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New Type of Difference Triple Sequence Spaces

Shyamal Debnath and Bimal Chandra Das

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Abstract. In this article we have introduced the difference triple sequence spaces $c_0^3(\Delta_p)$, $c^3(\Delta_p)$, $c^{3R}(\Delta_p)$, $l_\infty^3(\Delta_p)$ and $c^{3B}(\Delta_p)$, applying the difference operator Δ_p for a fixed $p \in \mathbf{N}$ on the sequence (x_{lmn}) and examine whether the spaces being symmetric, solid, convergence free etc. We have also obtained and proved some inclusion relation too.

1. Introduction and Preliminaries :

A triple sequence (real or complex) can be defined as a function $x : N \times N \times N \longrightarrow R(C)$, where N, R and C denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequences was introduced and investigated at the initial stage by Sahiner et. al. [2] and Dutta et. al. [4] and others. Recently Savas and Esi [7]have introduced statistical convergence of triple sequences on probabilistic normed space. Later on, Esi [1] have introduced statistical convergence of triple sequences in topological groups.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [8] as follows:

$Z(\Delta) = \{(x_n) \in w : (\Delta x_n) \in Z\}$, for $Z = c, c_0, l_\infty$, the spaces of convergent, null and bounded sequences, respectively, where $\Delta x_n = x_n - x_{n+1}$ for all $n \in N$. Tripathy and Esi[5] introduded the notion of difference sequence space as $\Delta_m x = (\Delta_m x_n) = x_n - x_{n+m}$ for all $n \in \mathbf{N}$ and $m \in \mathbf{N}$ be fixed. Later on it was studied by Tripathy and Sarma [6] and introduced difference double sequence spaces as follows: .

$Z(\Delta) = \{(x_{mn}) \in w : (\Delta x_{mn}) \in Z\}$, for $Z = c^2, c_0^2, l_\infty^2$, the spaces of convergent, null and bounded double sequences respectively, where $\Delta x_{mn} = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbf{N}$.

Definition 1.1 [2]: A triple sequence (x_{lmn}) is said to be convergent to L in Pringsheim′s sense if for every $\epsilon > 0$, there exists $\mathbf{N}(\epsilon) \in N$ such that .

$|x_{lmn} - L| < \epsilon$ whenever $l \geq \mathbf{N}$, $m \geq \mathbf{N}$, $n \geq \mathbf{N}$ and we write $lim_{l,m,n \longrightarrow \infty} x_{lmn} = L$.

Note: A triple sequence is convergent in Pringsheim′s sense may not be bounded [2].

Definition 1.2 [2]: A triple sequence (x_{lmn}) is said to be Cauchy sequence if for every $\epsilon > 0$, there exists $\mathbf{N}(\epsilon) \in N$ such that.

$|x_{lmn} - x_{pqr}| < \epsilon$ whenever $l \geq p \geq \mathbf{N}$, $m \geq q \geq \mathbf{N}$, $n \geq r \geq \mathbf{N}$.

Definition 1.3 [2]: A triple sequence (x_{lmn}) is said to be bounded if there exists $M > 0$, such that $|x_{lmn}| < M$ for all $l, m, n \in N$.

Definition 1.4 [4]: A triple sequence (x_{lmn}) is said to be converge regularly if it is convergent in Pringsheim′s sense and in addition the following limits holds:

$lim_{n \longrightarrow \infty} x_{lmn} = L_{lm}$ $(l, m \in N)$

$lim_{m \longrightarrow \infty} x_{lmn} = L_{ln}$ $(l, n \in N)$

$lim_{l \longrightarrow \infty} x_{lmn} = L_{mn}$ $(m, n \in N)$

Let w^3 denote the set of all triple sequence of real numbers. Then the class of triple sequences c_0^3, c^3, l_∞^3, c^{3R} and c^{3B} denotes the triple sequence spaces which are convergent to zero in Pringsheim′s sense, convergent in Pringsheim′s sense, bounded in Pringsheim′s sense, regularly convergent, bounded and convergent respectively.

These classes are all linear spaces.

It is obvious that $c_0^3 \subset c^3$, $c^{3R} \subset c^{3B} \subset l_\infty^3$ and the inclusion are strict.

Theorem 1.1: The spaces c_0^3, c^3, l_∞^3, c^{3R} and c^{3B} are complete normed linear spaces with the normed.

$\|x\| = sup_{l,m,n} |x_{lmn}| < \infty$

Proof: simple.

