ON TRACES IN ANALYTIC FUNCTION SPACES IN POLYBALLS

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Abstract We extend recent results on trace problem for functions holomorphic on polyballs. We give descriptions of traces for several concrete functional classes on polyballs defined with the help of Bergman metric ball. These results are new even in polydisk.

1 Introduction

The intention of this note to extend our recent results on traces of analytic function spaces in polyballs from [8], [9] and [10]. We will need first some definitions to formulate our main results.

Let $\mathbb{C}$ denote the set of complex numbers. Throughout the paper we fix a positive integer n and let $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$ denote the Euclidean space of complex dimension n. The open unit ball in $\mathbb{C}^n$ is the set $\mathbf{B} = \{z \in \mathbb{C}^n : |z| < 1\}$. The boundary of $\mathbf{B}$ will be denoted by $\mathbf{S}$, $\mathbf{S} = \{z \in \mathbb{C}^n : |z| = 1\}$.

As usual, we denote by $H(\mathbf{B})$ the class of all holomorphic functions on $\mathbf{B}$.

Let dv denote the volume measure on $\mathbf{B}$, normalized so that $v(\mathbf{B}) = 1$, and let $d\sigma$ denote the surface measure on $\mathbf{S}$ normalized so that $\sigma(\mathbf{S}) = 1$.

For $\alpha > -1$ the weighted Lebesgue measure dv_α is defined by

$$dv_\alpha = c_\alpha (1 - |z|^2)^\alpha dv(z), \tag{1.1}$$

where

$$c_\alpha = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)} \tag{1.2}$$

is a normalizing constant, so that $v_\alpha(\mathbf{B}) = 1$ (see [14]).

Let also $\widetilde{dv}_\beta(z) = dv_\beta(z_1) \ldots dv_\beta(z_m) = (1 - |z_1|^2)^\beta \cdots (1 - |z_n|^2)^\beta dv(z_1) \cdots dv(z_n)$. For $z \in \mathbf{B}$ and $r > 0$ the set $\mathcal{D}(z, r) = \{w \in \mathbf{B} : \beta(z, w) < r\}$ where β is a Bergman metric on $\mathbf{B}$, $\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}$ is called the Bergman metric ball at z (see [14]).

For $\alpha > -1$ and $p > 0$ the weighted Bergman space A_α^p consists of holomorphic functions f in $L^p(\mathbf{B}, dv_\alpha)$, that is,

$$A_\alpha^p(\mathbf{B}) = L^p(\mathbf{B}, dv_\alpha) \cap H(\mathbf{B}).$$

See [2] and [14] for more details of weighted Bergman spaces.

Let $m = 2, 3, \ldots$ be a natural number, $M \subset \mathbb{C}^n$ and $K \subset \mathbb{C}^{mn}$, $\mathbb{C}^{mn} = \mathbb{C}^n \times \cdots \times \mathbb{C}^n$, be a hyper surface. Let $X(M)$ be a class of functions on M, $Y(K)$ the same. We say $Trace Y(M^m) = X(M)$, $K = M^m$, $M^m = M \times \cdots \times M$, if for any $f \in Y(M^m)$, $f(w, \ldots, w) \in X(M)$, $w \in M$, and for any $g \in X(M)$, there exist a function $f \in Y(K)$ such that $f(w, \ldots, w) = g(w)$, $w \in M$. Traces of various functional spaces in $\mathbb{R}^n$ were described in [4] and [13]. In polydisk this problem is also known as a problem of diagonal map (see [2] and references there).

The intention of this paper is to consider the following natural Trace problem after polyballs:

Let M be a unit ball and let K be a polyball (product of m balls) in definition we gave above; let further $H(\mathbf{B} \times \cdots \times \mathbf{B})$ be a space of all holomorphic functions $f(z_1, \ldots, z_m)$, $z_j \in B$, $j = 1, \ldots, m$. Let further Y be a subspace of $H(\mathbf{B} \times \cdots \times \mathbf{B})$.

