximal with all other maximal ideals. In the next example, we show that a Marot Prüfer ring with exactly four regular maximal ideals can have the property that each regular nonunit is contained in at least two regular maximal ideals. As in Example 2.4, each of the regular maximal ideals is invertible.

Example 2.5. Let D be a Dedekind domain with class group $\mathbb{Z}$ such that there are four nonzero nonunits x_1, x_2, y_1, y_2 that satisfy the following: (i) $x_1 x_2 = y_1 y_2$, (ii) $x_1 D + x_2 D = D = y_1 D + y_2 D$, and (iii) $x_1 D + y_1 D = M_{1,1}$, $x_1 D + y_2 D = M_{1,2}$, $x_2 D + y_1 D = M_{2,1}$ and $x_2 D + y_2 D = M_{2,2}$ are distinct maximal ideals that are not principal. Let $\mathcal{P} = \mathrm{Max}(D)\backslash\{M_{1,1}, M_{1,2}, M_{2,1}, M_{2,2}\}$ and let $R = D + B$ be the ring of the form $A + B$ corresponding to D and $\mathcal{P}$.

(1) R is a Prüfer ring.

(2) $M_{1,1}R = M_{1,1} + B$, $M_{1,2}R = M_{1,2} + B$, $M_{2,1}R = M_{2,1} + B$ and $M_{2,2}R = M_{2,2} + B$ are invertible maximal ideals of R. These are the only regular prime ideals and none are principal.

(3) Each regular nonunit of R is contained in at least two (regular) maximal ideals. Thus for each $M_{i,j}$, the set of regular elements in $M_{i,j}R$ is contained in the union of the other three regular maximal ideals.

(4) R is a Marot ring.

(5) Let S be the complement of the union of any three of the maximal ideals $M_{1,1}R$, $M_{1,2}R$, $M_{2,1}R$ and $M_{2,2}R$. Then S contains no regular nonunits and $R = R_{(S)} \subsetneq R_{[S]}$.

Proof. By (i) and (ii), for each $i \in \{1, 2\}$, a maximal ideal that contains x_i contains exactly one of y_1 and y_2. Similarly, for each $j \in \{1, 2\}$, a maximal ideal that contains y_j contains exactly one of x_1 and x_2. Hence $M_{i,1}$ and $M_{i,2}$ are the only maximal ideals that contain x_i and $M_{1,j}$ and $M_{2,j}$ are the only maximal ideals that contain y_j.

Since the class group of D is $\mathbb{Z}$, no positive power of a $M_{i,j}$ is principal. Thus each principal ideal that is contained in at least one $M_{i,j}$ is contained in at least one other maximal ideal (for $x_i D$ and $y_j D$, each is in exactly one other maximal ideal).

Since D is a Dedekind domain, R is a Prüfer ring. All four of $M_{1,1}R$, $M_{1,2}R$, $M_{2,1}R$ and $M_{2,2}R$ are invertible and none are principal. Also note that for each positive integer n, $M_{i,j}^n$ can be generated by the set $\{x_i^n, y_j^n\}$. Thus $M_{i,j}^n R$ is generated by the regular elements it contains.

Let I be a nonzero proper ideal of D. Then $I = M_1^{k_1} M_2^{k_2} \cdots M_n^{k_n}$ for some (distinct) maximal ideals $M_1, M_2, \ldots, M_n$ and positive integers $k_1, k_2, \ldots, k_n$. If at least one M_m is in the set $\mathcal{P}$, then IR has a nonzero annihilator. On the other hand, if I is a product of powers of the $M_{i,j}$s, then no maximal ideal in $\mathcal{P}$ contains I and so in this case $IR = I + B$ is a regular (invertible) ideal of R. In addition, if $I = M_{1,1}^m M_{1,2}^n M_{2,1}^r M_{2,2}^s$ for some nonnegative integers m, n, r and s, then the product of the generating sets $\{x_1^m, y_1^m\}$, $\{x_1^n, y_2^n\}$, $\{x_2^r, y_1^r\}$ and $\{x_2^s, y_2^s\}$ provides a set of regular elements that generates $IR = I + B$. Hence R is a Marot ring.