Example 1.1 [2]: Let $x_{lmn} = \begin{cases} mn, & l = 3 \\ nl, & m = 5 \\ lm, & n = 7 \\ 8, & otherwise \end{cases}$

Then $(x_{lmn}) \to 8$ in Pringsheim′s sense but not bounded as well as not regularly convergent.

Example 1.2: Let $x_{lmn} = 1$, for all $l, m, n \in N$. Then (x_{lmn}) is convergent in Pringsheim′s sense, bounded and regularly convergent.

Definition 1.5 [4]: A triple sequence space E is said to be solid if $(\alpha_{lmn} x_{lmn}) \in E$ whenever $(x_{lmn}) \in E$ and for all sequences (α_{lmn}) of scalars with $|\alpha_{lmn}| \leq 1$, for all $l, m, n \in \mathbf{N}$.

Definition 1.6 [4]:A triple sequence space E is said to be monotone if it contains the canonical pre-images of all its step spaces.

Remark 1.1 [4]: A sequence space is solid implies that it is monotone.

Definition 1.7 [4]: A triple sequence space E is said to be convergence free if $(y_{lmn}) \in E$, whenever $(x_{lmn}) \in E$ and $x_{lmn} = 0$ implies $y_{lmn} = 0$.

Definition 1.8 [4]: A triple sequence space E is said to be symmetric if $(x_{lmn}) \in E$ implies $(x_{\pi(l)\pi(m)\pi(n)}) \in E$, where π is a permutation of $\mathbf{N} \times \mathbf{N} \times \mathbf{N}$.

Now we introduced the p^{th} order difference triple sequence spaces as follows:

$c_0^3(\Delta_p)$={$(x_{lmn}) \in w^3$: $(\Delta_p x_{lmn})$ is regularly null }

$c^3(\Delta_p)$={$(x_{lmn}) \in w^3$: $(\Delta_p x_{lmn})$ is convergent in Pringsheim's sense }

$c^{3R}(\Delta_p)$={$(x_{lmn}) \in w^3$: $(\Delta_p x_{lmn})$ is regularly convergent }

$l_\infty^3(\Delta_p)$={$(x_{lmn}) \in w^3$: $(\Delta_p x_{lmn})$ is bounded }

$c^{3B}(\Delta_p)$={$(x_{lmn}) \in w^3$: $(\Delta_p x_{lmn})$ is convergent in Pringsheim's sense and bounded }

Where $\Delta_p x_{lmn} = x_{lmn} - x_{lmn+p} - x_{lm+pn} + x_{lm+pn+p} - x_{l+pmn} + x_{l+pmn+p} + x_{l+pm+pn} - x_{l+pm+pn+p}$

when $p = 1$, the above spaces becomes $c_0^3(\Delta)$, $c^3(\Delta)$, $c^{3R}(\Delta)$, $l_\infty^3(\Delta)$ and $c^{3B}(\Delta)$, which was studied by Debnath, Sarma and Das [10].

2. Main Result:

Theorem 2.1: The classes of sequences $c_0^3(\Delta_p)$, $c^3(\Delta_p)$, $c^{3R}(\Delta_p)$, $l_\infty^3(\Delta_p)$, $c^{3B}(\Delta_p)$ are linear spaces.

Proof: Obvious.

Theorem 2.2: The classes of sequences $c_0^3(\Delta_p)$, $c^3(\Delta_p)$, $c^{3R}(\Delta_p)$, $l_\infty^3(\Delta_p)$ and $c^{3B}(\Delta_p)$ are complete normed linear spaces with the norm

$$\|x\| = sup_l|x_{lpp}| + sup_m|x_{pmp}| + sup_n|x_{ppn}| + sup_{l,m,n}|\Delta_p x_{lmn}| < \infty$$

Proof: Let (x^i) be a Cauchy sequence in $l_\infty^3(\Delta_p)$, where $x^i = (x^i_{lmn}) \in l_\infty^3(\Delta_p)$ for each $i \in \mathbf{N}$.
Then we have,
$\|x^i - x^j\| = sup_l|x^i_{lpp} - x^j_{lpp}| + sup_m|x^i_{pmp} - x^j_{pmp}| + sup_n|x^i_{ppn} - x^j_{ppn}| + sup_{l,m,n}|\Delta_p x^i_{lmn} - \Delta_p x^j_{lmn}| \to 0$

as $i, j \to \infty$

Therefore, $|x^i_{lmn} - x^j_{lmn}| \to 0$, for $i, j \to \infty$ and each $l, m, n \in \mathbf{N}$

Hence (x^i_{lmn})=$(x^1_{lmn}, x^2_{lmn}, x^3_{lmn}, \ldots\ldots\ldots\ldots\ldots\ldots)$ is a Cauchy sequence in R (Real numbers).

