

Some Sharper Estimations of Growth Relationships of Composite Entire Functions on the Basis of Their Maximum Terms

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Abstract. In this paper we discuss the growth rates of the maximum term of composition of entire functions with their corresponding left and right factors.

1 Introduction, Definitions and Notations.

Let f be an entire function defined in the open complex plane \mathbb{C} . The maximum term $\mu(r, f)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined by $\mu(r, f) = \max_{n \geq 0} (|a_n| r^n)$. We do not explain the standard definitions and notations in the theory of entire function as those are available in [7]. In the sequel the following two notations are used :

$$\log^{[k]} x = \log \left(\log^{[k-1]} x \right) \text{ for } k = 1, 2, 3, \dots ;$$

$$\log^{[0]} x = x$$

and

$$\exp^{[k]} x = \exp \left(\exp^{[k-1]} x \right) \text{ for } k = 1, 2, 3, \dots ;$$

$$\exp^{[0]} x = x.$$

To start our paper we just recall the following definitions:

Definition 1.1. The order ρ_f and lower order λ_f of an entire function f are defined as follows:

$$\rho_f = \limsup_{r \rightarrow \infty} \frac{\log^{[2]} M(r, f)}{\log r} \text{ and } \lambda_f = \liminf_{r \rightarrow \infty} \frac{\log^{[2]} M(r, f)}{\log r}.$$

Definition 1.2. [4] Let l be an integer ≥ 2 . The generalised order $\rho_f^{[l]}$ and generalised lower order $\lambda_f^{[l]}$ of an entire function f are defined as

$$\rho_f^{[l]} = \limsup_{r \rightarrow \infty} \frac{\log^{[l]} M(r, f)}{\log r} \text{ and } \lambda_f^{[l]} = \liminf_{r \rightarrow \infty} \frac{\log^{[l]} M(r, f)}{\log r}.$$

When $l = 2$, Definition 2 coincides with Definition 1.

Juneja, Kapoor and Bajpai[2] defined the (p, q) th order and (p, q) th lower order of an entire function f respectively as follows:

$$\rho_f(p, q) = \limsup_{r \rightarrow \infty} \frac{\log^{[p]} M(r, f)}{\log^{[q]} r} \quad \text{and} \quad \lambda_f(p, q) = \liminf_{r \rightarrow \infty} \frac{\log^{[p]} M(r, f)}{\log^{[q]} r},$$

where p, q are positive integers with $p > q$.

For $p = 2$ and $q = 1$, we respectively denote $\rho_f(2, 1)$ and $\lambda_f(2, 1)$ by ρ_f and λ_f .

Since for $0 \leq r < R$,

$$\mu(r, f) \leq M(r, f) \leq \frac{R}{R-r} \mu(R, f) \quad \{cf. [6]\}$$

it is easy to see that

$$\begin{aligned} \rho_f &= \limsup_{r \rightarrow \infty} \frac{\log^{[2]} \mu(r, f)}{\log r}, & \lambda_f &= \liminf_{r \rightarrow \infty} \frac{\log^{[2]} \mu(r, f)}{\log r}; \\ \rho_f^{[l]} &= \limsup_{r \rightarrow \infty} \frac{\log^{[l]} \mu(r, f)}{\log r}, & \lambda_f^{[l]} &= \liminf_{r \rightarrow \infty} \frac{\log^{[l]} \mu(r, f)}{\log r}; \end{aligned}$$

and

$$\rho_f(p, q) = \limsup_{r \rightarrow \infty} \frac{\log^{[p]} \mu(r, f)}{\log^{[q]} r}, \quad \lambda_f(p, q) = \liminf_{r \rightarrow \infty} \frac{\log^{[p]} \mu(r, f)}{\log^{[q]} r}.$$

Definition 1.3. Let “ a ” be a complex number, finite or infinite. The Nevanlinna’s deficiency of “ a ” with respect to a meromorphic function f are defined as

$$\delta(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r \rightarrow \infty} \frac{m(r, a; f)}{T(r, f)}.$$

In this paper we wish to prove some results relating to the growth rates of maximum terms of composition of two entire functions with their corresponding left and right factors on the basis of (p, q) th order ((p, q) th lower order) where p, q are positive integers with $p > q$.

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [5] Let f and g be any two entire functions with $g(0) = 0$. Then for all sufficiently large values of r ,

$$\mu(r, f \circ g) \geq \frac{1}{2} \mu \left(\frac{1}{8} \mu \left(\frac{r}{4}, g \right) - |g(0)|, f \right).$$

Lemma 2.2. [1] If f and g are any two entire functions then for all sufficiently large values of r ,

$$M(r, f \circ g) \leq M(M(r, g), f).$$

Lemma 2.3. [3] Let g be an entire function with $\lambda_g < \infty$ and assume that $a_i (i = 1, 2, \dots, n; n \leq \infty)$ are entire functions satisfying $T(r, a_i) = o\{T(r, g)\}$. If $\sum_{i=1}^n \delta(a_i, g) = 1$, then

$$\lim_{r \rightarrow \infty} \frac{T(r, g)}{\log M(r, g)} = \frac{1}{\pi}.$$

Lemma 2.4. Let f be an entire function with non zero finite generalised order $\rho_f^{[l]}$ (non zero finite generalised lower order $\lambda_f^{[l]}$). If $p-q = l-1$, then the (p, q) -th order $\rho_f(p, q)$ (lower (p, q) -th order $\lambda_f(p, q)$) of f will be equal to 1. If $p - q \neq l - 1$, then $\rho_f(p, q)$ ($\lambda_f(p, q)$) is either zero or infinity.

Proof. From the definition of generalised order of an entire function f we have for all sufficiently large values of r ,

$$\log^{[l]} \mu(r, f) \leq (\rho_f^{[l]} + \varepsilon) \log r \quad (2.1)$$

and for a sequence of values of r tending to infinity,

$$\log^{[l]} \mu(r, f) \geq (\rho_f^{[l]} - \varepsilon) \log r. \quad (2.2)$$

Next let a and b be any two positive integers.

Now from (2.1) we have for all sufficiently large values of r ,

$$\begin{aligned} \log^{[l+a]} \mu(r, f) &\leq \log^{[1+a]} r + O(1) \\ \text{i.e., } \frac{\log^{[l+a]} \mu(r, f)}{\log^{[1+b]} r} &\leq \frac{\log^{[1+a]} r + O(1)}{\log^{[1+b]} r}. \end{aligned} \quad (2.3)$$

If we take $l + a = p$ and $1 + b = q$, then $p - q = (l - 1) + (a - b)$.

