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MODULES THAT HAVE A SUPPLEMENT IN EVERY TORSION EXTENSION

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Abstract In this paper, over a commutative domain we define the concept of TE-modules, which is adapted from Zöschinger's modules with the property (E) over local (or, non-local) dedekind domains. In this paper, we provide some properties of these modules. We prove that a direct summand of a TE-module is a TE-module. We show that a class of TE-modules is closed under extensions. We also prove that, over a non-local ring, if every submodule of a module M is a TE-module, then it is cofinitely supplemented.

1 Introduction

Throughout this study, it is assumed that R is a commutative domain and all modules are unital left R-modules, unless otherwise stated. Let R be such a domain and let M be an R-module. The notation $(U \subset M)$ $U \subseteq M$ means that U is a (proper) submodule of M. A submodule $L \subseteq M$ is said to be *essential* in M, denoted as $L \trianglelefteq M$, if $L \cap U \neq 0$ for every nonzero submodule $U \subset M$. Dually, a proper submodule S of M is called *small* in M, denoted by $S << M$, if $M \neq S + L$ for every proper submodule L of M. By $Rad(M)$ we denote the radical of a module M, equivalently the sum of all small submodules of M as in [6]. If M has no maximal submodules, $M = Rad(M)$ and so we say the module M *radical*. In [7], a module M is said to be *coatomic* if $Rad(\frac{M}{K}) = \frac{M}{K}$ implies that $K = M$ for some submodule K of M, that is, every radical factor module of M is zero. It is well known that M is coatomic if and only if every proper submodule of M is contained in a maximal submodule of M.

A module M is said to be *injective* if it is a direct summand of every extension N. Here a modules N is *extensio*n of M provided $M \subseteq N$([4]).

As a proper generalization of direct summands of a module one defines *supplement submodules*. For U, V submodules of a module M, V is said to be a *supplement* of U in M or U is said to have a supplement V in M if it is minimal with respect to $M = U + V$, equivalently $M = U + V$ and $U \cap V \ll V$ [6].

Modules that have a supplement in every extension, i.e. modules with the property (E), was first introduced by H. Zöschinger in [8] as a proper generalization of injective modules. The author determined in the same paper the structure of modules with the property.

Also, in these recent papers [3] and [5], over an arbitrary ring two proper generalizations of modules with the property (E) are studied. Let $M \subseteq N$ be modules. If $\frac{N}{M}$ is (coatomic) finitely generated, N is called a *(coatomic) cofinite extension* of M. In [3], a module M is said to *have the property* (CE) if M has a supplement in every cofinite extension, and B. N. Türkmen [5] studies on modules that have a supplement in every coatomic extension and termed these modules E^**-modules*. Since finitely generated modules are coatomic, E^*-modules have the property (CE).

Let R be a commutative domain and M be an R-module. We denote by $T(M)$ the set of all elements m of M for which there exists a non-zero element r of R such that $rm = 0$, i.e. $Ann(m) \neq 0$. Then $T(M)$, which is a submodule of M, called *the torsion submodule* of M. If $M = T(M)$, then M is called a *torsion module* and M is called *torsion-free* provided $T(M) = 0$. For any module M, $\frac{M}{T(M)}$ is torsion-free.

For modules $M \subseteq N$ over a commutative domain, we say that N is a *torsion extension* of M if $\frac{N}{M}$ is torsion. M is called a TE*-module* if M has a supplement in every torsion extension N. In this study, we obtain various properties of these modules. We show that a class of TE-modules is closed under direct summands, extensions and finite direct sums. We prove that every submodule of a module is a TE-module if and only if it has ample supplements in every torsion extension. We also show that, over a non-local ring, if every submodule of a module M is a

TE-module, then it is cofinitely supplemented.