Whence by the completeness of R, it converges to x_{lmn} say, i.e., there exists

$lim\ x^i_{lmn} = x_{lmn}$ for each $l, m, n \in \mathbf{N}$

Further for each $\epsilon > 0$, there exists $\mathbf{N} = \mathbf{N}(\epsilon)$, such that for all $i, j \geq \mathbf{N}$, and for all $l, m, n \in \mathbf{N}$

$|x^i_{lpp} - x^j_{lpp}| < \epsilon$, $|x^i_{pmp} - x^j_{pmp}| < \epsilon$, $|x^i_{ppn} - x^j_{ppn}| < \epsilon$

$|\Delta_p x^i_{lmn} - \Delta_p x^j_{lmn}| = |(x^i_{l+pm+pn+p} - x^j_{l+pm+pn+p}) - (x^i_{l+pm+pn} - x^j_{l+pm+pn}) - (x^i_{l+pmn+p} - x^j_{l+pmn+p}) + (x^i_{l+pmn} - x^j_{l+pmn}) - (x^i_{lm+pn+p} - x^j_{lm+pn+p}) + (x^i_{lm+pn} - x^j_{lm+pn}) + (x^i_{lmn+p} - x^j_{lmn+p}) - (x^i_{lmn} - x^j_{lmn})| < \epsilon$

and

$lim_j\ |x^i_{lpp} - x^j_{lpp}| = |x^i_{lpp} - x_{lpp}| \leq \epsilon$,

$lim_j\ |x^i_{pmp} - x^j_{pmp}| = |x^i_{pmp} - x_{pmp}| \leq \epsilon$,

$lim_j\ |x^i_{ppn} - x^j_{ppn}| = |x^i_{ppn} - x_{ppn}| \leq \epsilon$,

Now

$lim_j|\Delta x^i_{lmn} - \Delta x^j_{lmn}| = |(x^i_{l+pm+pn+p} - x_{l+pm+pn+p}) - (x^i_{l+pm+pn} - x_{l+pm+pn}) - (x^i_{l+pmn+p} - x_{l+pmn+p}) + (x^i_{l+pmn} - x_{l+pmn}) - (x^i_{lm+pn+p} - x_{lm+pn+p}) + (x^i_{lm+pn} - x_{lm+pn}) + (x^i_{lmn+p} - x_{lmn+p}) - (x^i_{lmn} - x_{lmn})| \leq \epsilon$, for all $i \geq \mathbf{N}$

Since ϵ is not dependent on l, m, n

$sup_{l,m,n}\ |(x^i_{l+pm+pn+p} - x_{l+pm+pn+p}) - (x^i_{l+pm+pn} - x_{l+pm+pn}) - (x^i_{l+pmn+p} - x_{l+pmn+p}) + (x^i_{l+pmn} - x_{l+pmn}) - (x^i_{lm+pn+p} - x_{lm+pn+p}) + (x^i_{lm+pn} - x_{lm+pn}) + (x^i_{lmn+p} - x_{lmn+p}) - (x^i_{lmn} - x_{lmn})| \leq \epsilon,$

Consequently we have, $\|x^i_{lmn} - x_{lmn}\| \leq 4\epsilon$, for all $i \geq \mathbf{N}$

Hence we obtain $x^i_{lmn} \to x_{lmn}$ as $i \to \infty$ in $l^3_\infty(\Delta)$

Now we have to show that $(x_{lmn}) \in l^3_\infty(\Delta)$

$|x_{lmn} - x_{l+pm+pn+p}| = |x_{lmn} - x^N_{lmn} + x^N_{lmn} - x^N_{l+pm+pn+p} + x^N_{l+pm+pn+p} - x_{l+pm+pn+p}|$

$\leq |x^N_{lmn} - x_{l+pm+pn+p}| + \|x^N_{lmn} - x_{lmn}\| = O(1)$

This implies $x = (x_{lmn}) \in l^3_\infty(\Delta)$

Since $l^3_\infty(\Delta)$ is a linear space.

Hence $l^3_\infty(\Delta)$ is complete.

Similarly the others.