We discuss the following three cases:

Case I. Let $a = b$. Then from (2.3) we get for all sufficiently large values of r ,

$$\begin{aligned} \frac{\log^{[p]} \mu(r, f)}{\log^{[q]} r} &\leq 1 + \frac{O(1)}{\log^{[1+a]} r} \\ \text{i.e., } \limsup_{r \rightarrow \infty} \frac{\log^{[p]} \mu(r, f)}{\log^{[q]} r} &\leq 1. \end{aligned} \quad (2.4)$$

Similarly from (2.2) we have for a sequence of values of r tending to infinity,

$$\begin{aligned} \frac{\log^{[p]} \mu(r, f)}{\log^{[q]} r} &\geq 1 + \frac{O(1)}{\log^{[1+a]} r} \\ \text{i.e., } \limsup_{r \rightarrow \infty} \frac{\log^{[p]} \mu(r, f)}{\log^{[q]} r} &\geq 1. \end{aligned} \quad (2.5)$$

Now from (2.4) and (2.5) we have

$$\rho_f(p, q) = 1 \text{ when } p - q = l - 1.$$

Case II. Let $a > b$ (i.e., $p - q \neq l - 1$). Then from (2.3) we have for all sufficiently large values of r ,

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{\log^{[p]} \mu(r, f)}{\log^{[q]} r} &\leq 0 \\ \text{i.e., } \rho_f(p, q) &= 0 \text{ when } p - q \neq l - 1. \end{aligned}$$

Case III. Also let us choose a and b such that $a < b$ and $l + a > 1 + b$ (i.e., $p - q \neq l - 1$). Then from (2.2) it can be proved for a sequence of values of r tending to infinity that

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{\log^{[p]} \mu(r, f)}{\log^{[q]} r} &\geq \infty \\ \text{i.e., } \rho_f(p, q) &= \infty \text{ when } p - q \neq l - 1. \end{aligned}$$

Therefore combining Case II and Case III (not violating the condition $p > q$), it follows that $\rho_f(p, q)$ is either zero or infinity.

Similarly we may prove the conclusion for $\lambda_f(p, q)$.

This proves the lemma. \square

3 Theorems.

In this section we present the main results of the paper.

Theorem 3.1. *Let f, g, h and k be four entire functions such that $0 < \rho_f(p, q) < \infty$, $\lambda_h(a, b) > 0$, $0 < \rho_k^{[l]} < \infty$ and $\rho_g(m, n) < \rho_k^{[l]}$, where m, n, a, b, p, q are all positive integers with $m > n$; $a > b$; $p > q$ and $l > 1$. Then*

$$(i) \quad \limsup_{r \rightarrow \infty} \frac{\log^{[a]} \mu(r, h \circ k)}{\log^{[p]} \mu(r, f \circ g) + \log^{[m]} \mu(r, g)} = \infty$$

if $b = l - 1$ and $q \geq m$;

$$(ii) \quad \limsup_{r \rightarrow \infty} \frac{\log^{[a]} \mu(r, h \circ k)}{\log^{[p+m-q-1]} \mu(r, f \circ g) + \log^{[m]} \mu(r, g)} = \infty$$

if $b = l - 1$ and $q < m$;

$$(iii) \quad \limsup_{r \rightarrow \infty} \frac{\log^{[a]} \mu(r, h \circ k)}{\log^{[p]} \mu(r, f \circ g) + \log^{[m]} \mu(r, g)} = \infty$$

if $q \geq m$ and $l < b + 1 < n + l$;

$$(iv) \quad \limsup_{r \rightarrow \infty} \frac{\log^{[a]} \mu(r, h \circ k)}{\log^{[p]} \mu(r, f \circ g) + \log^{[m]} \mu(r, g)} \geq \frac{\rho_k^{[l]} \lambda_h(a, b)}{(\rho_f(p, q) + 1) \rho_g(m, n)}$$

if $q \geq m$, $b = l$ and $n = 1$;

$$(v) \quad \limsup_{r \rightarrow \infty} \frac{\log^{[a]} \mu(r, h \circ k)}{\log^{[p]} \mu(r, f \circ g) + \log^{[m]} \mu(r, g)} \geq \frac{\lambda_h(a, b)}{(\rho_f(p, q) + 1) \rho_g(m, n)}$$

if $q \geq m$ and $b - l + 1 = n > 2$;

$$(vi) \quad \limsup_{r \rightarrow \infty} \frac{\log^{[a]} \mu(r, h \circ k)}{\log^{[p+m+n-q-1]} \mu(r, f \circ g) + \log^{[m]} \mu(r, g)} = \infty$$

if $q < m$ and $l < b + 1 < n + l$;

$$(vii) \quad \limsup_{r \rightarrow \infty} \frac{\log^{[a]} \mu(r, h \circ k)}{\log^{[p+m+n-q-1]} \mu(r, f \circ g) + \log^{[m]} \mu(r, g)} \geq \frac{\rho_k^{[l]} \lambda_h(a, b)}{1 + \rho_g(m, n)}$$

if $q < m$, $b = l$ and $n = 2$

and

$$(viii) \quad \limsup_{r \rightarrow \infty} \frac{\log^{[a]} \mu(r, h \circ k)}{\log^{[p+m+n-q-1]} \mu(r, f \circ g) + \log^{[m]} \mu(r, g)} \geq \frac{\lambda_h(a, b)}{1 + \rho_g(m, n)}$$

if $q < m$ and $b - l + 1 = n > 2$.

Proof. Since for $0 \leq r < R$,

$$\mu(r, f) \leq M(r, f) \leq \frac{R}{R-r} \mu(R, f) \quad \{cf. [6]\}, \quad (3.1)$$

In view of Lemma 2.2 and the above inequality we obtain for all sufficiently large values of r that

$$\begin{aligned} \mu(r, f \circ g) &\leq M(r, f \circ g) \leq M(M(r, g), f) \\ \log^{[p]} \mu(r, f \circ g) &\leq \log^{[p]} M(M(r, g), f) \\ i.e., \log^{[p]} \mu(r, f \circ g) &\leq (\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g). \end{aligned} \quad (3.2)$$

Now the following cases may arise :

Case I. Let $q \geq m$. Then we have from (3.2) for all sufficiently large values of r ,

$$\log^{[p]} \mu(r, f \circ g) \leq (\rho_f(p, q) + \varepsilon) \log^{[m-1]} M(r, g). \quad (3.3)$$

Now from the definition of (m, n) th order of g we get for arbitrary positive ε and for all sufficiently large values of r ,

$$\log^{[m]} M(r, g) \leq (\rho_g(m, n) + \varepsilon) \log^{[n]} r \quad (3.4)$$

$$i.e., \log^{[m]} M(r, g) \leq (\rho_g(m, n) + \varepsilon) \log r. \quad (3.5)$$

Also for all sufficiently large values of r it follows from (3.5) that

$$\log^{[m-1]} M(r, g) \leq r^{(\rho_g(m, n) + \varepsilon)}. \quad (3.6)$$

So from (3.3) and (3.6) it follows for all sufficiently large values of r that

$$\log^{[p]} \mu(r, f \circ g) \leq (\rho_f(p, q) + \varepsilon) r^{(\rho_g(m, n) + \varepsilon)}. \quad (3.7)$$

