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# **RAD-***D*<sub>12</sub> **MODULES**

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Abstract Let R be a ring and M be a right R-module. M is called (*cofinitely*) Rad- $D_{12}$  if, for every (cofinite) submodule N of M, there exist a direct summand K of M and an epimorphism  $\psi : K \to \frac{M}{N}$  with  $ker(\alpha) \subseteq Rad(K)$ . In this paper, we provide various properties of Rad- $D_{12}$  modules and cofinitely Rad- $D_{12}$  modules. In particular, we characterize semiperfect rings, perfect rings and artinian serial rings using (cofinitely) Rad- $D_{12}$  modules. Moreover, we prove that every quasi-projective Rad- $D_{12}$  module is Rad- $\oplus$ -supplemented. Finally, we show that any factor module of a (cofinitely) Rad- $D_{12}$  module by a fully invariant submodule is (cofinitely) Rad- $D_{12}$ .

### 1 Introduction

Throughout this paper, it is assumed that R is an associative ring with identity and all modules are unital right R-modules. Let M be a module. A submodule N of an R-module M will be denoted by  $N \subseteq M$ . A submodule  $N \subseteq M$  is said to be *cofinite* if  $\frac{M}{N}$  is finitely generated. Maximal submodules are cofinite. Also, every submodule of a finitely generated module is cofinite. A submodule  $L \subseteq M$  is said to be *essential* in M, denoted as  $L \trianglelefteq M$ , if  $L \cap N \neq 0$  for every nonzero submodule  $N \subseteq M$ . M is said to be *uniform* if its submodules is essential in M, and it is said to be *extending* (or a CS-module) if every submodule of M is essential in a direct summand of M. Dually, a module M is called *lifting* (or  $D_1$ ) if, every submodule N of M contains a direct summand L of M such that  $M = L \oplus K$  and  $N \cap K$  is small in M. Here a submodule S of M is called *small* in M if  $S + K \neq M$  for every proper submodule K of M. In [5, 29.10], every right R-module is lifting if and only if R is a left and right artinian serial ring with  $J^2 = 0$ , where Jis the Jacobson radical of R. A module M is called *hollow* (or couniform) if every submodule is small in M. Hollow and semisimple modules are lifting. If M has a largest proper submodule, i.e. a proper submodule which contains all other proper submodules, then M is called *local* [23].

As a generalization of direct summands, a submodule V of M is called *a supplement* of a submodule U in M if M = U + V and  $U \cap V \ll V$  [23, pp. 348]. A module M is called *(cofinitely) supplemented* if every (cofinite) submodule has a supplement in M, and it is called *amply (cofinitely) supplemented* if, whenever (cofinite submodule N) M = N + K, N has a supplement  $V \subseteq K$  in M. Every right R-module is (cofinitely) supplemented if and only if R is right (semi)perfect [1, Theorem 2.13] and [23, 43.9]. It can be seen that M is lifting if and only if it is amply supplemented and every supplement in M is a direct summand of M. Following [18], M is said to be *cofinitely lifting* if it is amply cofinitely supplemented and supplements of a cofinite submodule of M is a direct summand of M.

Mohamed and Müller [12] call a module  $M \oplus$ -supplemented (according to [8],  $(D_{11})$ ) if every submodule N of M has a supplement that is a direct summand of M. Lifting modules are  $\oplus$ -supplemented. Zöschinger proved in [25] that every supplemented module over a dedekind domain is  $\oplus$ -supplemented. Moreover, it follows from [8, Theorem 1.1] a commutative ring R is an artinian principal ideal ring if and only if every right R-module is  $\oplus$ -supplemented. In [7], a module M is called  $\oplus$ -cofinitely supplemented if every cofinite submodule of M has a supplement that is a direct summand of M. For a module M, consider the following condition:

 $(D_{12})$  For every submodule N of M, there exist a direct summand K of M and an epimorphism  $\alpha: \frac{M}{K} \to \frac{M}{N}$  such that  $ker(\alpha) \ll \frac{M}{K}$ .

Modules with the property  $(D_{12})$  are extensively studied in [11]. In addition, it is proven in [11, Proposition 4.3] that every  $\oplus$ -supplemented module has the property  $(D_{12})$ . Wang [22] generalizes modules with  $(D_{12})$  to cofinitely  $(D_{12})$ -modules. A module M is called *cofinitely*  $(D_{12})$  if, for every cofinite submodule N of M, there exist a direct summand K of M and an epimorphism  $\alpha : \frac{M}{K} \to \frac{M}{N}$  such that  $ker(\alpha) \ll \frac{M}{K}$ . He obtained various properties of these modules in the same paper.