Theorem 2.3:

(i) $c^3_0(\Delta_p) \subset c^3(\Delta_p)$ and the inclusion is strict. .

(ii) $c^{3R}(\Delta_p) \subset c^3(\Delta_p)$ and the inclusion is strict.

(iii) $c^{3R}(\Delta_p) \subset c^{3B}(\Delta_p)$ and the inclusion is strict.

Proof: The inclusion being strict follows from the following example:

Example 2.1: We consider the sequence (x_{lmn}) defined by

(x_{lmn})= lmn, for all $l, m, n \in \mathbf{N}$

Then $x_{lmn} \in c^3(\Delta_p)$, but $x_{lmn} \notin c^3_0(\Delta_p)$

Hence the inclusion is strict.

Example 2.2: We consider the sequence defined by

$$x_{lmn} = \begin{cases} (-1)^n lmn, & \text{for } l = 1, m = 1, 2, 3, 4 \text{ for all } n \in \mathbf{N} \\ -2, & \text{otherwise} \end{cases}$$

Clearly $x_{lmn} \in c^3(\Delta_p)$, but the sequence $x_{lmn} \notin c^{3R}(\Delta_p)$

Hence the inclusion $c^{3R}(\Delta_p) \subset c^3(\Delta_p)$, is strict.

Example 2.3: We consider the sequence defined by

$$x_{lmn} = \begin{cases} 2, & \text{when } l = 1, m+n \text{ is even, for all } l, m, n \in \mathbf{N} \\ -1, & \text{when } l = 1, m+n \text{ is odd, for all } l, m, n \in \mathbf{N} \\ 1, & \text{otherwise} \end{cases}$$

$x_{lmn} \in c^{3B}(\Delta_p)$, but the sequence $x_{lmn} \notin c^{3R}(\Delta_p)$

Therefore, the inclusion is strict.

Theorem 2.4: The classes of sequences $c^3_0(\Delta_p)$, $c^3(\Delta_p)$, $c^{3R}(\Delta_p)$, $l^3_\infty(\Delta_p)$ and $c^{3B}(\Delta_p)$ are not solid for p odd.

Proof: This is clear from the following examples:

Example 2.4: We consider the sequence (x_{lmn}) defined by

(x_{lmn})= 3, for all $l, m, n \in \mathbf{N}$

Clearly the triple sequence $x_{lmn} \in c_0^3(\Delta_p)$, $c^3(\Delta_p)$, $c^{3R}(\Delta_p)$ and $c^{3B}(\Delta_p)$

Consider the sequence of scalars defined by

α_{lmn} = $(-1)^{l+m+n}$, for all $l, m, n \in$ **N**

Then the sequence $(\alpha_{lmn} x_{lmn})$ takes the following form

$\alpha_{lmn} x_{lmn}$ = 3.$(-1)^{l+m+n}$, for all $l, m, n \in$ **N**

Clearly $(\alpha_{lmn} x_{lmn}) \notin c_0^3(\Delta_p)$, $c^3(\Delta_p)$, $c^{3R}(\Delta_p)$ and $c^{3B}(\Delta_p)$ (for p odd)

Hence $c_0^3(\Delta_p)$, $c^3(\Delta_p)$, $c^{3R}(\Delta_p)$ and $c^{3B}(\Delta_p)$ are not solid.

Example 2.5: We consider the sequence (x_{lmn}) defined by

(x_{lmn})= lmn, for all $l, m, n \in$ **N**

Clearly the sequence $x_{lmn} \in l_\infty^3(\Delta_p)$

Consider the sequence of scalars defined by

α_{lmn} = $(-1)^{l+m+n}$, for all $l, m, n \in$ **N**

Then the sequence $(\alpha_{lmn} x_{lmn})$ takes the following form

$\alpha_{lmn} x_{lmn}$ = $(-1)^{l+m+n} lmn$, for all $l, m, n \in$ **N**

Clearly, when p odd, $(\alpha_{lmn} x_{lmn}) \notin l_\infty^3(\Delta_p)$,

Hence $l_\infty^3(\Delta_p)$ are not solid.

Theorem 2.5: The spaces $c_0^3(\Delta_p)$, $c^3(\Delta_p)$, $c^{3R}(\Delta_p)$ and $c^{3B}(\Delta_p)$ are not monotone for p odd.