Case II. Let $q < m$. Then we get from(3.2) for all sufficiently large values of r that

$$\log^{[p]} \mu(r, f \circ g) \leq (\rho_f(p, q) + \varepsilon) \exp^{[m-q]} \log^{[m]} M(r, g). \quad (3.8)$$

Again from (3.5) for all sufficiently large values of r ,

$$\begin{aligned} \exp^{[m-q]} \log^{[m]} M(r, g) &\leq \exp^{[m-q]} \log r^{(\rho_g(m, n) + \varepsilon)} \\ i.e., \exp^{[m-q]} \log^{[m]} M(r, g) &\leq \exp^{[m-q-1]} r^{(\rho_g(m, n) + \varepsilon)}. \end{aligned} \quad (3.9)$$

Now from (3.8) and (3.9) we obtain for all sufficiently large values of r that

$$\begin{aligned} \log^{[p]} \mu(r, f \circ g) &\leq (\rho_f(p, q) + \varepsilon) \exp^{[m-q-1]} r^{(\rho_g(m, n) + \varepsilon)} \\ i.e., \log^{[p+1]} \mu(r, f \circ g) &\leq \exp^{[m-q-2]} r^{(\rho_g(m, n) + \varepsilon)} \\ i.e., \log^{[p+m-q-1]} \mu(r, f \circ g) &\leq \log^{[m-q-2]} \exp^{[m-q-2]} r^{(\rho_g(m, n) + \varepsilon)} \\ i.e., \log^{[p+m-q-1]} \mu(r, f \circ g) &\leq r^{(\rho_g(m, n) + \varepsilon)}. \end{aligned} \quad (3.10)$$

Since $\rho_g(m, n) < \rho_k^{[l]}$, we can choose $\varepsilon(> 0)$ in such a way that

$$\rho_g(m, n) + \varepsilon < \rho_k^{[l]} - \varepsilon. \quad (3.11)$$

By Lemma 2.1 we obtain for a sequence of values of r tending to infinity,

$$\begin{aligned} \log^{[a]} \mu(r, h \circ k) &\geq \log^{[a]} \mu\left(\frac{1}{8} \mu\left(\frac{r}{4}, k\right), h\right) + O(1) \\ \text{i.e., } \log^{[a]} \mu(r, h \circ k) &\geq (\lambda_h(a, b) - \varepsilon) \log^{[b]} \mu\left(\frac{r}{4}, k\right) + O(1) \\ \text{i.e., } \log^{[a]} \mu(r, h \circ k) &\geq (\lambda_h(a, b) - \varepsilon) \log^{[b-l+1]} \log^{[l-1]} \mu\left(\frac{r}{4}, k\right) + O(1). \end{aligned} \quad (3.12)$$

Now the following two cases may arise :

Case III. Let $b = l - 1$. Then from (3.12) we get for a sequence of values of r tending to infinity that

$$\log^{[a]} \mu(r, h \circ k) \geq (\lambda_h(a, b) - \varepsilon) \left(\frac{r}{4}\right)^{(\rho_k^{[l]} - \varepsilon)} + O(1). \quad (3.13)$$

Case IV. Let $b - l + 1 = d > 0$. Then from (3.12) it follows for a sequence of values of r tending to infinity that

$$\log^{[a]} \mu(r, h \circ k) \geq (\lambda_h(a, b) - \varepsilon) \log^{[d]} \left(\frac{r}{4}\right)^{(\rho_k^{[l]} - \varepsilon)}. \quad (3.14)$$

Now from the definition of (m, n) th order of g we have for arbitrary positive ε and for all sufficiently large values of r ,

$$\log^{[m]} \mu(r, g) \leq (\rho_g(m, n) + \varepsilon) \log^{[n]} r. \quad (3.15)$$

Let $q \geq m$. Then we have from (3.2) and (3.4) for all sufficiently large values of r ,

$$\log^{[p]} \mu(r, f \circ g) \leq (\rho_f(p, q) + \varepsilon) (\rho_g(m, n) + \varepsilon) \log^{[n]} r. \quad (3.16)$$

Now if $b = l - 1$ and $q \geq m$, we get from (3.7), (3.13), (3.15) and in view of (3.11) for a sequence of values of r tending to infinity,

$$\begin{aligned} &\frac{\log^{[a]} \mu(r, h \circ k)}{\log^{[p]} \mu(r, f \circ g) + \log^{[m]} \mu(r, g)} \\ &\geq \frac{(\lambda_h(a, b) - \varepsilon) \left(\frac{r}{4}\right)^{(\rho_k^{[l]} - \varepsilon)} + O(1)}{(\rho_f(p, q) + \varepsilon) r^{(\rho_g(m, n) + \varepsilon)} + (\rho_g(m, n) + \varepsilon) \log^{[n]} r} \\ \text{i.e., } \limsup_{r \rightarrow \infty} &\frac{\log^{[a]} \mu(r, h \circ k)}{\log^{[p]} \mu(r, f \circ g) + \log^{[m]} \mu(r, g)} = \infty, \end{aligned}$$

which proves the first part of the theorem.

Again we obtain from (3.10), (3.11), (3.13) and (3.15), for a sequence of values of r tending to infinity when $b = l - 1$ and $q < m$

$$\begin{aligned} &\frac{\log^{[a]} \mu(r, h \circ k)}{\log^{[p+m-q-1]} \mu(r, f \circ g) + \log^{[m]} \mu(r, g)} \geq \frac{(\lambda_h(a, b) - \varepsilon) \left(\frac{r}{4}\right)^{(\rho_k^{[l]} - \varepsilon)} + O(1)}{r^{(\rho_g(m, n) + \varepsilon)} + (\rho_g(m, n) + \varepsilon) \log^{[n]} r} \\ \text{i.e., } \limsup_{r \rightarrow \infty} &\frac{\log^{[a]} \mu(r, h \circ k)}{\log^{[p+m-q-1]} \mu(r, f \circ g) + \log^{[m]} \mu(r, g)} = \infty. \end{aligned}$$

This proves the second part of the theorem.