We will denote by Rad(M), namely radical of M, the sum of all small submodules of a module M. We say that a submodule V of a module M is *Rad-supplement* (in [24], *generalized supplement*) of a submodule U in M if M = U + V and  $U \cap V \subseteq Rad(V)$  as in [5, pp. 100]. Clearly, we have the following diagram on submodules.

### direct summand $\implies$ supplement $\implies$ Rad-supplement

Motivated by the above definitions, we say a module M is (cofinitely) Rad-supplemented if every (cofinite) submodule of M has a Rad-supplement in M, and M is (cofinitely) Rad- $\oplus$ supplemented if every (cofinite) submodule of M has a Rad-supplement that is a direct summand in M as in these papers [4], [13] and [19]. Characterizations of those modules are studied in the same papers. From [13], we will use  $cgs^{\oplus}$  instead of cofinitely Rad- $\oplus$ -supplemented.

Recall from [1] that a module M has the property  $(P^*)$  if, for every submodule N of M, M has the decomposition  $M = L \oplus K$  such that  $L \subseteq N$  and  $N \cap K \subseteq Rad(K)$ . Modules with the property  $(P^*)$  is a dual notion of modules with the property (P) which is a generalization of extending modules [1]. Clearly lifting modules have the property  $(P^*)$ . Also, by [19, Proposition 2.9], a projective module with the property  $(P^*)$  is lifting.

Talebi et.al. call a module  $M \operatorname{Rad-} D_{12}$  if, for every submodule N of M, there exist a direct summand K of M and an epimorphism  $\psi : K \to \frac{M}{N}$  with  $\ker(\alpha) \subseteq \operatorname{Rad}(K)$ . Some properties of Rad- $D_{12}$  modules are given in [17]. It is shown in [17, Proposition 2.1] that every Rad- $\oplus$ supplemented module is Rad- $D_{12}$ . It is of obvious interest to study characterizations of Rad- $D_{12}$ modules by rings. In Theorem 2.9, we will prove that a ring R is right perfect if and only if every projective right R-module is Rad- $D_{12}$ . In particular, we shall show in Theorem 2.16 that every right R-module is Rad- $D_{12}$  if and only if a commutative ring R is an artinian serial ring.

Let M be a module. We call a module M cofinitely  $Rad-D_{12}$  if for every cofinite submodule N of M, there exist a direct summand K of M and an epimorphism  $\alpha : K \to \frac{M}{N}$  such that  $ker(\alpha) \subseteq Rad(K)$ . We will investigate various properties of cofinitely  $Rad-D_{12}$  modules in section 2.

Under given definitions, we clearly have the following implication on modules:



In this paper, we give a new characterization of semiperfect rings via cofinitely Rad- $D_{12}$  modules. Every non-radical indecomposable cofinitely Rad- $D_{12}$ -module is  $\omega$ -local. We show that if every right *R*-module is cofinitely Rad- $D_{12}$ , then *R* is a noetherian serial ring.

## 2 (Cofinitely) Rad-D<sub>12</sub> Modules

In this section, we will give characterizations of (semi)perfect rings in terms of (cofinitely) Rad- $D_{12}$ . In particular, we will determine commutative rings whose modules are Rad- $D_{12}$ .

Recall from [23] that an epimorphism  $f: P \longrightarrow M$  is called a *cover* if  $Ker(f) \ll P$ , and a cover f is called a *projective cover* if P is a projective module. In the spirit of [23], a module M is said to be *semiperfect* if every factor module of M has a projective cover. Every semiperfect module is supplemented. A ring R is called *semiperfect*, if every finitely generated right (or left) *R*-module has a projective cover, and a ring *R* is called *right perfect* if every right *R*-module has a projective cover.

The proof of the next result is taken from [17, Proposition 2.1], but is given for the sake of completeness.

### **Proposition 2.1.** *Every* $cgs^{\oplus}$ *-module is cofinitely* Rad- $D_{12}$ *.*

*Proof.* Let N be a cofinite submodule of M. Since M is  $cgs^{\oplus}$ , then there exist direct summands K and K' of M such that  $M = N + K = K \oplus K'$  and  $N \cap K \subseteq Rad(K)$ . Now we have the epimorphism g from K to  $\frac{M}{N}$  which is defined by  $k \to k + N$  with  $ker(g) = N \cap K \subseteq Rad(K)$ . Hence M is a cofinitely Rad- $D_{12}$  module. 