Let S be the complement of the union of any three of $M_{1,1}R$, $M_{1,2}R$, $M_{2,1}R$ and $M_{2,2}R$. As noted above, a nonzero element $r \in M_{i,j}$ is contained in at least one other maximal ideal. Thus if (r, b) is a regular element of R, then r is in no maximal ideal $N \in \mathcal{P}$ and (r, b) is in at least two of $M_{1,1}R$, $M_{1,2}R$, $M_{2,1}R$ and $M_{2,2}R$. It follows that S contains no regular nonunits of R and therefore $R_{(S)} = R$. On the other hand, S has a nonempty intersection with the fourth regular maximal ideal. It follows that $R_{[S]}$ contains the dual of this invertible ideal and thus $R_{[S]} \supsetneq R_{(S)} = R$. □

In (the proof of) [3, Theorem 7], Claborn gives a way to take an arbitrary Krull domain D' (not a field) and produce a Dedekind domain D with the same ideal class group where D is a certain localization of the polynomial ring $D'[\mathrm{z}_1, \mathrm{z}_2, \ldots]$. In his construction, each height one prime of D' extends to a maximal ideal of D (and so pairs of elements $a, b \in D'$ are comaximal in D if and only if no height one prime of D' contains both a and b). Thus if we start with the Krull domain $D' = K[\mathrm{x}_1, \mathrm{x}_2, \mathrm{y}_1, \mathrm{y}_2]/(\mathrm{x}_1\mathrm{x}_2 - \mathrm{y}_1\mathrm{y}_2)$ where $\mathrm{x}_1, \mathrm{x}_2, \mathrm{y}_1, \mathrm{y}_2$ are indeterminates over a field K, then the resulting Dedekind domain D has class group $\mathbb{Z}$ (see, for example, the proof of [3, Proposition 6] and [4, Pages 65–66]]). The respective images of $\mathrm{x}_1, \mathrm{x}_2, \mathrm{y}_1$ and y_2 satisfy the desired restrictions for the elements x_1, x_2, y_1 and y_2. Specifically, if we let x_i denote the image of x_i in D and y_j denote the image of y_j, then we have $x_1 x_2 = y_1 y_2$, $x_1 D + x_2 D = D = y_1 D + y_2 D$ and none of the four (distinct) maximal ideals $M_{1,1} = x_1 D + y_1 D$, $M_{1,2} = x_1 D + y_2 D$, $M_{2,1} = x_2 D + y_1 D$ and $M_{2,2} = x_2 D + y_2 D$ are principal.

There are alternate constructions for the base ring D' in the previous paragraph. For example, let $D' = K[\mathrm{x}, \mathrm{y}, \mathrm{xw}, \mathrm{yw}]$ where x, y and w be indeterminates over K. Then the (surjective) homomorphism $\varphi : K[\mathrm{x}_1, \mathrm{x}_2, \mathrm{y}_1, \mathrm{y}_2] \to D'$ defined by $\varphi(f(\mathrm{x}_1, \mathrm{x}_2, \mathrm{y}_1, \mathrm{y}_2) = f(\mathrm{x}, \mathrm{yw}, \mathrm{y}, \mathrm{xw})$ has kernel the principal ideal $(\mathrm{x}_1\mathrm{x}_2 - \mathrm{y}_1\mathrm{y}_2)$. In the proof of the next example (which is simply a more specific one than Example 2.5), we make a few specific choices with regard to the maximal ideals $M_{i,j}$ in defining the set $\mathcal{P}$. The end result, $R = D + B$, is (still) a Prüfer Marot ring with four regular maximal ideals $M_{1,1}R = M_{1,1} + B$, $M_{1,2}R = M_{1,2} + B$, $M_{2,1}R = M_{2,1} + B$ and $M_{2,2}R = M_{2,2} + B$ (again all invertible and none principal). Moreover, we will have that $(x_1, 0) \in M_{1,1}R \bigcap M_{1,2}R$ and $(x_2, 0) \in M_{2,1}R \bigcap M_{2,2}R$ are comaximal regular elements, as are $(y_1, 0) \in M_{1,1}R \bigcap M_{2,1}R$ and $(y_2, 0) \in M_{1,2}R \bigcap M_{2,2}R$, but there are no comaximal regular elements $(f, b) \in M_{1,1}R \bigcap M_{2,2}R$ and $(g, c) \in M_{1,2}R \bigcap M_{2,1}R$.