Bergman classes on polyballs $A^p(\mathbf{B}^m, dv_{\alpha_1} \cdots dv_{\alpha_m})$ consists of functions f in $H(\mathbb{B}^m)$, such that,

$$\int_{\mathbf{B}} \cdots \int_{\mathbf{B}} |f(z_1, \ldots, z_m)|^p (1 - |z_1|)^{\alpha_1} \cdots (1 - |z_m|)^{\alpha_m} dv(z_1) \cdots dv(z_m) < \infty,$$

$p \in (0, \infty)$, $\alpha_j > -1$, $j = 1, \dots, m$.

The problem we would like to study and solve in this work is the following:
Find the complete description of Trace Y in a sense of our definition for several concrete functional classes. We observe that for $n = 1$ this problem completely coincide with the well-known problem of diagonal map. The last problem of description of diagonal of various subspaces of $H(\mathbf{D}^n)$ of spaces of all holomorphic functions in the polydisk was studied earler by many authors (see [2], [3], [5], [6], [12] and references there). With the help of Bergman metric ball in $\mathbf{B}$ we introduce new holomorphic functional classes on polyballs and describe completely their Traces via classical Bergman spaces in the unit ball $\mathbf{B}$ of $\mathbb{C}^n$.

In our previous paper [8] we completely described traces of weighted Bergman classes on polyballs for all values of $p \in (0, \infty)$ and traces of some analytic Bloch type spaces on polyballs expanding known theorems on diagonal map in polydisk (see [2], [5] and references there). Some results of this paper are new even for $n = 1$ (polydisk case). Basic properties of a known so-called r-lattice in the Bergman metric that can be found in [14] and estimates of expanded Bergman projection in the unit ball are essential for our proofs.

Trace theorems even for $n = 1$ (case of polydisk) have numerous applications in the theory of holomorphic functions (see for example [1], [2], [7]).

Throughout the paper, we write C (sometimes with indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed.

Proofs of all our theorems are heavily based on properties of r-lattice $\{a_k\}$ in a Bergman metric (see [14]). Proofs are also based on same ideas as in [8], [9], [10] and [11] with some more delicate estimates. In particular we will use the following lemmas.

Lemma 1.1. *[14] There exists a positive integer N such that for any $0 < r \le 1$ we can find a sequence $\{a_k\}$ in $\mathbf{B}$ with the following properties:*
(1) $\mathbf{B} = \bigcup_k \mathcal{D}(a_k, r)$;
(2) The sets $\mathcal{D}(a_k, \frac{r}{4})$ are mutually disjoint;
(3) Each point $z \in \mathbf{B}$ belongs to at most N of the sets $\mathcal{D}(a_k, 4r)$

Lemma 1.2. *[14] For each $r > 0$ there exists a positive constant C_r such that*

$$C_r^{-1} \le \frac{1 - |a|^2}{1 - |z|^2} \le C_r, \qquad C_r^{-1} \le \frac{1 - |a|^2}{|1 - \langle z, a \rangle|} \le C_r,$$

for all a and z such that $\beta(a, z) < r$. Moreover, if r is bounded above, then we may choose C_r independent of r.

Lemma 1.3. *[14] Let $0 < p \le 1$ and $\alpha > -1$. Then*

$$\int_{\mathbf{B}} |f(z)| (1 - |z|^2)^{\frac{n+1+\alpha}{p} - (n+1)} dv(z) \le \frac{\|f\|_{p,\alpha}}{c_\alpha}$$

for all $f \in A_\alpha^p$ where c_α is the constant defined in (1.2).