The following example shows that a cofinitely Rad- $D_{12}$  module need not  $cgs^{\oplus}$ .

**Example 2.2.** (See [11, Examples 4.5§4.6]) Let *R* be a local artinian ring with radical *J* such that  $J^2 = 0, Q = \frac{R}{T}$  is commutative, dim(QJ) = 2 and dim(JQ) = 1. Consider the indecomposable injective right *R*-module  $U = \begin{bmatrix} \frac{(R \oplus R)}{D} \end{bmatrix}$  with J = Ru + Rv and  $D = \{(ur, -vr) \mid r \in R\}$ . Now let  $S = \frac{R}{J}$ , the simple *R*-module, and  $M = U \oplus S$ . By [11, Example 4.6], *M* is cofinitely Rad- $D_{12}$ , but not  $cgs^{\oplus}$ .

Recall from [23] that an *R*-module *M* is called *quasi-projective* if, for every *R*-module *K*, every R-epimorphism  $\xi: M \to K$ , and every R-homomorphism  $f: M \to K$ , there is an  $\gamma \in End_R(M)$  such that  $\xi \circ \gamma = f$ . Now we prove that every quasi-projective (cofinitely) Rad- $D_{12}$  module is Rad- $\oplus$ -supplemented ( $cgs^{\oplus}$ ).

**Theorem 2.3.** Let M be a quasi-projective module.

- (i) If M is Rad- $D_{12}$ , then M is a Rad- $\oplus$ -supplemented module.
- (ii) If M is cofinitely Rad- $D_{12}$ , then M is a  $cgs^{\oplus}$ -module.

*Proof.* (1) Let N be a submodule of M. Then there exist a direct summand K of M and an epimorphism  $\alpha: K \to \frac{M}{N}$  with  $ker(\alpha) \subseteq Rad(K)$ . Let  $\pi: M \to \frac{M}{N}$  be the natural epimorphism. Since M is a quasi-projective, we have the homomorphism  $h: M \to K$  with  $\pi = \alpha \circ h$ . Since K is M-projective, h splits. Hence there is a direct summand K' of M with  $h|_{K'}: K' \cong K$ . So  $\pi|_{K'}$  is an epimorphism. Therefore M = K' + N and  $N \cap K' = ker(\pi|_{K'}) \subseteq Rad(M)$ . Since K' is a direct summand of  $M, N \cap K' \subseteq Rad(K')$ . Thus M is Rad- $\oplus$ -supplemented. 

(2) The proof can be made similar to (1).

Clearly, every cofinitely  $(D_{12})$ -module is cofinitely Rad- $D_{12}$ . But the converse is not always true the following example shows. Recall from [4] that a module M is called  $\omega$ -local if it has a unique maximal submodule. It is clear that a module is  $\omega$ -local if and only if its radical is maximal.

**Example 2.4.** (See [16, Theorem 4.3 and Remark 4.4]) Let M be a biuniform module and let S = End(M). Assume that P is a projective S-module with dim(P) = (1,0). Then P is an indecomposable  $\omega$ -local module. Since dim(P) = (1,0), we conclude that P is not finitely generated. Hence, P is a  $cgs^{\oplus}$ -module but not  $\oplus$ -cofinitely supplemented. Thus, P is cofinitely Rad- $D_{12}$  but not cofinitely  $(D_{12})$ .

**Example 2.5.** (See [9, 11.3]) Let R denote the ring K[[x]] of all power series  $\sum_{i=0}^{\infty} k_i x^i$  in an indeterminate x and with coefficients from a field K which is a local ring. Note that R is a semiperfect ring that is not perfect. Then by [7, Theorem 2.9] and [10, Corollary 2.11], the free (projective) R-module  $R^{(N)}$  is  $\oplus$ -cofinitely supplemented but not  $\oplus$ -supplemented. It follows that  $R^{(N)}$  is  $cgs^{\oplus}$ . By [19, Theorem 2.2],  $R^{(N)}$  is not Rad- $\oplus$ -supplemented. Therefore  $R^{(N)}$  is cofinitely Rad- $D_{12}$  but not Rad- $D_{12}$  by Theorem 2.3.

**Theorem 2.6.** For a ring R, R is semiperfect if and only if every free right R-module is cofinitely Rad- $D_{12}$ .

*Proof.*  $(\Rightarrow)$  Suppose that a ring R is semiperfect. By [13, Theorem 2.4], every free right Rmodule is  $cgs^{\oplus}$ . Then by Proposition 2.1, every free right *R*-module is cofinitely Rad- $D_{12}$ .