When $b > l - 1$ and $q \geq m$, from (3.14), (3.15), and (3.16) we get for a sequence of values of r tending to infinity,

$$\begin{aligned} & \frac{\log^{[a]} \mu(r, h \circ k)}{\log^{[p]} \mu(r, f \circ g) + \log^{[m]} \mu(r, g)} \\ & \geq \frac{(\lambda_h(a, b) - \varepsilon) \log^{[a]} \left(\frac{r}{4}\right)^{(\rho_k^{[l]} - \varepsilon)}}{(\rho_f(p, q) + \varepsilon)(\rho_g(m, n) + \varepsilon) \log^{[n]} r + (\rho_g(m, n) + \varepsilon) \log^{[n]} r} \\ \text{i.e., } \limsup_{r \rightarrow \infty} & \frac{\log^{[a]} \mu(r, h \circ k)}{\log^{[p]} \mu(r, f \circ g) + \log^{[m]} \mu(r, g)} = \infty \text{ if } l < b + 1 < n + l; \end{aligned}$$

again

$$\begin{aligned} & \limsup_{r \rightarrow \infty} \frac{\log^{[a]} \mu(r, h \circ k)}{\log^{[p]} \mu(r, f \circ g) + \log^{[m]} \mu(r, g)} \\ & \geq \frac{\rho_k^{[l]} \lambda_h(a, b)}{(\rho_f(p, q) + 1) \rho_g(m, n)} \text{ if } b = l, n = 1; \end{aligned}$$

and also

$$\begin{aligned} & \limsup_{r \rightarrow \infty} \frac{\log^{[a]} \mu(r, h \circ k)}{\log^{[p]} \mu(r, f \circ g) + \log^{[m]} \mu(r, g)} \\ & \geq \frac{\lambda_h(a, b)}{(\rho_f(p, q) + 1) \rho_g(m, n)} \text{ if } b - l + 1 = n > 2. \end{aligned}$$

This respectively proves the third, fourth and fifth part of the theorem.

Again when $b > l - 1$ and $q < m$, combining (3.10), (3.14) and (3.15) we obtain for a sequence of values of r tending to infinity,

$$\begin{aligned} & \frac{\log^{[a]} \mu(r, h \circ k)}{\log^{[p+m+n-q-1]} \mu(r, f \circ g) + \log^{[m]} \mu(r, g)} \\ & \geq \frac{(\lambda_h(a, b) - \varepsilon) \log^{[a]} \left(\frac{r}{4}\right)^{(\rho_k^{[l]} - \varepsilon)}}{\log^{[n]} r + (\rho_g(m, n) + \varepsilon) \log^{[n]} r} \\ \text{i.e., } \limsup_{r \rightarrow \infty} & \frac{\log^{[a]} \mu(r, h \circ k)}{\log^{[p+m+n-q-1]} \mu(r, f \circ g) + \log^{[m]} \mu(r, g)} = \infty \text{ if } l < b + 1 < n + l; \end{aligned}$$

also

$$\begin{aligned} & \limsup_{r \rightarrow \infty} \frac{\log^{[a]} \mu(r, h \circ k)}{\log^{[p+m+n-q-1]} \mu(r, f \circ g) + \log^{[m]} \mu(r, g)} \\ & \geq \frac{\rho_k^{[l]} \lambda_h(a, b)}{1 + \rho_g(m, n)} \text{ if } b = l, n = 1 \end{aligned}$$

and again

$$\begin{aligned} & \limsup_{r \rightarrow \infty} \frac{\log^{[a]} \mu(r, h \circ k)}{\log^{[p+m+n-q-1]} \mu(r, f \circ g) + \log^{[m]} \mu(r, g)} \\ & \geq \frac{\lambda_h(a, b)}{1 + \rho_g(m, n)} \text{ if } b - l + 1 = n > 2, \end{aligned}$$

from which the sixth, seventh and eighth part of the theorem follows respectively. \square

Remark 3.2. The condition $\rho_g(m, n) < \rho_k$ and $\rho_f(p, q) < \infty$ in Theorem 3.1 are essential as we see in the following examples:

Example 3.3. Let

$$f = g = h = k = \exp z.$$

Also let

$$p = m = a = 2 \text{ and } q = n = b = 1.$$

Then

$$\rho_f = 1, \rho_g = 1 = \rho_k \text{ and } \lambda_h = 1.$$

Now in view of the inequality $\mu(r, f) \leq M(r, f) \leq \frac{R}{R-r} \mu(R, f)$ {cf. [6]} and $T(r, g) \leq \log^+ M(r, g)$, we get that

$$\begin{aligned} \log \mu(r, h \circ k) &\leq \log M(r, h \circ k) \\ &\leq 3T(2r, h \circ k) \sim \frac{3 \exp(2r)}{(4\pi^3 r)^{\frac{1}{2}}} \end{aligned}$$

$$\text{i.e., } \log^{[2]} \mu(r, h \circ k) \leq 2r - \frac{1}{2} \log r + O(1)$$

and

$$\begin{aligned} \log \mu(r, f \circ g) &\geq \log M\left(\frac{r}{2}, f \circ g\right) + O(1) \\ &\geq T\left(\frac{r}{2}, f \circ g\right) + O(1) \sim \frac{\exp\left(\frac{r}{2}\right)}{(2\pi^3 \frac{r}{2})^{\frac{1}{2}}} \end{aligned}$$

$$\text{i.e., } \log^{[2]} \mu(r, f \circ g) \geq \frac{r}{2} - \frac{1}{2} \log r + O(1).$$

So

$$\limsup_{r \rightarrow \infty} \frac{\log^{[2]} \mu(r, h \circ k)}{\log^{[2]} \mu(r, f \circ g) + \log^{[2]} \mu(r, g)} \leq \limsup_{r \rightarrow \infty} \frac{2r - \frac{1}{2} \log r + O(1)}{\frac{r}{2} - \frac{1}{2} \log r + O(1) + \log \frac{r}{2}}$$

$$\begin{aligned} &\text{i.e., } \limsup_{r \rightarrow \infty} \frac{\log^{[2]} \mu(r, h \circ k)}{\log^{[2]} \mu(r, f \circ g) + \log^{[2]} \mu(r, g)} \\ &\leq \limsup_{r \rightarrow \infty} \frac{2r - \frac{1}{2} \log r + O(1)}{\frac{r}{2} + \frac{1}{2} \log r + O(1)} = 1, \end{aligned}$$

which is contrary to Theorem 3.1.

Example 3.4. Let

$$f = \exp^{[2]} z, g = h = \exp z \text{ and } k = \exp(z^2)$$

and

$$p = m = a = 2 \text{ and } q = n = b = 1.$$

Then

$$\rho_f = \infty, \rho_g = 1 < 2 = \rho_k \text{ and } \lambda_h = 1.$$

Now

$$\begin{aligned} \log \mu(r, h \circ k) &\leq \log M(r, h \circ k) = \log \exp^{[2]}(r^2) \\ \text{i.e., } \log \mu(r, h \circ k) &\leq \exp(r^2), \\ \text{and } \log \mu(r, f \circ g) &\geq \log M\left(\frac{r}{2}, f \circ g\right) \\ \text{i.e., } \log \mu(r, f \circ g) &\geq \log \exp^{[3]}\left(\frac{r}{2}\right) = \exp^{[2]}\left(\frac{r}{2}\right) \\ \text{and } \log^{[2]} \mu(r, g) &\geq \log^{[2]} M\left(\frac{r}{2}, g\right) = 1. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\log^{[2]} \mu(r, h \circ k)}{\log^{[2]} \mu(r, f \circ g) + \log^{[2]} \mu(r, g)} &\leq \frac{\log \exp(r^2)}{\log \exp^{[2]}\left(\frac{r}{2}\right) + O(1) + \log^{[2]} \exp r} \\ \text{i.e., } \frac{\log^{[2]} \mu(r, h \circ k)}{\log^{[2]} \mu(r, f \circ g) + \log^{[2]} \mu(r, g)} &\leq \frac{r^2}{\exp\left(\frac{r}{2}\right) + \log r + O(1)} \\ \text{i.e., } \limsup_{r \rightarrow \infty} \frac{\log^{[2]} \mu(r, h \circ k)}{\log^{[2]} \mu(r, f \circ g) + \log^{[2]} \mu(r, g)} &= 0, \end{aligned}$$

which is contrary to Theorem 3.1.