2 Main results

Let $\alpha_j > -1$, $\beta_j > -1$, $\beta > -1$, $0 < p < \infty$, $j = 1, \dots m$. We define Herz type spaces of analytic functions in polyballs

$$K_{\alpha,\beta}^{p,q} = \left\{ f \in H(\mathbf{B}^m) : \int_{\mathbf{B}} \cdots \int_{\mathbf{B}} \left(\int_{\mathcal{D}(z_1,r)} \cdots \int_{\mathcal{D}(z_m,r)} |f(z)|^p d\widetilde{v}_\alpha(z) \right)^{q/p} d\widetilde{v}_\beta(z) < \infty \right\},$$

$$\tilde{K}_{\alpha,\beta}^{p,q} = \left\{ f \in H(\mathbf{B}^m) : \int_{\mathbf{B}} \left(\int_{\mathcal{D}(z,r)} \cdots \int_{\mathcal{D}(z,r)} |f(z)|^p d\widetilde{v}_\alpha(z) \right)^{q/p} dv_\beta(z) < \infty \right\}.$$

We now introduce the mixed norm classes in polyballs

$$A_{\alpha_1,\dots,\alpha_m}^{p_1,\dots,p_m}(\mathbf{B}^m) = \left\{ f \in H(\mathbf{B}^m) : \|f\|_{A_{\alpha_1,\dots,\alpha_m}^{p_1,\dots,p_m}} := \left(\int_{\mathbf{B}} (1 - |z_m|)^{\alpha_m} \left(\int_{\mathbf{B}} (1 - |z_{m-1}|)^{\alpha_{m-1}} \cdots \right. \right. \right.$$

$$\cdots \int_{\mathbf{B}} |f(z_1,\ldots,z_m)|^{p_1}(1-|z_1|)^{\alpha_1} dv(z_1))^{\frac{p_2}{p_1}} \cdots dv(z_{m-1}))^{\frac{p_m}{p_{m-1}}} dv(z_m))^{\frac{1}{p_m}} < \infty\},$$

where $0 < p_i < \infty$, $\alpha_i > -1$, $i = 1,\ldots,m$.

The first result for $p_j > 1$, $j = 1,\ldots,m$ can be seen in [11].

Theorem 2.1. *Let* $\gamma = \alpha_m + \sum_{j=1}^{m-1}(n+1+\alpha_j)\cdot\frac{p_m}{p_j}$, $\alpha_j > -1$, $0 < p_j \le 1$, $j = 1,\ldots,m$. *Then*

$$Trace\left(A_{\alpha_1,\ldots,\alpha_m}^{p_1,\ldots,p_m}(\mathbf{B}^m)\right) = A_\gamma^{p_m}(\mathbf{B}).$$

Theorems 2.2, 2.3 and 2.4 for $p = q$ were proved in [8], [9], [10] and [11] by same authors.

Theorem 2.2. *Let* $0 < p \le q \le 1$, $t_j > -1$, $\beta_j > -1$, $j = 1,\ldots,m$, $\alpha > -1$ *and*

$$\alpha = \sum_{j=1}^{m}\left[(n+1+\beta_j)\cdot\frac{p}{q} + (n+1+t_j)\right] - (n+1).$$

Then

$$Trace\left(K_{\beta,t}^{q,p}(\mathbf{B}^m)\right) = A_\alpha^p(\mathbf{B}).$$

Theorem 2.3. *Let* $1 < q$, $p \le q$, $t_j > -1$, $\beta_j > -1$, $j = 1,\ldots,m$, $\alpha > -1$ *and*

$$\alpha = \sum_{j=1}^{m}\left[(n+1+\beta_j)\cdot\frac{p}{q} + (n+1+t_j)\right] - (n+1).$$

Then

$$Trace\left(K_{\beta,t}^{q,p}(\mathbf{B}^m)\right) = A_\alpha^p(\mathbf{B}).$$

Theorem 2.4. *Let* $q > 1$ *or* $q \le 1$, $p \le q$, $t > -1$, $\beta_j > -1$, $j = 1,\ldots,m$, $\alpha > -1$ *and*

$$\alpha = \sum_{j=1}^{m}\left[(n+1+\beta_j)\cdot\frac{p}{q}\right] + t.$$

Then

$$Trace\left(\tilde{K}_{\beta_j,t}^{q,p}(\mathbf{B}^m)\right) = A_\alpha^p(\mathbf{B}).$$

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