 $(\Leftarrow)$  Since every free right *R*-module is cofinitely Rad- $D_{12}$ , it is  $cgs^{\oplus}$  by Theorem 2.3. It follows from [13, Theorem 2.4] that R is semiperfect.  **Proposition 2.7.** Let M be a cofinitely Rad- $D_{12}$  module. If  $Rad(M) \ll M$ , then M is a cofinitely  $(D_{12})$ -module.

*Proof.* Let N be a cofinite submodule of M. Since M is cofinitely Rad- $D_{12}$ , there exist a direct summand K of M and an epimorphism  $\alpha : K \to \frac{M}{N}$  such that  $ker(\alpha) \ll M$ . Since K is a direct summand of M,  $ker(\alpha) \ll K$ . Hence M is a cofinitely  $(D_{12})$ -module.

A module M is called *coatomic* if every proper submodule is contained in a maximal submodule of M. Every coatomic module has a small radical. Using the above proposition, we obtain the following corollary.

**Corollary 2.8.** *Every coatomic cofinitely*  $Rad-D_{12}$  *module is cofinitely*  $(D_{12})$ *.* 

**Theorem 2.9.** For a ring R, R is right perfect if and only if every projective right R-module is  $Rad-D_{12}$ .

*Proof.* The proof follows from Theorem 2.3(1) and [19, Corollary 2.3].

A module M is called *radical* if Rad(M) = M.

**Proposition 2.10.** Let M be a non-radical indecomposable module. Suppose that M is a cofinitely Rad- $D_{12}$  module. Then M is  $\omega$ -local.

*Proof.* Suppose that  $Rad(M) \neq M$ . Then M contains a maximal submodule N. By the hypothesis, there exist a direct summand K of M and an epimorphism  $\alpha : K \to \frac{M}{N}$  with  $ker(\alpha) \subseteq Rad(K)$ . Note that  $K \neq 0$ . Since M is indecomposable, K = M. Therefore  $\alpha : M \to \frac{M}{N}$  is an epimorphism with  $ker(\alpha) \subseteq Rad(M)$ . It follows that  $\frac{M}{ker(\alpha)} \cong \frac{M}{N}$ . Since N is a maximal submodule of M,  $ker(\alpha)$  is a maximal submodule of M. But  $ker(\alpha) \subseteq Rad(M)$ . Thus Rad(M) is a maximal submodule of M. Hence M is  $\omega$ -local.

**Corollary 2.11.** Every finitely generated indecomposable, (cofinitely)  $Rad-D_{12}$  module is local.

In [15, 1.4] a module M is called *uniserial* if its lattice of submodules is a chain. By [5, 2.17], a module M is uniserial if and only if every submodule of M is hollow. A module M is said to be *serial* if M is a direct sum of uniserial modules. A commutative ring R is called *uniserial* if the module  $_RR$  (or  $R_R$ ) is uniserial, and the ring is called *serial* if the module  $_RR$  (or  $R_R$ ) is serial.

Recall from [5, 1.5] that a module M is uniform if and only if every non-zero submodule of M is indecomposable.

**Proposition 2.12.** Let *M* be a uniform module over a local commutative ring *R*. Then the following statements are equivalent.

- (i) M is uniserial.
- (ii) Every submodule of M is cofinitely Rad- $D_{12}$ .

*Proof.*  $(i) \Rightarrow (ii)$  Clear.

 $(ii) \Rightarrow (i)$  Let N be a finitely generated submodule of M. By (2), N is Rad- $D_{12}$ . Since N is indecomposable, applying Corollary 2.11 we obtain that N is local. It follows from [5, 2.17] that M is uniserial.

By E(M) we denote the injective hull of a module M. Note that the injective hull of a simple module is uniform.

**Corollary 2.13.** Let R be a local commutative ring. Suppose that M is the module  $E(\frac{R}{Rad(R)})$ , and every submodule of M is cofinitely Rad-D<sub>12</sub>. Then, R is uniserial.

*Proof.* Since M is uniform, the hypothesis implies that M is uniserial by Proposition 2.12. It follows from [20, 6.2] that R is uniserial.

**Lemma 2.14.** (See [8, Theorem 1.1], [19, Corollary 2.15]) Let R be a commutative ring. Then the following statements are equivalent.

- (*i*) *R* is an artinian serial ring.
- (ii) Every R-module is  $\oplus$ -supplemented.
- (iii) Every R-module is Rad-⊕-supplemented.