Remark 3.5. The condition $\rho_g(m, n) < \lambda_k$ in Theorem 3.1 is necessary which is true in general only if $\rho_f(p, q) > 0$ otherwise the condition $\rho_g(m, n) < \rho_k$ will be violated. The following example ensures this comment:

Example 3.6. Let

$$f = h = k = \exp z \text{ and } g = \exp(z^3).$$

Also let

$$p = 3, m = a = 2 \text{ and } q = n = b = 1.$$

Then

$$\bar{\rho}_f = \rho_f(3, 1) = 0 < \infty, \rho_g = 3 > 1 = \rho_k \text{ and } \lambda_h = 1.$$

Now

$$\begin{aligned} \log \mu(r, h \circ k) &\geq \log M\left(\frac{r}{2}, h \circ k\right) + O(1) \\ &\geq T\left(\frac{r}{2}, h \circ k\right) + O(1) \sim \frac{\exp\left(\frac{r}{2}\right)}{(2\pi^3 \frac{r}{2})^{\frac{1}{2}}} \\ \text{i.e., } \log^{[2]} \mu(r, h \circ k) &\geq \frac{r}{2} - \frac{1}{2} \log r + O(1). \end{aligned}$$

$$\begin{aligned} \log \mu(r, f \circ g) &\leq \log M(r, f \circ g) = \exp r^3 \\ \text{i.e., } \log^{[2]} \mu(r, f \circ g) &\leq 3 \log r. \\ \text{and } \log^{[2]} \mu(r, g) &\leq \log^{[2]} M(r, g) = 3 \log r. \end{aligned}$$

Therefore

$$\frac{\log^{[2]} \mu(r, h \circ k)}{\log^{[2]} \mu(r, f \circ g) + \log^{[2]} \mu(r, g)} \geq \frac{\frac{r}{2} - \frac{1}{2} \log r + O(1)}{6 \log r}$$

$$i.e., \lim_{r \rightarrow \infty} \frac{\log^{[2]} \mu(r, h \circ k)}{\log^{[2]} \mu(r, f \circ g) + \log^{[2]} \mu(r, g)} = \infty.$$

Theorem 3.7. Let f and g be any two entire functions such that $\lambda_f(p, q)$ and λ_g are both finite and p, q are any two positive integers with $p > q$. Also suppose that there exist entire functions $a_i (i = 1, 2, \dots, n; n \leq \infty)$ satisfying

(A) $T(r, a_i) = o\{T(r, g)\}$ as $r \rightarrow \infty$ and

(B) $\sum_{i=1}^n \delta(a_i, g) = 1$. Then for any $R > 2r$,

$$\liminf_{r \rightarrow \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log \mu(R, g)} \leq \frac{3\lambda_f(p, q)}{\pi}.$$

Proof. In view of (3.1) we have from Lemma 2.2, for all sufficiently large values of r ,

$$\log^{[p]} \mu(r, f \circ g) \leq \log^{[p]} M(M(r, g), f). \quad (3.17)$$

Since $\varepsilon (> 0)$ is arbitrary and $T(r, g) \leq \log^+ M(r, g) \leq 3T(2r, g)$, from (3.17) and in view of (3.1) we get for a sequence of values of r tending to infinity that

$$\begin{aligned} \log^{[p]} \mu(r, f \circ g) &\leq (\lambda_f(p, q) + \varepsilon) \log^{[q]} M(r, g) \\ i.e., \log^{[p]} \mu(r, f \circ g) &\leq (\lambda_f(p, q) + \varepsilon) \log M(r, g) \\ i.e., \log^{[p]} \mu(r, f \circ g) &\leq 3(\lambda_f(p, q) + \varepsilon)T(2r, g) \\ i.e., \frac{\log^{[p]} \mu(r, f \circ g)}{\log \mu(R, g)} &\leq \frac{3(\lambda_f(p, q) + \varepsilon)T(2r, g)}{\log M(2r, g)}. \end{aligned} \quad (3.18)$$

Since $\varepsilon (> 0)$ is arbitrary, by Lemma 2.3 it follows from (3.18) that

$$\liminf_{r \rightarrow \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log \mu(R, g)} \leq \frac{3\lambda_f(p, q)}{\pi}.$$

This proves the theorem. □

Theorem 3.8. Let f and g be any two entire functions such that $\rho_f(p, q)$ and λ_g are both finite and p, q are any two positive integers with $p > q$. Also suppose that there exist entire functions $a_i (i = 1, 2, \dots, n; n \leq \infty)$ satisfying

(A) $T(r, a_i) = o\{T(r, g)\}$ as $r \rightarrow \infty$ and

(B) $\sum_{i=1}^n \delta(a_i, g) = 1$. Then for any $R > 2r$,

$$\limsup_{r \rightarrow \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log \mu(R, g)} \leq \frac{3\rho_f(p, q)}{\pi}.$$

The proof of Theorem 3.8 is omitted as it can be carried out in the line of Theorem 3.7.

Theorem 3.9. Let f be an entire function such that $\rho_f(p, q) < \infty$, where p, q are positive integers with $p > q > 1$. Also let g be entire. If $\lambda_{f \circ g}(p, q) = \infty$ then for every positive number β ,

$$\lim_{r \rightarrow \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[p]} \mu(r^\beta, f)} = \infty.$$

Proof. Let us suppose that the conclusion of the theorem does not hold. Then we can find a constant $\alpha > 0$ such that for a sequence of values of r tending to infinity

$$\log^{[p]} \mu(r, f \circ g) \leq \alpha \log^{[p]} \mu(r^\beta, f). \quad (3.19)$$

Again for $q > 1$, from the definition of $\rho_f(p, q)$ it follows that for all sufficiently large values of r

$$\begin{aligned} \log^{[p]} \mu(r^\beta, f) &\leq (\rho_f(p, q) + \varepsilon) \log^{[q]}(r^\beta) \\ \text{i.e., } \log^{[p]} \mu(r^\beta, f) &\leq (\rho_f(p, q) + \varepsilon) \log^{[q]} r + O(1). \end{aligned} \quad (3.20)$$

Thus from (3.19) and (3.20) we have for a sequence of values of r tending to infinity that

$$\begin{aligned} \log^{[p]} \mu(r, f \circ g) &\leq \alpha (\rho_f(p, q) + \varepsilon) \log^{[q]} r + O(1) \\ \text{i.e., } \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[q]} r} &\leq \frac{\alpha (\rho_f(p, q) + \varepsilon) \log^{[q]} r + O(1)}{\log^{[q]} r} \\ \text{i.e., } \liminf_{r \rightarrow \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[q]} r} &= \lambda_{f \circ g}(p, q) < \infty. \end{aligned}$$

This is a contradiction.