2 TE-Modules

Zöschinger showed in [8, Lemma 2.1] that every module with small radical over a local dedekind domain is the direct sum of a finitely generated free module and a bounded module. In [8], he generalized the concept of modules with small radical to radical supplemented modules. M is called *radical supplemented* if $Rad(M)$ has a supplement in M, and gave the various properties of radical supplemented modules over a local dedekind domain.

Clearly, every module with the property (E) is a TE-module, but the following examples show that a TE-module need not be a module with the property (E).

Recall that a module M is called *hollow* if every proper submodule of M is small in M. A finitely generated hollow module is said to be *local*. A ring R is said to be *local* if ${}_RR$ is a local module. The following fact is due to Zöschinger.

Lemma 2.1. *(see [8, Lemma 5.5]) Let R be a non-local dedekind domain and M be an R-module. Then, M is a TE-module if and only if the torsion submodule $T(M)$ of M is radical supplemented and $\frac{M}{T(M)}$ is injective.*

Corollary 2.2. *Over a non-local dedekind domain every torsion radical supplemented module is a TE-module.*

Proof. It follows from Lemma 2.1. □

Example 2.3. Consider the $\mathbb{Z}$-module $N = \prod_{p\in\Gamma} \frac{\mathbb{Z}}{p\mathbb{Z}}$, where Γ is the set of all distinct prime elements of $\mathbb{Z}$. By [2, Lemma 2.9], the torsion submodule $T(N) = \bigoplus_{p\in\Gamma} \frac{\mathbb{Z}}{p\mathbb{Z}}$ is semisimple. Put $M = T(N)$. Since semisimple modules are radical supplemented, M is radical supplemented. It follows from Corollary 2.2 that M is a TE-module. However, M hasn't the property (E) by [2, Example 2.11].

By a *valuation ring* (also called a *chain ring*) we mean a commutative ring R whose ideals are totally ordered by inclusions. Equivalently, if $a, b \in R$, then either $a \in Rb$ or $b \in Ra$. A valuation ring that is a domain will be called a *valuation domain*. A valuation ring R is called *maximal* if ${}_RR$ is linearly compact, i.e., every family of cosets $\{a_i + L_i | i \in I\}$ with the finite intersection property has a non-empty intersection.

Example 2.4. Let R be the localization ring $\mathbb{Z}_{(p)}$ of the ring $\mathbb{Z}$ of integers at a prime ideal $p\mathbb{Z} \neq 0$. Then, the completion of $\mathbb{Z}_{(p)}$, the ring $J_{(p)}$ of p-adic integers, is a maximal valution domain which is not field. By [8, Corollary 1 and Theorem 3.5], the local ring $\mathbb{Z}_{(p)}$ is a TE-module, which hasn't the property (E).

Proposition 2.5. *Every direct summand of a TE-module is a TE-module.*

Proof. Let M be a TE-module and $M = M_1 \oplus M_2$. Let N be any torsion extension of M_1. Consider the canonical embedding $\phi : M \to N'$, where N' is the external direct sum $N \oplus M_2$. Since $\frac{N}{M_1}$ is torsion, N' is a torsion extension of $\phi(M)$. By the hypothesis, $\phi(M)$ has a supplement V in N', that is, $N' = \phi(M) + V$ and $\phi(M) \cap V \ll N'$. For the projection $\pi : N' \to N$, we have that $M_1 + \pi(V) = N$. Also since $Ker(\pi) \subseteq \phi(M)$, $\pi(\phi(M) \cap V) = \pi(\phi(M)) \cap \pi(V) = M_1 \cap \pi(V) \ll N$. Hence $\pi(V)$ is a supplement of M_1 in N. □

In the following example, we show that, in general, a submodule of a TE-module need not be TE.

Example 2.6. Let M be the left $\mathbb{Z}$-module $\mathbb{Q}$, where $\mathbb{Q}$ is the quotient field of the commutative domain $\mathbb{Z}$. Since M is injective, it is a TE-module. On the other hand, the $\mathbb{Z}$-submodule $\mathbb{Z}$ is not TE by Lemma 2.1.