By Lemma 2.14, every module over an artinian serial ring is Rad- $D_{12}$ . Now we show that the converse of this fact is true in the following Theorem. Firstly, we have:

**Proposition 2.15.** Let R be a commutative ring. If every right R-module is cofinitely Rad- $D_{12}$ , then R is a serial ring.

*Proof.* Let M be a free R-module. By the hypothesis, M is cofinitely Rad- $D_{12}$ . It follows from Theorem 2.6 that R is semiperfect. Note that  $R = R_1 \oplus R_2 \oplus ... \oplus R_n$  such that the ring  $R_i$  is local for all  $1 \le i \le n$  with  $n \in \mathbb{N}$  ([23, 42.6]). For all  $1 \le i \le n$ ,  $R_i$  is commutative and every  $R_i$ -module is cofinitely Rad- $D_{12}$  by assumption. Using Corollary 2.13, we get  $R_i$  is uniserial. Thus R is a serial ring.

**Theorem 2.16.** The following statements are equivalent for a commutative ring R.

- (*i*) *R* is an artinian serial ring.
- (ii) Every R-module is Rad- $D_{12}$ .

*Proof.*  $(i) \Rightarrow (ii)$  Clear.

 $(ii) \Rightarrow (i)$  Applying Theorem 2.9, we obtain that R is perfect. It follows from ([23, 42.6]) that we can write  $R = R_1 \oplus R_2 \oplus ... \oplus R_n$ , where each  $R_i$  is a local perfect ring for all  $1 \le i \le n$ . By Corollary 2.13 and the hypothesis, it can be seen easily that each  $R_i$  is noetherian. Therefore, R is a serial noetherian ring as a finite direct sum of uniserial noetherian rings  $R_i$ . Applying [9, 11.6.4(c)], we deduce that R is an artinian serial ring.

Let M be a module.  $U \subseteq M$  is called QSL in M if  $\frac{(A+U)}{U}$  is a direct summand of  $\frac{M}{U}$ , then there exists a direct summand P of M such that  $P \subseteq A$  and A + U = P + U [3]. M is said to be *cofinitely weak Rad-supplemented* if every cofinite submodule U of M has a weak Radsupplement in M, i.e. there exists a submodule V of M such that M = U + V and  $U \cap V \subseteq Rad(M)$  [6].

**Proposition 2.17.** Let M be a cofinitely weak Rad-supplemented module with Rad(M) QSL in M. Then M is cofinitely Rad- $D_{12}$ .

*Proof.* Let N be a cofinite submodule of M. Then  $\frac{M}{N}$  is finitely generated, and so

$$\frac{\frac{M}{Rad(M)}}{\frac{N+Rad(M)}{Rad(M)}} \cong \frac{\frac{M}{N}}{\frac{N+Rad(M)}{N}}$$

is finitely generated. Thus  $\frac{N+Rad(M)}{Rad(M)}$  is a cofinite submodule of  $\frac{M}{Rad(M)}$ . By [6, Corollary 2.5],  $\frac{N+Rad(M)}{Rad(M)}$  is a direct summand of  $\frac{M}{Rad(M)}$ . Since Rad(M) is QSL in M, there exists a decomposition  $M = K \oplus L$  such that  $K \subseteq N$  and N + Rad(M) = K + Rad(M). Now consider the epimorphism  $\alpha : L \to \frac{M}{N}$  defined by  $\alpha(l) = l + N$  ( $l \in L$ ). Since  $M = K \oplus L$ , then  $Rad(M) = Rad(K) \oplus Rad(L)$ . It follows that N + Rad(L) = K + Rad(L) and, so  $L \cap N + Rad(L) = L \cap K + Rad(L) = Rad(L)$ . Note that  $Ker(\alpha) = L \cap N \subseteq Rad(L)$ . Hence M is cofinitely Rad- $D_{12}$ .

A module M is called *refinable* if for any submodules U, V of M with M = U + V, there exists a direct summand U' of M with  $U' \subseteq U$  and M = U' + V [5, 11.26]. It is easy to see that M is *refinable* if and only if every submodule of M is QSL.

**Corollary 2.18.** Let M be a cofinitely weak Rad-supplemented refinable module. Then M is cofinitely Rad- $D_{12}$ .

*Proof.* Clear by Proposition 2.17.

**Proposition 2.19.** Let M be a cofinitely Rad- $D_{12}$  module. If  $Rad(M) \neq M$ , then M has a non-zero  $\omega$ -local direct summand.