This proves the theorem. \square

Remark 3.10. Theorem 3.9 is also valid with “limit superior” instead of “limit” if $\lambda_{f \circ g}(p, q) = \infty$ is replaced by $\rho_{f \circ g}(p, q) = \infty$ and the other conditions remaining the same.

Corollary 3.11. Under the assumptions of Remark 3.10,

$$\limsup_{r \rightarrow \infty} \frac{\log^{[p-1]} \mu(r, f \circ g)}{\log^{[p-1]} \mu(r^\beta, f)} = \infty.$$

Proof. From Remark 3.10, we obtain for all sufficiently large values of r and for $K > 1$,

$$\begin{aligned} \log^{[p]} \mu(r, f \circ g) &> K \log^{[p]} \mu(r^\beta, f) \\ \text{i.e., } \log^{[p-1]} \mu(r, f \circ g) &> \left\{ \log^{[p-1]} \mu(r^\beta, f) \right\}^K, \end{aligned}$$

from which the corollary follows. \square

Corollary 3.12. Under the same conditions of Theorem 3.9 if $q = 1$

$$\lim_{r \rightarrow \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[p]} \mu(r^\beta, f)} = \infty.$$

Corollary 3.13. Under the same conditions of Remark 3.10 if $q = 1$

$$\limsup_{r \rightarrow \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[p]} \mu(r^\beta, f)} = \infty.$$

Remark 3.14. The condition $\lambda_{f \circ g}(p, 1) = \infty$ in Corollary 3.12 is necessary as we see in the following example.

Example 3.15. Let $f = \exp z$, $g = z$ and $p = 2$, $q = 1$, $\beta = 1$.
Then $\rho_f(p, 1) = \lambda_{f \circ g}(p, 1) = 1$.

Now

$$\log \mu(r, f \circ g) \leq \log M(r, f \circ g) = \log M(r, \exp z) = r$$

and

$$\log \mu(r, f \circ g) \geq \log M\left(\frac{r}{2}, f \circ g\right) = \log M\left(\frac{r}{2}, \exp z\right) = \frac{r}{2}.$$

Then

$$\begin{aligned} & \lim_{r \rightarrow \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[p]} \mu(r^\beta, f)} \\ & \leq \lim_{r \rightarrow \infty} \frac{\log^{[2]} \mu(r, f \circ g)}{\log^{[2]} \mu(r, f)} \\ & \leq \lim_{r \rightarrow \infty} \frac{\log r}{\log \frac{r}{2}} \\ & \leq 1 \neq \infty, \text{ which is contrary to Corollary 3.12.} \end{aligned}$$

Remark 3.16. Considering $f = \exp z$, $g = z$ and $p = 2$, $q = 1$, $\beta = 1$ one can easily verify that the condition $\rho_{f \circ g}(p, 1) = \infty$ in Corollary 3.13 is essential.

Theorem 3.17. Let f and g be any two entire functions such that $\rho_g(m, n) = 1 < \lambda_f(p, q) \leq \rho_f(p, q) < \infty$, where p, q, m, n are positive integers with $p > q$ and $m - n = 1$. Then for any $R > r$,

$$\begin{aligned} (i) \quad & \lim_{r \rightarrow \infty} \frac{\left\{ \log^{[p]} \mu(\exp^{[n-1]} r, f \circ g) \right\}^2}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f) \log^{[q]} \mu(\exp^{[n-1]} R, g)} \\ & = 0 \text{ if } q \geq m \end{aligned}$$

and

$$\begin{aligned} (ii) \quad & \lim_{r \rightarrow \infty} \frac{\log^{[p+m-q-1]} \mu(\exp^{[n-1]} r, f \circ g) \log^{[p]} \mu(\exp^{[n-1]} r, f \circ g)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f) \log^{[q]} \mu(\exp^{[n-1]} R, g)} \\ & = 0 \text{ if } q < m. \end{aligned}$$

Proof. From the definition of (p, q) th lower order of f we have for arbitrary positive ε and for all sufficiently large values of r ,

$$\begin{aligned} \log^{[p]} \mu(\exp^{[q-1]} r, f) & \geq (\lambda_f(p, q) - \varepsilon) \log^{[q]} \exp^{[q-1]} r \\ \text{i.e., } \log^{[p]} \mu(\exp^{[q-1]} r, f) & \geq (\lambda_f(p, q) - \varepsilon) \log r \\ \text{i.e., } \log^{[p-1]} \mu(\exp^{[q-1]} r, f) & \geq r^{(\lambda_f(p, q) - \varepsilon)}. \end{aligned} \quad (3.21)$$

Again in view of (3.1) we obtain from Lemma 2.2, for all sufficiently large values of r that

$$\begin{aligned} \log^{[p]} \mu(\exp^{[n-1]} r, f \circ g) & \leq \log^{[p]} M(M(\exp^{[n-1]} r, g), f) \\ \text{i.e., } \log^{[p]} \mu(\exp^{[n-1]} r, f \circ g) & \\ & \leq (\rho_f(p, q) + \varepsilon) \log^{[q]} M(\exp^{[n-1]} r, g). \end{aligned} \quad (3.22)$$

Now the following two cases may arise :

Case I. Let $q \geq m$. Then from (3.22), we get for all sufficiently large values of r that

$$\begin{aligned} & \log^{[p]} \mu(\exp^{[n-1]} r, f \circ g) \\ & \leq (\rho_f(p, q + \varepsilon) \log^{[m-1]} M(\exp^{[n-1]} r, g)). \end{aligned} \quad (3.23)$$

Now for all sufficiently large values of r ,

$$\begin{aligned} & \log^{[m]} M(\exp^{[n-1]} r, g) \leq (\rho_g(m, n) + \varepsilon) \log r \\ \text{i.e., } & \log^{[m-1]} M(\exp^{[n-1]} r, g) \leq r^{\rho_g(m, n) + \varepsilon}. \end{aligned} \quad (3.24)$$

From (3.23) and (3.24) it follows for all sufficiently large values of r ,

$$\log^{[p]} \mu(\exp^{[n-1]} r, f \circ g) \leq (\rho_f(p, q + \varepsilon) r^{\rho_g(m, n) + \varepsilon}). \quad (3.25)$$

Case II. Let $q < m$. Then from (3.22) we have for all sufficiently large values of r that

$$\begin{aligned} & \log^{[p]} \mu(\exp^{[n-1]} r, f \circ g) \\ & \leq (\rho_f(p, q) + \varepsilon) \exp^{[m-q]} \log^{[m]} M(\exp^{[n-1]} r, g). \end{aligned} \quad (3.26)$$