Example 2.6. Let $D' = K[\mathrm{x}, \mathrm{y}, \mathrm{xw}, \mathrm{yw}]$. This is a Krull domain with class group $\mathbb{Z}$. To construct the corresponding Dedekind domain D, we make explicit use of the construction in [3]. Let $D'' = D'[\mathrm{z}_1, \mathrm{z}_2, \ldots]$ where $\{\mathrm{z}_n\}$ is a countably infinite set of indeterminates over $K(\mathrm{x}, \mathrm{y}, \mathrm{w})$. For a well-chosen set N consisting of prime elements of D'', we obtain a Dedekind domain D with a corresponding set of maximal ideals $\mathcal{P}$ such that $R = A + B$ has exactly four regular maximal ideals $M_{1,1} + B$, $M_{1,2} + B$, $M_{2,1} + B$ and $M_{2,2} + B$ where $M_{i,j} = x_i D + y_j D$ under the identification $x_1 = \mathrm{x}$, $x_2 = \mathrm{yw}$, $y_1 = \mathrm{y}$ and $y_2 = \mathrm{xw}$. Of these four, $(x_1, 0) \in M_{1,1}R \bigcap M_{1,2}R$ and $(x_2, 0) \in M_{2,1}R \bigcap M_{2,2}R$ are regular comaximal elements, as are $(y_1, 0) \in M_{1,1}R \bigcap M_{2,1}R$ and $(y_2, 0) \in M_{1,2}R \bigcap M_{2,2}R$, but there are no comaximal regular elements $f \in M_{1,1}R \bigcap M_{2,2}R$ and $g \in M_{1,2}R \bigcap M_{2,1}R$.

Proof. For each height two prime Q of D'', choose a nonzero element $a \in Q$ and then choose a $b \in Q$ such that no height one prime contains both a and b. Both a and b involve at most finitely many Z_is (and each is in only finitely many height one primes of D''), so for sufficiently large n, $a\mathrm{Z}_n + b$ is a prime element of $D'[\mathrm{Z}_1, \mathrm{Z}_2, \ldots, \mathrm{Z}_n]$ (and of D'') [see, for example, [5, Lemma 45.7] and [14, Proposition 8]]. For X, Y, XW and YW, no height one prime of D' contains both X and YW and none contains both Y and XW. Let N denote the multiplicative set generated by the appropriately defined polynomials "$a\mathrm{Z}_n + b$." We obtain a Dedekind domain $D = D''_N$ (it is a one-dimensional Krull domain since each prime Q of D'' with $ht(Q) > 1$, has a nonempty intersection with N).

Since D' is a Krull domain, each height one prime of D' extends to a height one prime of D'' (that has empty intersection with N), and thus to a maximal ideal of D. By the construction of N, all other nonzero primes of D' blow up in D. Also, in D', it is clear that the set $S' = \{a\mathrm{X}^m\mathrm{Y}^n(\mathrm{XW})^r(\mathrm{YW})^s \mid a \in K^*,\ m, n, r, s$ nonnegative integers$\}$ is a saturated multiplicative subset of D'. In addition, S' is saturated in D''. While some height one primes of D'' (not extended from height one primes of D') have nonempty intersection with N, there are others that don't. In D, the saturation of the set S' defined above is $S = S'_N = \{u\mathrm{X}^m\mathrm{Y}^m(\mathrm{XW})^r(\mathrm{YW})^s \mid$ u a unit of $D = D''_N$ and m, n, r, s nonnegative integers$\}$.

With regard to the notation of Example 2.5 (and the desired conclusion), we identify X with x_1, Y with y_1, YW with x_2 and XW with y_2. Then $M_{1,1} = \mathrm{X}D + \mathrm{Y}D$, $M_{1,2} = \mathrm{X}D + \mathrm{XW}D$, $M_{2,1} = \mathrm{YW}D + \mathrm{Y}D$ and $M_{2,2,} = \mathrm{YW}D + \mathrm{XW}D$. If M is a maximal ideal not among these four, then $M \bigcap S = \emptyset$ since a height one prime of D'' that contains at least one of X, Y, YW and XW contains exactly two of these elements (but not both X and YW, and not both Y and XW).

