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Total Graphs Associated to a Commutative Ring

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Abstract. Let R be a commutative ring with nonzero unity. Let $Z(R)$ be the set of all zero-divisors of R. The total graph of R, denoted by $T(\Gamma(R))$, is the simple graph with vertex set R and two distinct vertices x and y are adjacent if their sum $x + y \in Z(R)$. Several authors presented various generalizations for $T(\Gamma(R))$. This article surveys research conducted on $T(\Gamma(R))$ and its generalizations. A historical review of literature is given. Further properties of $T(\Gamma(R))$ are also studied. Many open problems are presented for further research.

1 Introduction

The study of graphs associated with algebraic structures dates back to 1878 when Arthur Cayley introduced Cayley Graph for finite groups [12]. In 1988 Beck [18] defined the zero-divisor graph of a commutative ring R. The vertices of Beck's graph are all elements of R where two vertices are adjacent if their product is a zero divisor of R. Anderson and Livingston [11] modified the definition of zero divisor graph by restricting the vertices to the non-zero zero divisors of the ring R. This graph is denoted by $\Gamma(R)$. The reader may refer to [7] and [23] for survey on the zero divisor graphs. The interplay between ring theory and graph theory has been the impetus to define and investigate other many graphs associated with algebraic structures.

Twenty years later after Beck's graphs, Andersen and Badawi [8] introduced a new graph associated with a commutative ring R with a non-zero unity. They called this graph the total graph of R.

Definition 1.1. [8] Let R be a commuatative ring with a non-zero unity, let $Z(R)$ be the set of all zero divisors in R. The total graph of R is the simple graph with vertex set R and two distinct vertices x and y are adjacent if their sum $x + y \in Z(R)$. This graph is denoted by $T(\Gamma(R))$.

Several induced subgraphs of $T(\Gamma(R))$ are also studied in the literature. The graphs $\mathrm{Reg}(\Gamma(R))$, $Z(\Gamma(R))$, $Z_0(\Gamma(R))$ and $T_0(\Gamma(R))$ are defined to be the induced subgraphs of $T(\Gamma(R))$ with vertex sets $Reg(R)$, $Z(R)$, $Z(R) \setminus \{0\}$ and $R \setminus \{0\}$, respectively, where $Reg(R)$ is the set of all regular elements of R and $Z(R)$ is the set of all zero divisors of R.

The definition of $T(\Gamma(R))$ brings back to our minds the infamous Cayley graph $Cay(R, Z(R)^*)$, where $Z(R)^* = Z(R) \setminus \{0\}$.

Definition 1.2. Let R be a commuatative ring with a non-zero unity, let $Z(R)$ be the set of all zero divisors in R. The Cayley graph, $Cay(R, Z(R)^*)$, is the simple graph with vertex set R and two distinct vertices x and y are adjacent if $x - y \in Z(R)$.

For more on Cayley graphs, the reader may refer to [19]. Despite the similarity in the definitions of the graphs $T(\Gamma(R))$ and $Cay(R, Z(R)^*)$, the two graphs could have totally different graph theoretic properties. In [34], the authors characterize finite rings R for which $T(\Gamma(R))$ is isomorphic to $Cay(R, Z(R)^*)$, in particular, they provide the following theorem.

Theorem 1.3. [34] Let R be a finite commutative ring. Then the two graphs $T(\Gamma(R))$ and $Cay(R, Z(R)^*)$ are isomorphic if and only if at least one of the following conditions is true:

(i) $R = R_1 \times R_2 \times \ldots \times R_k, k \geq 1$, where each R_i is a local ring of an even order;

(ii) $R = R_1 \times R_2 \times \ldots \times R_k, k \geq 2$, where each R_i is a local ring and $\min\{|R_i/Z(R_i)|, i = 1, 2, ..., k\} = 2$.

In 2010, Ashrafi et.al [13] introduced the unit graph of a ring R, denoted by $G(R)$, as follows.

Definition 1.4. [13] Let R be a commuatative ring with a non-zero unity, let $U(R)$ be the set of all unit elements in R. The unit graph of R is the simple graph with vertex set R and two distinct vertices x and y are adjacent if their sum $x + y \in U(R)$. This graph is denoted by $G(R)$.

Clearly, if R is finite, then $R = Z(R) \cup U(R)$, thus, in this case, $G(R)$ is the complement of $T(\Gamma(R))$ denoted by $\overline{T(\Gamma(R))}$. However, if R is infinite then $G(R)$ is just a subgraph of $\overline{T(\Gamma(R))}$. For any commutative ring R, $\overline{T(\Gamma(R))}$ is investigated in [22].