Now for all sufficiently large values of r ,

$$\begin{aligned} & \log^{[m]} M(\exp^{[n-1]} r, g) \leq (\rho_g(m, n) + \varepsilon) \log^{[n]} \exp^{[n-1]} r \\ \text{i.e., } & \log^{[m]} M(\exp^{[n-1]} r, g) \leq (\rho_g(m, n) + \varepsilon) \log r \\ \text{i.e., } & \exp^{[m-q]} \log^{[m]} M(\exp^{[n-1]} r, g) \leq \exp^{[m-q]} \log r^{(\rho_g(m, n) + \varepsilon)} \\ \text{i.e., } & \exp^{[m-q]} \log^{[m]} M(\exp^{[n-1]} r, g) \leq \exp^{[m-q-1]} r^{(\rho_g(m, n) + \varepsilon)}. \end{aligned} \quad (3.27)$$

Now from (3.26) and (3.27), we get for all sufficiently large values of r that

$$\begin{aligned} & \log^{[p]} \mu(\exp^{[n-1]} r, f \circ g) \\ & \leq (\rho_f(p, q) + \varepsilon) \exp^{[m-q-1]} r^{(\rho_g(m, n) + \varepsilon)} \\ \text{i.e., } & \log^{[p+1]} \mu(\exp^{[n-1]} r, f \circ g) \leq \exp^{[m-q-2]} r^{(\rho_g(m, n) + \varepsilon)} \\ \text{i.e., } & \log^{[p+m-q-1]} \mu(\exp^{[n-1]} r, f \circ g) \\ & \leq \log^{[m-q-2]} \exp^{[m-q-2]} r^{(\rho_g(m, n) + \varepsilon)} \\ \text{i.e., } & \log^{[p+m-q-1]} \mu(\exp^{[n-1]} r, f \circ g) \leq r^{(\rho_g(m, n) + \varepsilon)}. \end{aligned} \quad (3.28)$$

As $\rho_g(m, n) < \lambda_f(p, q)$, we can choose $\varepsilon (> 0)$ in such a way that

$$\rho_g(m, n) < \lambda_f(p, q) - \varepsilon. \quad (3.29)$$

Now combining (3.25) of Case I and (3.21) we have for all sufficiently large values of r ,

$$\frac{\log^{[p]} \mu(\exp^{[n-1]} r, f \circ g)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} \leq \frac{(\rho_f(p, q + \varepsilon) r^{(\rho_g(m, n) + \varepsilon)})}{r^{(\lambda_f(p, q) - \varepsilon)}}.$$

In view of (3.29), we get from above that

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{\log^{[p]} \mu(\exp^{[n-1]} r, f \circ g)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} &= 0 \\ \text{i.e., } \lim_{r \rightarrow \infty} \frac{\log^{[p]} \mu(\exp^{[n-1]} r, f \circ g)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} &= 0. \end{aligned} \quad (3.30)$$

Again combining (3.28) of Case II and (3.21) it follows for all sufficiently large values of r that

$$\frac{\log^{[p+m-q-1]} \mu(\exp^{[n-1]} r, f \circ g)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} \leq \frac{r^{(\rho_g(m,n)+\varepsilon)}}{r^{(\lambda_f(p,q)-\varepsilon)}}.$$

Now in view of (3.29) we obtain from above that

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{\log^{[p+m-q-1]} \mu(\exp^{[n-1]} r, f \circ g)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} &= 0 \\ \text{i.e., } \lim_{r \rightarrow \infty} \frac{\log^{[p+m-q-1]} \mu(\exp^{[n-1]} r, f \circ g)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} &= 0. \end{aligned} \quad (3.31)$$

Since $M(r, g) \leq \frac{R}{R-r} \mu(R, f)$, from (3.22) we have for all sufficiently large values of r that

$$\begin{aligned} \log^{[p]} \mu(\exp^{[n-1]} r, f \circ g) &\leq (\rho_f(p, q + \varepsilon) \log^{[q]} \mu(\exp^{[n-1]} R, g)) \\ \text{i.e., } \frac{\log^{[p]} \mu(\exp^{[n-1]} r, f \circ g)}{\log^{[q]} \mu(\exp^{[n-1]} R, g)} &\leq (\rho_f(p, q + \varepsilon)). \end{aligned}$$

Since $\varepsilon(> 0)$ is arbitrary, we get from above that

$$\limsup_{r \rightarrow \infty} \frac{\log^{[p]} \mu(\exp^{[n-1]} r, f \circ g)}{\log^{[q]} \mu(\exp^{[n-1]} R, g)} \leq \rho_f(p, q). \quad (3.32)$$

From (3.30) and (3.32) we obtain for all sufficiently large values of r that

$$\begin{aligned} &\limsup_{r \rightarrow \infty} \frac{\left\{ \log^{[p]} \mu(\exp^{[n-1]} r, f \circ g) \right\}^2}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f) \log^{[q]} \mu(\exp^{[n-1]} R, g)} \\ &= \lim_{r \rightarrow \infty} \frac{\log^{[p]} \mu(\exp^{[n-1]} r, f \circ g)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} \cdot \limsup_{r \rightarrow \infty} \frac{\log^{[p]} \mu(\exp^{[n-1]} r, f \circ g)}{\log^{[q]} \mu(\exp^{[n-1]} R, g)} \\ &\leq 0 \cdot \rho_f(p, q) = 0. \end{aligned}$$

This proves the first part of the theorem.

Again from (3.31) and (3.32) we get for all sufficiently large values of r that

$$\begin{aligned} &\limsup_{r \rightarrow \infty} \frac{\log^{[p+m-q-1]} \mu(\exp^{[n-1]} r, f \circ g) \log^{[p]} \mu(\exp^{[n-1]} r, f \circ g)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f) \log^{[q]} \mu(\exp^{[n-1]} R, g)} \\ &= \lim_{r \rightarrow \infty} \frac{\log^{[p+m-q-1]} \mu(\exp^{[n-1]} r, f \circ g)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f)} \cdot \limsup_{r \rightarrow \infty} \frac{\log^{[p]} \mu(\exp^{[n-1]} r, f \circ g)}{\log^{[q]} \mu(\exp^{[n-1]} R, g)} \\ &\leq 0 \cdot \rho_f(p, q) = 0 \end{aligned}$$

$$i.e., \lim_{r \rightarrow \infty} \frac{\log^{[p+m-q-1]} \mu(\exp^{[n-1]} r, f \circ g) \log^{[p]} \mu(\exp^{[n-1]} r, f \circ g)}{\log^{[p-1]} \mu(\exp^{[q-1]} r, f) \log^{[q]} \mu(\exp^{[n-1]} R, g)} = 0.$$