Let $\mathcal{P} = \mathrm{Max}(D)\backslash\{M_{1,1}, M_{1,2}, M_{2,1}, M_{2,2}\}$ and let $R = D + B$ be the ring of form $A + B$ corresponding to D and $\mathcal{P}$. Since the set S is saturated, if $h \in D\backslash S$, then $h \in M$ for some $M \in \mathcal{P}$. It follows that (h, b) is a zero divisor of R for each $b \in B$. Thus a necessary condition for (r, a) to be a regular element of R is for r to be in the set S. We have $(\mathrm{X}, 0) \in M_{1,1}R \bigcap M_{1,2}R$ and $(\mathrm{YW}, 0) \in M_{2,1}R \bigcap M_{2,2}$ are comaximal regular elements, as are $(\mathrm{Y}, 0) \in M_{1,1} \bigcap M_{2,1}$ and $(\mathrm{XW}, 0) \in M_{1,2}R \bigcap M_{2,2}R$. The set of regular elements $A = \{(\mathrm{XYW}, 0), (\mathrm{XXW}, 0), (\mathrm{YYW}, 0)\}$ is a generating set for $M_{1,1}R \bigcap M_{2,2}R$ and the set $B = \{(\mathrm{XY}, 0), (\mathrm{XYW}, 0), (\mathrm{XWYW}, 0)\}$ fills the same role for $M_{1,2}R \bigcap M_{2,1}R$. Each member of A is contained in at least one of $M_{1,2}R$ and $M_{2,1}R$, and each member of B is contained in at least one of $M_{1,1}R$ and $M_{2,2}R$. As each monomial in $M_{1,1}R \bigcap M_{2,2}R$ is a multiple of at least one member of A, and each monomial in $M_{1,2}R \bigcap M_{2,1}R$ is a multiple of at least one member of B, there are no regular elements $f \in M_{1,1}R \bigcap M_{2,2}R$ and $g \in M_{1,2}R \bigcap M_{2,1}R$ that are comaximal. □

One of many characterizations of a Krull domain is that an integral domain D with quotient field $K \neq D$ is a Krull domain if and only if there is a family of discrete rank one valuation domains $\{W_\alpha\}$ such that (i) $\bigcap W_\alpha = D$ and (ii) for each nonzero element of K, there are at most finitely many W_α where the element is not a unit of W_α. A Krull ring R can be defined in an analogous manner using a family of discrete rank one valuation rings of $T(R)$. For such a family $\{V_\alpha\}$, (i) is simply that $R = \bigcap V_\alpha$. For (ii), a zero divisor of $T(R)$ cannot be a unit in in some V_α, but a regular element of $T(R)$ can. So we replace (ii) with each regular element $t \in T(R)$ is a unit except in at most finitely many V_αs. If R is a Krull ring with defining family of valuations $\{V_\alpha\}$, then there is a family of regular prime ideals $\{P_\alpha\}$ such that $V_\alpha = R_{[P_\alpha]}$ with $[P_\alpha]R_{[P_\alpha]}$ the set of elements with positive value and no regular prime ideal of R is properly contained in a P_α (see, for example, [10, Remark 6] and [1, Propositions 2.10 and 2.11]). Portelli and Spangher showed that if R is an additively regular Krull ring and $\{V_1, V_2, \ldots, V_n\}$ is a finite subset of the family $\{V_\alpha\}$, then for each $\overline{k} = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$, there is a regular element $t \in T(R)$ such that $\mathrm{v}_i(t) = k_i$ for $1 \leq i \leq n$ and $\mathrm{v}_\alpha(t) \geq 0$ for all other v_αs [13, Proposition 49]. If we omit the restriction on t being regular, Osmanagic showed if $R \neq T(R)$ is simply a Krull ring, then for a given finite subset $\{V_1, V_2, \ldots, V_n\}$ of the family $\{V_\alpha\}$ and corresponding n-tuple $\overline{k} = (k_1, k_2, \cdots, k_n) \in \mathbb{Z}^n$, there is an element $t \in T(R)$ such that $\mathrm{v}_i(t) = k_i$ and $\mathrm{v}_\alpha(t) \geq 0$ for all other v_α [12, Theorem 2.1]. Yet another use of the Marot ring R in Example 2.4 is to show that there need not be such a regular element for a Marot Krull ring defined by a finite set of discrete rank one valuation rings (in this case, just two!).