Proof: The proof is clear from the following example:

Example 2.6: Consider the sequence (x_{lmn}) defined by

(x_{lmn})= 1, for all $l, m, n \in$ **N**

It is clear that $x_{lmn} \in c_0^3(\Delta_p)$, $c^3(\Delta_p)$, $c^{3R}(\Delta_p)$ and $c^{3B}(\Delta_p)$

Now we consider the sequence (y_{lmn}) in its pre-image space defined by

$$y_{lmn} = \begin{cases} 1, & \text{for all } l+m+n \text{ is odd, for all } l,m,n \in \mathbf{N} \\ 0, & \text{otherwise} \end{cases}$$

Clearly when p odd, $(y_{lmn}) \notin c_0^3(\Delta_p)$, $c^3(\Delta_p)$, $c^{3R}(\Delta_p)$ and $c^{3B}(\Delta_p)$

Hence the spaces $c_0^3(\Delta_p)$, $c^3(\Delta_p)$, $c^{3R}(\Delta_p)$ and $c^{3B}(\Delta_p)$ are not monotone.

Theorem 2.6: The classes of sequences $c_0^3(\Delta_p)$, $c^3(\Delta_p)$, $c^{3R}(\Delta_p)$ and $c^{3B}(\Delta_p)$ are not symmetric when p odd.

Proof: The proof is followed by the following example:

Example 2.7: Consider the triple sequence (x_{lmn}) defined by

(x_{lmn})= m, for all $m \in$ **N**

Clearly the sequence $x_{lmn} \in c_0^3(\Delta_p)$, $c^3(\Delta_p)$, $c^{3R}(\Delta_p)$ and $c^{3B}(\Delta_p)$

Consider a rearrange sequence (y_{lmn}) of (x_{lmn}) defined by

$$y_{lmn} = \begin{cases} m+1 \text{ , for } m=l, n \text{ is even} \\ m-1 \text{ , for } m=l+1, n \text{ is even} \\ m \text{ , otherwise} \end{cases}$$

Clearly $(y_{lmn}) \notin c_0^3(\Delta_p)$, $c^3(\Delta_p)$, $c^{3R}(\Delta_p)$ and $c^{3B}(\Delta_p)$ (for p odd)

Hence $c_0^3(\Delta_p)$, $c^3(\Delta_p)$, $c^{3R}(\Delta_p)$ and $c^{3B}(\Delta_p)$ are not symmetric.

Theorem 2.7: When p is odd, the classes of sequences $c_0^3(\Delta_p)$, $c^3(\Delta_p)$, $c^{3R}(\Delta_p)$, $l_\infty^3(\Delta_p)$ and $c^{3B}(\Delta_p)$ are not convergent free.

Proof:We provide an example to prove the result.

Example 2.8: Consider the sequence defined by

$$x_{lmn} = \begin{cases} 0, & \text{if } n=1, \text{ for all } l,m \in \mathbf{N} \\ 2, & \text{otherwise} \end{cases}$$

Clearly the triple sequence $x_{lmn} \in c_0^3(\Delta_p)$, $c^3(\Delta_p)$, $c^{3R}(\Delta_p)$, $l_\infty^3(\Delta_p)$ and $c^{3B}(\Delta_p)$

Let the sequence (y_{lmn}) be defined by

$$y_{lmn} = \begin{cases} 0, & \text{if } n \text{ is odd, for all } l,m \in \mathbf{N} \\ lmn, & \text{otherwise} \end{cases}$$

Clearly $(y_{lmn}) \notin c_0^3(\Delta_p)$, $c^3(\Delta_p)$, $c^{3R}(\Delta_p)$, $l_\infty^3(\Delta_p)$ and $c^{3B}(\Delta_p)$, (for p odd)

Hence $c_0^3(\Delta_p)$, $c^3(\Delta_p)$, $c^{3R}(\Delta_p)$, $l_\infty^3(\Delta_p)$ and $c^{3B}(\Delta_p)$, are not convergence free.

Theorem 2.8: The classes of sequences $c_0^3(\Delta_p)$, $c^3(\Delta_p)$, $c^{3R}(\Delta_p)$, $l_\infty^3(\Delta_p)$ and $c^{3B}(\Delta_p)$ are all sequence algebra.

Proof: It is obvious.

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Author information

Shyamal Debnath, Department of Mathematics, Tripura University (A Central University), Suryamaninagar-799022, Agartala, india.
E-mail: shyamalnitamath@gmail.com

Bimal Chandra Das, Department of Mathematics, Govt. Degree College, Kamalpur, Dhalai, Tripura, India.
E-mail: bcdas3744@gmail.com

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