Thus the second part of the theorem is established. \square

Theorem 3.18. *Let f and g be any two entire functions such that $\rho_f(p, q) < \infty$ and $\rho_{f \circ g}(a, b) < \infty$ where p, q, a, b are all positive integers with $p > q$ and $a > b$. Also let $\lambda_g < \infty$. Then for any two positive integers m, n with $m - n = 1$, $m > 2$ and any $R > r$,*

$$\begin{aligned} & \limsup_{r \rightarrow \infty} \frac{\log^{[p]} \mu(\exp^{[n-1]} r, f \circ g) \cdot \log^{[a]} \mu(\exp^{[b-1]} r, f \circ g)}{\log^{[q]} \mu(\exp^{[n-1]} R, g) \cdot \log^{[m]} \mu(\exp^{[n-1]} r, g)} \\ & \leq \rho_f(p, q) \rho_{f \circ g}(a, b). \end{aligned}$$

Proof. For all sufficiently large values of r we have

$$\begin{aligned} \log^{[a]} \mu(\exp^{[b-1]} r, f \circ g) & \leq (\rho_{f \circ g}(a, b) + \varepsilon) \log^{[b]} \exp^{[b-1]} r \\ i.e., \log^{[a]} \mu(\exp^{[b-1]} r, f \circ g) & \leq (\rho_{f \circ g}(a, b) + \varepsilon) \log r. \end{aligned} \quad (3.33)$$

Again for all sufficiently large values of r , it follows that

$$\begin{aligned} \log^{[m]} \mu(\exp^{[n-1]} r, g) & \geq (\lambda_g(m, n) - \varepsilon) \log^{[n]} \exp^{[n-1]} r \\ i.e., \log^{[m]} \mu(\exp^{[n-1]} r, g) & \geq (\lambda_g(m, n) - \varepsilon) \log r. \end{aligned} \quad (3.34)$$

Now combining (3.33) and (3.34) we have for all sufficiently large values of r that

$$\frac{\log^{[a]} \mu(\exp^{[b-1]} r, f \circ g)}{\log^{[m]} \mu(\exp^{[n-1]} r, g)} \leq \frac{\rho_{f \circ g}(a, b) + \varepsilon}{\lambda_g(m, n) - \varepsilon}.$$

As $\varepsilon (> 0)$ is arbitrary, we get from above that

$$\limsup_{r \rightarrow \infty} \frac{\log^{[a]} \mu(\exp^{[b-1]} r, f \circ g)}{\log^{[m]} \mu(\exp^{[n-1]} r, g)} \leq \frac{\rho_{f \circ g}(a, b)}{\lambda_g(m, n)}. \quad (3.35)$$

Now from (3.32) and (3.35) we obtain that

$$\begin{aligned} & \limsup_{r \rightarrow \infty} \frac{\log^{[p]} \mu(\exp^{[n-1]} r, f \circ g) \cdot \log^{[a]} \mu(\exp^{[b-1]} r, f \circ g)}{\log^{[q]} \mu(\exp^{[n-1]} R, g) \cdot \log^{[m]} \mu(\exp^{[n-1]} r, g)} \\ & \leq \limsup_{r \rightarrow \infty} \frac{\log^{[a]} \mu(\exp^{[b-1]} r, f \circ g)}{\log^{[m]} \mu(\exp^{[n-1]} r, g)} \cdot \limsup_{r \rightarrow \infty} \frac{\log^{[p]} \mu(\exp^{[n-1]} r, f \circ g)}{\log^{[q]} \mu(\exp^{[n-1]} R, g)} \\ & \leq \frac{\rho_{f \circ g}(a, b) \rho_f(p, q)}{\lambda_g(m, n)}. \end{aligned}$$

Thus in view of Lemma 2.4, the theorem follows from above. \square

Corollary 3.19. *Under the same conditions of Theorem 3.18 when $m = 2$,*

$$\begin{aligned} & \limsup_{r \rightarrow \infty} \frac{\log^{[p]} \mu(r, f \circ g) \log^{[a]} \mu(\exp^{[b-1]} r, f \circ g)}{\log^{[q]} \mu(R, g) \log^{[2]} \mu(r, g)} \\ & \leq \frac{\rho_{f \circ g}(a, b) \rho_f(p, q)}{\lambda_g}. \end{aligned}$$

Theorem 3.20. Let f and g be any two entire functions such that $\lambda_f(p, q)$ and λ_g are both finite and p, q are any two positive integers with $p > q$. Then for any $R > r$,

$$(i) \liminf_{r \rightarrow \infty} \frac{\log^{[p+1]} \mu(r, f \circ g)}{\log^{[q+1]} \mu(R, g)} \leq 1$$

and

$$(ii) \liminf_{r \rightarrow \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[q]} \mu(R, g)} \leq \lambda_f(p, q).$$

Proof. In view of (3.1) we get from (3.17) for a sequence of values of r tending to infinity that

$$\begin{aligned} \log^{[p]} \mu(r, f \circ g) &\leq (\lambda_f(p, q) + \varepsilon) \log^{[q]} M(r, g) \\ \text{i.e., } \log^{[p]} \mu(r, f \circ g) &\leq (\lambda_f(p, q) + \varepsilon) \log^{[q]} \mu(R, g) \\ \text{i.e., } \log^{[p+1]} \mu(r, f \circ g) &\leq \log^{[q+1]} \mu(R, g) + O(1) \end{aligned} \quad (3.36)$$

$$\text{i.e., } \frac{\log^{[p+1]} \mu(r, f \circ g)}{\log^{[q+1]} \mu(R, g)} \leq \frac{\log^{[q+1]} \mu(R, g) + O(1)}{\log^{[q+1]} \mu(R, g)}. \quad (3.37)$$

Also from (3.36), it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[q]} \mu(R, g)} \leq \frac{(\lambda_f(p, q) + \varepsilon) \log^{[q]} \mu(R, g)}{\log^{[q]} \mu(R, g)}. \quad (3.38)$$

Since $\varepsilon (> 0)$ is arbitrary, it follows from (3.37) that

$$\liminf_{r \rightarrow \infty} \frac{\log^{[p+1]} \mu(r, f \circ g)}{\log^{[q+1]} \mu(R, g)} \leq 1.$$

This proves the first part of the theorem.

As $\varepsilon (> 0)$ is arbitrary, we obtain from (3.38) that

$$\liminf_{r \rightarrow \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[q]} \mu(R, g)} \leq \lambda_f(p, q).$$

Thus the second part of the theorem follows. \square

In the line of Theorem 3.20, the following theorem may be deduced.

Theorem 3.21. Let f and g be any two entire functions such that $\rho_f(p, q)$ and λ_g are both finite and p, q are any two positive integers with $p > q$. Then for any $R > r$,

$$(i) \limsup_{r \rightarrow \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[q]} \mu(R, g)} \leq \rho_f(p, q)$$

and

$$(ii) \limsup_{r \rightarrow \infty} \frac{\log^{[p+1]} \mu(r, f \circ g)}{\log^{[q+1]} \mu(R, g)} \leq 1.$$

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