Let $U \subseteq M$ be modules. The submodule U has *ample supplements* in M if every submodule V of M with $M = U + V$ contains a supplement V' of U in M in [8]. Following [8], M said to have *the property* (EE) if M has ample supplements in every extension. Clearly, every linearly compact module has the property (EE).

Proposition 2.7. *For a module M, the following statements are equivalent.*

(i) *Every submodule of M is a TE-module.*

(ii) M has ample supplements in every torsion extension.

Proof. (1) $\Longrightarrow$ (2). Suppose that every submodule of M is TE. For any torsion extension N of M, let $N = M + K$ for some submodule K of N. Note that

$$\frac{N}{M} \cong \frac{K}{M \cap K}$$

is torsion. Since $M \cap K$ is a TE module, there exists a submodule L of K such that $(M \cap K) + L = K$ and $(M \cap K) \cap L = M \cap L \ll K$. Note that $N = M + K = M + ((M \cap K) + L)) = M + L$. It follows that L is a supplement of M in N.

(2) $\Longrightarrow$ (1). Let M_1 be any submodule of M. For any torsion extension N of M_1, let $F = \frac{M \oplus N}{H}$, where the submodule H is the set of all elements $(m', -m')$ of $M \oplus N$ with $m' \in M_1$. Consider these monomorphism $\gamma : M \to F$ via $\gamma(m) = (m, 0) + H$ and $\psi : N \to F$ via $\psi(n) = (0, n) + H$ for all $m \in M, n \in N$. For inclusion homomorphisms $\iota_1 : M_1 \to N$ and $\iota_2 : M_1 \to M$, we can draw the following pushout:

$$\begin{array}{ccc} M_1 & \xrightarrow{\iota_1} & N \\ \Big\downarrow{\scriptstyle \iota_2} & & \Big\downarrow{\scriptstyle \psi} \\ M & \xrightarrow{\gamma} & F \end{array}$$

It follows that $F = Im(\gamma) + Im(\psi)$. Now we define $\Psi : F \longrightarrow \frac{N}{M_1}$ by $\Psi((m, n) + H) = n + M_1$ for all $(m, n) + H \in F$. Then Ψ is an epimorphism. Note that

$$Ker(\Psi) = Im(\gamma)$$

and so

$$\frac{N}{M_1} \cong \frac{F}{Im(\gamma)}$$

is torsion. By (2), $Im(\gamma)$ has ample supplements in every torsion extension because $Im(\gamma)$ is a monomorphism. So there exists a supplement V of $Im(\gamma)$ in F such that $V \leq Im(\psi)$, i.e. $F = Im(\gamma) + V$ and $Im(\gamma) \cap V \ll F$. Then, we obtain that $N = \psi^{-1}(Im(\gamma)) + \psi^{-1}(V) = M_1 + \psi^{-1}(V)$ and $M_1 \cap \psi^{-1}(V) \ll N$. Hence, $\psi^{-1}(V)$ is a supplement of M_1 in N. □

Theorem 2.8. *Let*

$$0 \longrightarrow K \xrightarrow{f} M \xrightarrow{g} L \longrightarrow 0$$

be a short exact sequence. If K and L are TE-modules, then M is a TE-module.

Proof. Without restriction of generality we will assume that $K \leq M$. Let N be a torsion extension of M. For $K \leq M \leq N$,