*Proof.* Let N be a maximal submodule of M. Then N is a cofinite submodule of M. Since M is a cofinitely Rad- $D_{12}$  module, there exist a direct summand K of M and an epimorphism  $\alpha : K \to \frac{M}{N}$  such that  $ker(\alpha) \subseteq Rad(K)$ . Clearly,  $K \neq 0$  and  $ker(\alpha)$  is a maximal submodule of K. Therefore  $ker(\alpha) = Rad(K)$  and hence K is a non-zero  $\omega$ -local direct summand of M.

Recall from [23] that an *R*-module *M* has the summand sum property (SSP) if the sum of two direct summands of *M* is again a direct summand of *M*, and a submodule *U* of an *R*-module *M* is called *fully invariant* if f(U) is contained in *U* for every *R*-endomorphism *f* of *M*. Let *M* be an *R*-module and let  $\tau$  be a preradical for the category of *R*-modules. Then Rad(M), P(M) and  $\tau(M)$  are fully invariant submodules of *M*. An *R*-module *M* is called a (*weak*) duo module if every (direct summand) submodule of *M* is fully invariant. Note that weak duo modules have SSP (See [14]).

The following Example shows a cofinitely Rad- $D_{12}$  module that contains a direct summand which is not cofinitely Rad- $D_{12}$ .

**Example 2.20.** Consider the right *R*-module  $M = U \oplus S$  in Example 2.2. The module *M* is cofinitely Rad- $D_{12}$ , but the submodule *U* is not cofinitely Rad- $D_{12}$ .

**Theorem 2.21.** Let  $M = M_1 \oplus M_2$ . Then  $M_2$  is cofinitely Rad- $D_{12}$  if and only if for every cofinite submodule N of M containing  $M_1$ , there exist a direct summand K of  $M_2$  and an epimorphism  $\varphi: M \to \frac{M}{N}$  such that K is a direct summand Rad-supplement of  $ker(\varphi)$  in M.

*Proof.* Suppose that  $M_2$  is a cofinitely Rad- $D_{12}$  module. Let N be a cofinite submodule of M with  $M_1 \subseteq N$ . Consider the submodule  $N \cap M_2$  of  $M_2$ . Since  $\frac{M_2}{N \cap M_2} \cong \frac{M}{N}$ ,  $N \cap M_2$  is a cofinite submodule of  $M_2$ . Then there exist a direct summand K of  $M_2$  and an epimorphism  $\alpha : K \to \frac{M_2}{N \cap M_2}$  such that  $ker(\alpha) = N \cap K \subseteq Rad(K)$ . Note that  $M = N + M_2$  and K is a direct summand of M. Let  $M = K \oplus K'$  for some submodule K' of M. Consider the projection map  $\xi : M \to K$  and the isomorphism  $\beta : \frac{M_2}{N \cap M_2} \to \frac{M}{N}$  defined by  $\beta(x + N \cap M_2) = x + N$ . Thus  $\beta \circ \alpha \circ \xi : M \to \frac{M}{N}$  is an epimorphism. Let  $\varphi = \beta \circ \alpha \circ \xi$ . Clearly, we have  $ker(\varphi) = N + K' = ker(\alpha) \oplus K'$ . Therefore  $M = K + ker(\varphi)$ . Moreover  $K \cap ker(\varphi) = K \cap N = ker(\alpha) \subseteq Rad(K)$ .

Conversely, suppose that every cofinite submodule of M containing  $M_1$  has the stated property. Let H be a cofinite submodule of  $M_2$ . Consider the submodule  $H \oplus M_1$  of M. Since  $\frac{M}{H \oplus M_1} \cong \frac{M_2}{H}$  is finitely generated,  $H \oplus M_1$  is a cofinite submodule of M. By the hypothesis, there exist a direct summand K of  $M_2$  and an epimorphism  $\mu : M \to \frac{M}{H \oplus M_1}$  such that  $M = K + ker(\mu)$  and  $K \cap ker(\mu) \subseteq Rad(K)$ . Let  $g : K \to \frac{M}{H \oplus M_1}$  be the restriction of  $\mu$  to K. Consider the isomorphism  $\eta : \frac{M}{H \oplus M_1} \to \frac{M_2}{H}$  defined by  $\eta(m_1 + m_2 + (H \oplus M_1)) = m_2 + H$ . Therefore  $\eta \circ g : K \to \frac{M_2}{H}$  is an epimorphism. Let  $\kappa = \eta \circ g$ . Clearly,  $ker(\kappa) \subseteq Rad(K)$ . Hence  $M_2$  is a cofinitely Rad- $D_{12}$  module.