Example 2.7. Let D, $\mathcal{P}$ and R be as in Example 2.4.

(1) R is a one-dimensional Marot Krull ring with two regular maximal ideals, $MR = M + B$ and $NR = N + B$ where $M = 2D + \sqrt{10}D$ and $N = 5D + \sqrt{10}D$. So the defining family

of discrete rank one valuations rings is $\{V_1 = R_{(MR)}, V_2 = R_{(NR)}\}$.

(2) No regular element of R corresponds to the ordered pair $(1,0) \in \mathbb{Z} \oplus \mathbb{Z}$.

Proof. Since D is a Krull domain, R is a Krull ring [9, Theorem 8.6]. Also R is one-dimensional since D is one-dimensional. The only regular prime ideals of R are MR and NR. Since R is Marot, $R_{(MR)} = R_{[MR]}$ and $R_{(NR)} = R_{[NR]}$ and each of these is a discrete rank one valuation ring since D_M and D_N are discrete rank one valuation domains (Theorem 2.3). In addition, $R = R_{(MR)} \bigcap R_{(NR)}$. Suppose $(r,b) \in T(R)$ is such that $\mathrm{v}_1((r,b)) = 1$ and $\mathrm{v}_2((r,b)) = 0$, then $r \in M \backslash N$ and $rD_M = MD_M$. Since M is not principal, there is a maximal ideal $P \in \mathcal{P}$ such that $r \in P$. It follows that (r,b) is a zero divisor of R. Hence no regular element of R corresponds to the ordered pair $(1,0) \in \mathbb{Z} \oplus \mathbb{Z}$. The analogous conclusion holds for $(0,1)$. □

3 Additively Regular Rings

As we saw in Example 2.5, even if S is the complement of the union of finitely many maximal ideals in a Marot ring R, $R_{(S)}$ need not be equal to $R_{[S]}$. However, if the ring R is additively regular, then $R_{(S)} = R_{[S]}$ whenever S is the complement of the union of finitely many prime ideals (see Corollary 3.3 below). As with localizing at a multiplicative set, when forming the corresponding large ring of quotients, one may assume the set S is multiplicatively closed since $tc \in R$ for some c in the saturation of S implies there is an element $b \in R$ such that $cb \in S$ and then we clearly have $tcb \in R$. However, if c is a regular element in the saturation of S, it may be that the only elements that multiply c into S are zero divisors (in the saturation). The ring R in the next example is additively regular with exactly four maximal ideals, each regular, but there is a multiplicative set S with saturation S' (in R) such that $R_{(S')}$ is strictly larger than $R_{(S)}$ (and $R_{(S)} \supsetneq R$).

Example 3.1. Let $R = D \oplus D$ where $D = \mathbb{Z}_{(2)} \bigcap \mathbb{Z}_{(3)}$. Next let S be the multiplicative subset of R generated by $(2,2)$ and $(3,0)$ and let S' be the saturation of S in R.

(1) Each regular element of R has the form (c,d) where both c and d are nonzero and a nonzero zero divisor either has the form $(f,0)$ or $(0,f)$ for some nonzero $f \in D$. The sums $(f,0) + (0,1)(c,d) = (f,d)$ and $(0,f) + (1,0)(c,d) = (c,f)$ are regular elements of R. Hence R is additively regular (see [6, Lemma 9] for a much more general result).

(2) Since D has only two maximal ideals, R has four maximal ideals and each of these is regular.

(3) $(a,b) \in S$ if and only if either $a = b = 2^n$ for some positive integer n or $(a,b) = (a,0)$ with $a = 2^n 3^m$ for some nonnegative integer n and positive integer m.

(4) $\{(2^n, 2^n) \mid n \geq 1\}$ is the complete set of regular elements in S.