$$\frac{N}{M} \cong \frac{\frac{N}{K}}{\frac{M}{K}}$$

is torsion, and so $\frac{M}{K}$ is a torsion extension of $\frac{N}{K}$. Since $L \cong \frac{M}{K}$ is a TE-module, there exists a submodule $\frac{V}{K}$ of $\frac{N}{K}$ such that $\frac{M}{K} + \frac{V}{K} = \frac{N}{K}$ and $\frac{(M \cap V)}{K} << \frac{V}{K}$. Note that $N = M + V$. Then $\frac{V}{K}$ is torsion and K has a supplement K' in V, i.e. $V = K + K'$ and $K \cap K' << K'$ because K is a TE-module. Now we have $N = M + V = M + K'$. Suppose that $M + X = N$ for some submodule X of K'. It follows that $\frac{M}{K} + \frac{(X+K)}{K} = \frac{N}{K}$, hence $\frac{(X+K)}{K} = \frac{V}{K}$ by the minimality of $\frac{V}{K}$. Then we have $V = X + K$ and so $X = K'$ by the minimality of K'. Thus K' is a supplement of M in N. Therefore M is a TE-module. □

Corollary 2.9. *Let M_i $(i = 1, 2, ..., n)$ be any finitely collection of TE-modules and $M = M_1 \oplus M_2 \oplus ... \oplus M_n$. Then, M is a TE-module.*

Proof. To prove that M is a TE-module it is sufficient by induction on n to prove this is the case when $n = 2$. Thus suppose $M = M_1 \oplus M_2$. By using the following short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$$

we have that M is a TE-module by Theorem 2.8. □

Lemma 2.10. *Let R be a ring which is not local. If M is a simple R-module, then it is torsion.*

Proof. Since M is simple, we can write $M = Rm$ for every nonzero elements $m \in M$. So we get $\frac{R}{I} \cong Rm$, where I is the ideal $Ann(m)$ of R. By the assumption, we deduce that $Ann(m) \neq 0$. Therefore, there exists a nonzero element $r \in R$ such that $rm = 0$. Hence, M is torsion. □

By $Soc(M)$ we denote the sum of all simple submodules of a module M.

Proposition 2.11. *Let R be a ring which is not local. Then, $Soc(M) \subseteq T(M)$ for every R-module M.*

Proof. Let M be an R-module. If $Soc(M) = 0$, then it is clear. Suppose that $Soc(M) \neq 0$. If m is a nonzero element of $Soc(M)$, there exist nonzero elements $m_1, m_2, ..., m_n$ of M such that $m = m_1 + m_2 + ... + m_n$, where each $(1 \leq i \leq n)$ Rm_i is a simple submodule of M. By Lemma 2.10, we can write $(1 \leq i \leq n)$ $r_i m_i = 0$ for some nonzero $r_i \in R$. Put $r = r_1 r_2 ... r_n$. Since R is a domain, we have $r \neq 0$. So $rm = 0$. Hence, $m \in T(M)$. □

A module M is called *cofinitely supplemented* if every cofinite submodule U of M (i.e. $\frac{M}{U}$ is finitely generated) has a supplement in M [1].

Corollary 2.12. *Let R be a ring which is not local and let M be an R-module. If every submodule of M is a TE-module, it is cofinitely supplemented.*

Proof. By [1, Theorem 2.8], it sufficies to show that every maximal submodule of M has a supplement in M. Let U be any maximal submodule of M. Then, $\frac{M}{U}$ is simple, and so it is torsion by Lemma 2.10. By the hypothesis, U has a supplement in M. Thus, M is cofinitely supplemented. □

Lemma 2.13. *Let M be a TE-module and N be a torsion extension of M such that $Rad(N) = 0$. Then, M is a direct summand of N.*

Proof. By the hypothesis, M has a supplement in N, say V. Since $M \cap V << V$, it follows from [6] that $M \cap V \subseteq Rad(N) = 0$. Hence, $N = M \oplus V$. □

In [6] a commutative ring R is said to be a *V-ring* if every simple left R-module is injective. It is well known that a ring R is a left V-ring if and only if $Rad(M) = 0$ for every left R-module M. The next result can be directly obtained from Lemma 2.13.

Corollary 2.14. *Let M be a TE-module over a V-ring R. Then, M is a direct summand of N with torsion $\frac{N}{M}$.*

Remark 2.15. By Example 2.4, every left R-module is not a TE-module even though R is a local dedekind domain.

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