**Theorem 2.22.** Let  $\{M_i\}_{i \in I}$  be any family of cofinitely Rad- $D_{12}$  modules on a ring R and  $M = \bigoplus_{i \in I} M_i$ . If every cofinite submodule of M is fully invariant, then M is a cofinitely Rad- $D_{12}$  module.

*Proof.* Let N be a cofinite submodule of M. Since N is fully invariant, we have  $N = \bigoplus_{i \in I} (N \cap M_i)$ . Since  $\frac{M}{N} \cong \bigoplus_{i \in I} \frac{M_i}{N \cap M_i}$ , for every  $i \in I$ ,  $N \cap M_i$  is a cofinite submodule of  $M_i$ . Then there exist a direct summand  $K_i$  of  $M_i$  and an epimorphism  $\alpha_i : K_i \to \frac{M_i}{N \cap M_i}$  with  $ker(\alpha_i) \subseteq Rad(K_i)$ . Now we define the homomorphism  $\alpha : \bigoplus_{i \in I} K_i \to \bigoplus_{i \in I} \frac{M_i}{N \cap M_i}$  by  $k_{i_1} + \ldots + k_{i_n} \mapsto \alpha_{i_1}(k_{i_1}) + \ldots + \alpha_{i_n}(k_{i_n})$  with  $k_{i_j} \in K_{i_j}$  for every  $j = 1, 2, \ldots, n$ . It is not hard to check that  $\alpha$  is an epimorphism with  $ker(\alpha) \subseteq Rad(\bigoplus_{i \in I} K_i)$  and  $\bigoplus_{i \in I} K_i$  is a direct summand of M. It follows that M is cofinitely Rad- $D_{12}$ .

**Proposition 2.23.** Let M be a cofinitely Rad- $D_{12}$  module with the property SSP. Suppose that L is a direct summand of M. Then,  $\frac{M}{L}$  is a cofinitely Rad- $D_{12}$  module.

*Proof.* Let M be a cofinitely Rad- $D_{12}$  module and  $\frac{N}{L}$  be a cofinite submodule of  $\frac{M}{L}$ . Then N is a cofinite submodule of M. Since M is a cofinitely Rad- $D_{12}$  module, there exist a direct summand K of M and an epimorphism  $\alpha : K \to \frac{M}{N}$  with  $ker(\alpha) \subseteq Rad(K)$ . Since M has the property SSP, K + L is a direct summand of M. Therefore there exists a submodule X of M such that  $M = (K + L) \oplus X$ . Note that  $\frac{M}{L} = \frac{K+L}{L} \oplus \frac{X+L}{L}$ . Because  $\frac{K+L}{L} \cap \frac{X+L}{L} \subseteq \frac{X \cap (K+L)+L \cap (K+L+X)}{L} = \frac{L}{L}$ . Since  $\frac{M}{K} \cong \frac{M}{N}$ , we can define the homomorphism  $\alpha' : \frac{K+L}{L} \to \frac{M}{N}$  by  $k+l+L = k+L \mapsto \alpha(k)$  with  $k \in K$ ,  $l \in L$ . It's easy to see that  $\alpha'$  is an epimorphism with  $ker(\alpha') \subseteq Rad(\frac{K+L}{L})$  and  $\frac{K+L}{L}$  is a direct summand of  $\frac{M}{L}$ . Hence  $\frac{M}{L}$  is a cofinitely Rad- $D_{12}$  module.

**Theorem 2.24.** Let M be a (cofinitely) Rad- $D_{12}$  module. If L is a fully invariant submodule of M, then  $\frac{M}{L}$  is a (cofinitely) Rad- $D_{12}$  module.

*Proof.* Let  $\frac{N}{L}$  be a (cofinite) submodule of  $\frac{M}{L}$ . Then N is a (cofinite) submodule of M. Since M is a (cofinitely) Rad- $D_{12}$  module, there exist a direct summand K of M and an epimorphism  $\alpha : K \to \frac{M}{N}$  with  $ker(\alpha) \subseteq Rad(K)$ . It follows that there exists a submodule K' of M such that  $M = K \oplus K'$ . Since L is a fully invariant submodule of M,  $L = (L \cap K) \oplus (L \cap K')$ . It is clear that  $\frac{M}{L} = \frac{K+L}{L} \oplus \frac{K'+L}{L}$ . Since  $\frac{M}{L} \cong \frac{M}{N}$ , we can define the homomorphism  $\beta : \frac{K+L}{L} \to \frac{M}{N}$  by  $k + L \mapsto \beta(k + L) = \alpha(k)$  with  $k \in K$ . Then  $\beta$  is an epimorphism and  $ker(\beta) \subseteq Rad(\frac{K+L}{L})$ . Hence  $\frac{M}{L}$  is a (cofinitely) Rad- $D_{12}$  module.