(5) $R_{(S)} = \{(p,q) \in \mathbb{Q} \oplus \mathbb{Q} \mid 2^n p, 2^n q \in D \text{ for some } n \geq 1\}$.

(6) The regular element $(3,3)$ is in S' since $(3^2,0) = (3,3)(3,0) \in S$. Thus $(1/3, 1/3) \in R_{(S')} \backslash R_{(S)}$.

Theorem 3.2. *Let R be an additively regular ring and let S be a saturated multiplicatively closed subset of R that misses at least one regular prime ideal. If there is a regular element r that is contained in each regular prime that is maximal with respect to missing S, then $R_{(S)} = R_{[S]}$.*

Proof. First note that each regular nonunit in $R \backslash S$ is contained in at least one regular prime that is maximal with respect to missing S. Also, since S is saturated, it is the complement of the union of those prime ideals Q such that $Q \bigcap S = \emptyset$. Assume $r \in R \backslash S$ is a regular element that is contained in each regular prime that is maximal with respect to missing S. For each $t \in R_{[S]}$, there is an element $d \in S$ such that $dt \in R$ and a regular element $b \in R$ such that $bt \in R$. Since R is additively regular, there is an element $f \in R$ such that $d + frb$ is regular. If N is a prime that is maximal with respect to missing S and N is not regular, then $d + frb$ is not in N. Also if P is a regular prime that is maximal with respect to missing S, then $frb \in P$ but d is not. Hence $d + frb \in S$. As $t(d + frb) \in R$, $t \in R_{(S)}$ and we have $R_{(S)} = R_{[S]}$. □

Corollary 3.3. *Let R be an additively regular ring. If S is a saturated multiplicatively closed subset of R such that there are only finitely many regular primes that are maximal with respect to missing S, then $R_{(S)} = R_{[S]}$. In particular, $R_{(S)} = R_{[S]}$ whenever S is the complement of the union of a finite set of prime ideals.*

Proof. This follows easily from Theorem 3.2 and the fact that the intersection of finitely many regular primes is regular. □

Statement (1) in the next theorem is a special case of [13, Proposition 8].

Theorem 3.4. *Let $M_1, M_2, \ldots, M_n$ be regular maximal ideals of an additively regular ring R and let $S = R \backslash \bigcup_{i=1}^{n} M_i$.*

(1) If I is a regular ideal that is not contained in the union $\bigcup M_i$, then I contains a regular element that is contained in no M_i.

(2) The only regular ideals of R that survive in $R_{(S)}$ are those that are contained in at least one M_i.

(3) If M is a regular maximal ideal that is not one of the M_is, then $MR_{(S)} = R_{(S)}$ and M contains a regular element that is contained in S.

Proof. We revisit the proof of [13, Proposition 8]. To start, let I be a regular ideal of R that is not contained in $\bigcup M_i$. Then there is an element $c \in I$ such that $c \notin \bigcup M_i$. The ideal $Q = I \bigcap M_1 \bigcap M_2 \bigcap \cdots \bigcap M_n$ is regular, so we let b be a regular element in Q. Since R is additively regular, there is an element $s \in R$ such that $c + sb$ is regular. Since $b \in Q$ and c is in no M_i, $c + sb \in I$ is a regular element that is contained in the set S. Thus it is a unit of $R_{(S)}$. It follows that $1 \in IR_{(S)}$. This takes care of all three statements. □

The Marot ring in Example 2.5 shows it is possible that none of the statements in the previous theorem hold if R is Marot but not additively regular, even if R has only finitely many regular maximal ideals.

Theorem 3.5. *Let R be an additively regular ring. If R has only finitely many regular maximal ideals, then each invertible ideal is principal.*

Proof. Assume R has only finitely many regular maximal ideals (it can have infinitely many maximal ideals that are not regular) and let I be an invertible ideal. Let $M_1, M_2, \ldots, M_n$ be the regular maximal ideals of R. Since R is additively regular, $M_i R_{(M_i)}$ is the unique regular maximal ideal of $R_{(M_i)}$. Hence by Theorem 2.1 and its proof, $IR_{(M_i)}$ is a principal regular ideal of $R_{(M_i)}$ and there is a regular element $a_i \in I$ such that $a_i R_{(M_i)} = IR_{(M_i)}$. The ideal $J = a_1 R + a_2 R + \cdots + a_n R$ is regular and contained in I. In addition $JR_{(M)} = R_{(M)} = IR_{(M)}$ for each maximal ideal M that is not regular (in this case $R_{(M)} = T(R)$) and $JR_{(M_i)} = IR_{(M_i)}$ for each M_i. It follows that $J = I$.