Since the definition of the total graph $T(\Gamma(R))$ in 2008, several groups of authors conducted rigorous research to extend the results obtained in [8] to more general contexts. In 2011, Pucanović defined The total graph of a module as follows

Definition 1.5. [30] Let R be a commutative ring with identity, let M be an R-module, let $T(M) = \{m \in M : rm = 0 \text{ for some } r \in R^*\}$ be the set of its torsion elements. The total graph of a module, $T\Gamma(M)$, is defined to be the graph with vertex set M and two distinct vertices $m_1, m_2 \in M$ are adjacent if $m_1 + m_2 \in T(M)$.

Observe that if $M = R$, then $T(M) = Z(R)$ and hence, the resultant graph will be $T(\Gamma(R))$.

In 2012, Barati et al. [17] introduced the graph $\Gamma_S(R)$ associated to a ring R and a multiplicatively closed subset S of R (i.e, S is closed under multiplication).

Definition 1.6. [17] The graph $\Gamma_S(R)$ is defined to be the simple graph with all elements of R as vertices, and two distinct vertices x and y of R are adjacent if and only if $x + y \in S$.

Obviously, both $Z(R)$ and $U(R)$ are multiplicatively closed subsets of R, so if we take S to be $Z(R)$, then we get the total graph of R. Besides, if $S = U(R)$ then the resultant graph will be $G(R)$. Thus, in this sense, $\Gamma_S(R)$ generalizes both the total and the unit graphs of R. A multiplicatively closed subset S of R is called saturated if $xy \in S$ implies that $x \in S$ and $y \in S$.

Let R be a commutative ring with non-zero identity. Let I be a proper ideal of R. Let $S(I)$ be the set of elements of R that are not prime to I. An element $a \in R$ is said to be prime to I if $ar \in I$, for $r \in R$ implies that $r \in I$, see [24]. In 2012, Abbasi [2] presented another generalization to the total graph in the following definition

Definition 1.7. [2] The total graph of a commutative ring R with respect to proper ideal I is the graph whose vertices are all elements of R and two distinct vertices $x, y \in R$ are adjacent if $x + y \in S(I)$. This graph is denoted by $T(\Gamma_I(R))$.

In the this defintion, if we set $I = \{0\}$, then $T(\Gamma_I(R)) = T(\Gamma(R))$. The set $S(I)$ is not in general an ideal of R. However, when $S(I)$ is an ideal of R, then it is a prime ideal of R. The graphs $S(\Gamma_I(R))$ and $\overline{S}(\Gamma_I(R))$ are defined to be the (induced) subgraphs of $T(\Gamma_I(R))$ with vertex sets $S(I)$, and $R \setminus S(I)$ respectively.

Let R be a commutative ring with non-zero identity. A nonempty proper multiplicatively closed subset H of R is said to be a multiplicative-prime subset of R if $ab \in H$ for some $a, b \in R$, implies that either $a \in H$ or $b \in H$. In 2013, Anderson and Badawi [10] introduced the generalized total graph as follows:

Definition 1.8. [10] let H be a multiplicative-prime subset of a commutative ring R, the generalized total graph of R, denoted by $GT_H(R)$, is the graph with vertex set $V(GT_H(R)) = R$ and edge set $E(GT_H(R)) = \{xy : x, y \in R \text{ and } x + y \in H, \text{ where } x \neq y\}$.

Note that when $H = Z(R)$, then $GT_H(R)$ is just $T(\Gamma(R))$. Moreover, if H is the union of all the maximal ideals of R, then $\overline{GT_H(R)}$ is the unit graph $G(R)$. If R is an integral domain, then $T(\Gamma(R))$ is the union of one copy of K_1 and copies $K_{1,1}$. On the other hand, the graph $GT_H(R)$ for an integral domain R is much more interesting. However, as we will see, $GT_H(R)$ and $T(\Gamma(R))$ have many common structural properties as one may expect. Let $A \subseteq R$, then $GT_H(A)$ is defined to be the subgraph of $GT_H(R)$ induced by the set A.

One can easily see that H is a multiplicative-prime subset of R if and only if $R \setminus H$ is a saturated multiplicatively closed subset of R. On the other hand, by Theorem 2, page 2 of [26] a subset S of R is saturated if and only if $R \setminus S$ is a union of some prime ideals. Therefore, H is a multiplicative-prime subset of R if and only if H is a union of prime ideals. Observe that if S is saturated then $\Gamma_S(R)$ is the complement of the graph $GT_{R\setminus S}(R)$.

Other graphs which extend the concept of total graph include the L-total graph of an L-module over an L-commutative ring [15], the total torsion element graph of a module M over a commutative ring R [16], and The total graph of a module with respect to multiplicative-prime subsets [33].

2 Some Structural Properties

All total graphs of a commutative ring R in this paper, except $T\Gamma(M)$ whose vertex set is M, have the same vertex set R. In each graph, two distinct vertices x and y are adjacent if $x + y$ belongs to a particular subset T of R. The study of each such graph breaks naturally into two cases depending on whether T satisfies a specific condition, the table below shows the set T in each graph and the corresponding condition on T.