### References

- I. Al-Khazzi and P.F. Smith, Modules with Chain Conditions on Superfluous Submodules, *Communica*tions in Algebra, 19(8), 2331-2351, (1991).
- [2] R. Alizade, G. Bilhan and P. F. Smith, Modules whose maximal submodules have supplements, *Comm. Algebra*, 29(6), 2389-2405 (2001).
- [3] M. Alkan, On  $\tau$ -Lifting and  $\tau$ -Semiperfect Modules, *Turkish J. Math.*, volume=**33**, 117-139, (2009).
- [4] E. Büyükaşık and C. Lomp, On a Recent Generalization of Semiperfect Rings, Bulletin of The Australian Mathematical Society, 78(2), 317-325, (2008).
- [5] J. Clark, C. Lomp, N. Vanaja and R. Wisbauer, Lifting Modules. Supplements and Projectivity in Module Theory, *Frontiers in Mathematics, Birkhäuser-Basel*, 406, (2006).
- [6] S.K. Choubey, B.M. Pandeya, A.J. Gupta and H. Ranjan, Cofinitely Weak Rad-Supplemented Modules, *Research Journal of Pure Algebra*, 2(9), 259-262, (2012).
- [7] H. Çalışıcı and A. Pancar, ⊕-Cofinitely Supplemented Modules, Chech. Math. J., 54(129), 1083-1088, (2004).
- [8] A. Idelhadj and R. Tribak, Modules for Which Every Submodule Has a Supplement That Is a Direct Summand, *The Arabian Journal for Sciences and Engineering*, 25(2C), 179-189, (2000).
- [9] F. Kasch, Modules and Rings, publisher=Academic Press Inc., (1982).
- [10] D. Keskin, P.F. Smith and W. Xue, Rings Whose Modules Are ⊕-Supplemented, *Journal of Algebra*, 218, 470-487, (1999).
- [11] D. Keskin and W. Xue, Generalizations of Lifting Modules, Acta Math. Hungar, 91(3), 253-261, (2001).
- [12] S. H. Mohamed and B. J. Müller, Continuous and Discrete modules, series=London Math. Soc. LNS 147 Cambridge University-Cambridge, 190, (1990).
- [13] B. Nişancı Türkmen and A. Pancar, On Generalization of ⊕-Cofinitely Supplemented Modules, Ukrainian Mathematical Journal, 62(2), 2003-209, (2010).
- [14] A. Ç. Özcan, A. Harmacı and P.F. Smith, Duo Modules, *Glasgow Math. J.*, 48, 533-545, (2006).
- [15] G. Puninski, Serial Rings, Kluwer Academic Publishers, Dordrecht, Boston, London, ISBN 0-7923-7187-9, (2001).
- [16] G. Puninski, Projective Modules over the Endomorphism Ring of a Biuniform Module, J. Pure Appl. Algebra, 188, 227-246, (2004).
- [17] Y. Talebi, A.R.M. Hamzekolaee and D. Keskin Tütüncü, On Rad-D<sub>12</sub> Modules, An. Şt. Univ. Ovidius Constanta, 21(1), 201-208, (2013).
- [18] R. Tribak, On Cofinitely Lifting and Cofinitely Weak Lifting Modules, *Communications in Algebra*, 36(12), 4448-4460, (2008).
- [19] E. Türkmen, Rad-⊕-Supplemented Modules, An. Şt. Univ. Ovidius Constanta, 21(1), 225-238, (2013).
- [20] D. W. Sharpe and P. Vamos, Injective Modules, *Lectures in Pure Mathematics University of Sheffield-The Great Britain*, (1972).
- [21] Y. Wang and N. Ding, Generalized Supplemented Modules, Taiwanese J. Math., 6, 1589-1601, (2006).
- [22] Y. Wang, Cofinitely (D<sub>12</sub>)-Modules, JP Journal of Algebra, Number Theory and Applications, 27(2), 143-149, (2012).
- [23] R. Wisbauer, Foundations of modules and rings, Gordon and Breach, (1991).
- [24] W. Xue, Characterizations of Semiperfect and Perfect Modules, *Publicacions Matematiques*, 40(1), 115-125, (1996).
- [25] H. Zöschinger, Komplementierte moduln über Dedekindringen, Jounal of Algebra, 29, 42-56 (1974).

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