For the remainder of the proof we use induction on n (and continue with the notation above).

Theorem 2.1 takes care of the case $n = 1$. Suppose $n = 2$. From the argument above there is a pair of regular elements $a_1, a_2 \in I$ such that $I = a_1 R + a_2 R$, $a_1 R_{(M_1)} = IR_{(M_1)}$ and $a_2 R_{(M_2)} = IR_{(M_2)}$. We may further assume that $a_1 R_{(M_2)} \subsetneq a_2 R_{(M_2)}$ and $a_2 R_{(M_1)} \subsetneq a_1 R_{(M_1)}$ (otherwise we can choose $a_1 = a_2$). Thus there are elements $b_1, b_2, t_1, t_2 \in R$ with t_1 a regular element in $R \backslash M_1$ and t_2 a regular element in $R \backslash M_2$ such that $t_1 a_2 = b_1 a_1$ and $t_2 a_1 = b_2 a_2$. Since all four of a_1, a_2, t_1 and t_2 are regular, so are b_1 and b_2. As we have assumed $a_2 R_{(M_1)} \subsetneq a_1 R_{(M_1)}$ and $a_1 R_{(M_2)} \subsetneq a_2 R_{(M_2)}$, it must be that $b_1 \in M_1$ and $b_2 \in M_2$.

Let $d = t_1 a_1 + t_2 a_2$. Since $a_1 a_2$ is regular, there is an element $s \in R$ such that $g = d + s a_1 a_2$ is regular. We claim that g generates I. It is clear that $g \in I$. Thus it suffices to show $gR_{(M_1)} = IR_{(M_1)}$ and $gR_{(M_2)} = IR_{(M_2)}$. Note that $t_1 g = t_1^2 a_1 + t_1 t_2 a_2 + t_1 s a_1 a_2 = t_1^2 a_1 + t_2 b_1 a_1 + t_1 s a_1 a_2 = (t_1^2 + t_2 b_1 + t_1 s a_2) a_1$. Similarly, $t_2 g = (t_2^2 + t_1 b_2 + t_2 s a_1) a_2$. Since t_1, t_2 and g are regular, so are $t_1^2 + t_2 b_1 + t_1 s a_2$ and $t_2^2 + t_1 b_2 + t_2 s a_1$. For the element $t_1^2 + t_2 b_1 + t_1 s a_2$, both b_1 and a_2 are in M_1 while t_1 is not. Hence $t_1^2 + t_2 b_1 + t_1 s a_2$ is a unit of $R_{(M_1)}$ and we have $gR_{(M_1)} = a_1 R_{(M_1)} = IR_{(M_1)}$. Similarly, $t_2^2 + t_1 b_2 + t_2 s a_1$ is a unit of $R_{(M_2)}$ since $t_2 \in R \backslash M_2$ while both b_2 and a_1 are in M_2. Therefore $gR = I$.

Now assume that each invertible ideal of an additively regular ring is principal if the ring has $n-1$ or fewer regular maximal ideals (for some $n \geq 3$).

Assume R has exactly n regular maximal ideals (with $n \geq 3$). Without loss of generality we may assume $I \subseteq M_n$. Thus the element a_n defined above is in M_n. Let $Q_n = \bigcap_{i \neq n} M_i$ and $S_n = R \backslash \bigcup_{i \neq n} M_i$. Then $R_{(S_n)}$ is an additively regular ring with exactly $n-1$ regular maximal ideals (Theorem 3.4). By the induction hypothesis, $IR_{(S_n)}$ is a (regular) principal ideal. Hence there is a regular element $c \in I$ such that $IR_{(S_n)} = cR_{(S_n)}$. Thus $IR_{(M_i)} = cR_{(M_i)}$ for each $1 \leq i < n$. If we also have $cR_{(M_n)} = IR_{(M_n)}$, then $I = cR$. Hence we may assume $cR_{(M_n)} \subsetneq IR_{(M_n)}$. From above, we have $IR_{(M_n)} = a_n R_{(M_n)}$ and so we may further assume that $a_n R_{(S_n)} \subsetneq IR_{(S_n)} = cR_{(S_n)}$ and $cR_{(M_n)} \subsetneq a_n R_{(M_n)}$.

We have regular elements $t_1, t_n, b_1, b_n \in R$ with $t_1 \in S_n$ and $t_n \in R \backslash M_n$ such that $t_1 a_n = b_1 c$ and $t_n c = b_n a_n$. As we have assumed $I \subseteq M_n$ and $cR_{(M_n)} \subsetneq a_n R_{(M_n)}$, both c and b_n are in M_n.

Since Q_n is a regular ideal (in a Marot ring) that is comaximal with M_n, there is a regular element $z \in Q_n \backslash M_n$. Note that z is a unit in $R_{(M_n)}$.

Let $h = t_1 c + t_n z a_n$. Since R is additively regular and zca_n is regular, there is an element $s \in R$ such that $g = h + szca_n$ is regular. It suffices to show that $IR_{(S_n)} = gR_{(S_n)}$ and $IR_{(M_n)} = gR_{(M_n)}$.

Since t_1 is a regular element of S_n, it is a unit of $R_{(S_n)}$. Also $t_1 g = t_1 h + t_1 szca_n = t_1^2 c + t_1 t_n z a_n + t_1 szca_n = t_1^2 c + t_1 b_1 zc + t_1 szca_n = (t_1^2 + t_1 b_1 z + t_1 sza_n)c$. Since both t_1 and g are regular, so is $(t_1^2 + t_1 b_1 z + t_1 sza_n)$. In addition, $t_1^2 + t_1 b_1 z + t_1 sza_n$ is in S_n since $t_1^2 \in S_n$ and $z \in Q_n$. Thus $t_1^2 + t_1 b_1 z + t_1 sza_n$ is a unit of $R_{(S_n)}$ and therefore $IR_{(S_n)} = cR_{(S_n)} = gR_{(S_n)}$.

An analogous argument works for $IR_{(M_n)}$. The element $t_n^2 z$ is a regular element outside of M_n. In addition, $t_n g = t_n^2 z a_n + t_n t_1 c + t_n szca_n = t_n^2 z a_n + t_n b_n a_n + t_n szca_n = (t_n^2 z + t_n b_n + t_n szc) a_n$ which implies $t_n^2 z + t_n b_n + t_n szc$ is a regular element. As $t_n b_n + t_n szc$ is in M_n and $t_n^2 z$ is not, $t_n^2 z + t_n b_n + t_n szc$ is not in M_n and thus it is a unit of $R_{(M_n)}$. It follows that $IR_{(M_n)} = a_n R_{(M_n)} = gR_{(M_n)}$ and therefore $I = gR$. □

Corollary 3.6. *If R is an additively regular ring with only finitely many regular maximal ideals, then it is a Prüfer ring if and only if it is a regular Bézout ring.*

We close with a few questions about Marot rings. For all, assume R is a Marot ring (that is not additively regular).

Q1: If P_1, P_2 and P_3 are incomparable regular prime ideals, is there a regular element $r \in P_1$ that is not in $P_2 \bigcup P_3$?

Q2: If P_1, P_2 and P_3 are pairwise comaximal regular prime ideals, is there a regular element $r \in P_1$ that is not in $P_2 \bigcup P_3$?

Q3: If M_1, M_2 and M_3 are distinct regular maximal ideals, is there a regular element $r \in M_1$ that is not in $M_2 \bigcup M_3$? If the answer is "not necessarily," is the answer "yes" when M_1, M_2 and M_3 are the only regular maximal ideals.

Q4: If M_1, M_2 and M_3 are distinct regular maximal ideals, do there exist regular elements $r \in M_1$ and $s \in M_2 \bigcap M_3$ such that $r + s = 1$ (or at least $rR + sR = R$)? If the answer is "not necessarily," is the answer "yes" when M_1, M_2 and M_3 are the only regular maximal ideals.

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