The study of	depends on whether
$T(\Gamma(R))$	$Z(R)$ is an ideal or not
$T\Gamma(M)$	$T(M)$ is a proper submodules or not
$\Gamma_S(R)$	S is an ideal or not
$T(\Gamma_I(R))$	$S(I)$ is an ideal or not
$GT_H(R)$	H is a prime ideal or not

2.1 The Total Graph $T(\Gamma(R))$

Two main well known graphs reveals to be building blocks for the total graph $T(\Gamma(R))$. A complete graph on n vertices is a graph in which each pair of distinct vertices are adjacent. this graph is denoted by K_n. A complete bipartite graph is a graph whose vertices can be partitioned into two subsets V_1 and V_2 and two distinct vertices are adjacent if and only if one vertex belongs to V_1 and the other belongs to V_2. If $|V_1| = m$ and $|V_2| = n$, this graph is denoted by $K_{m,n}$.

The study of $T(\Gamma(R))$ falls naturally into two cases depending on whether $Z(R)$ is an ideal of R or not. If R is a commutative ring such that $Z(R)$ is an ideal of R. Then, clearly, in this case, $Z(\Gamma(R))$ is a complete (induced) subgraph of $T(\Gamma(R))$. Andersen and Badawi [8] proved that Reg$(\Gamma(R))$ is the union of complete graphs if $2 \in Z(R)$, otherwise, Reg$(\Gamma(R))$ is the union of complete bipartite graphs. Summing up, they obtained a perfect description of $T(\Gamma(R))$, when $Z(R)$ is an ideal of R, in the following theorem

Theorem 2.1. [8] Let R be a commutative ring such that $Z(R)$ is an ideal of R. Let $|Z(R)| = \lambda$, $|R/Z(R)| = \mu$. Then

$$T(\Gamma(R)) = \begin{cases} \bigcup\limits_{i=1}^{\mu} K_\lambda, & \text{if } 2 \in Z(R); \\ K_\lambda \cup \left(\bigcup\limits_{i=1}^{\frac{\mu-1}{2}} K_{\lambda,\lambda} \right), & \text{if } 2 \in R \setminus Z(R). \end{cases}$$

A graph is connected when there is a path between every pair of its vertices. A graph that is not connected is disconnected. A graph with no or one vertex is connected. An edgeless graph with two or more vertices is totally disconnected.

Theorem 2.2. [8] Let R be a commutative ring such that $Z(R)$ is an ideal of R. Then

(i) Reg$(\Gamma(R))$ is complete if and only if either $R/Z(R) \cong \mathbb{Z}_2$ or $R \cong \mathbb{Z}_3$.

(ii) Reg$(\Gamma(R))$ is connected if and only if either $R/Z(R) \cong \mathbb{Z}_2$ or $R/Z(R) \cong \mathbb{Z}_3$.

(iii) Reg$(\Gamma(R))$ is totally disconnected if and only if R is an integral domain with char$(R) = 2$.

Theorem 2.3. [8] Let R be a commutative ring such that $Z(R)$ is an ideal of R. Then the following statements are equivalent.

(i) $\text{Reg}(\Gamma(R))$ is connected.

(ii) Either $x + y \in Z(R)$ or $x - y \in Z(R)$ for every $x, y \in \text{Reg}(R)$.

(iii) Either $x + y \in Z(R)$ or $x + 2y \in Z(R)$ for every $x, y \in \text{Reg}(R)$. In particular, either $2x \in Z(R)$ or $3x \in Z(R)$ (but not both) for every $x \in \text{Reg}(R)$.

(iv) Either $R/Z(R) \cong \mathbb{Z}_2$ or $R/Z(R) \cong \mathbb{Z}_3$

The case when $Z(R)$ is not an ideal of R is much more complicated. In this case, $Z(\Gamma(R))$ is connected and is not disjoint from $\text{Reg}(\Gamma(R))$ [8].

Theorem 2.4. [8] Let R be a commutative ring such that $Z(R)$ is not an ideal of R. Then $\text{Reg}(\Gamma(R))$ is connected implies that $T(\Gamma(R))$ is connected.

Theorem 2.5. [31] Let R be a commutative ring such that $Z(R)$ is not an ideal of R, then $T(\Gamma(R[x]))$ is connected if and only if $T(\Gamma(R))$ is connected.

We say that a ring R is generated by a subset T of R, denoted by $(T) = R$ if $R = (t_1, ..., t_n)$ for some $t_1, ..., t_n \in T$.

Theorem 2.6. [8] Let R be a commutative ring such that $Z(R)$ is not an ideal of R, then $T(\Gamma(R))$ is connected if and only if $(Z(R)) = R$. In particlar, if R is a finite ring and $Z(R)$ is not an ideal then $T(\Gamma(R))$ is connected.

Theorem 2.7. [28] Let R be a commutative ring, then for any vertex $u \in V(T(\Gamma(R)))$,

$$\deg(u) = \begin{cases} |Z(R)| - 1, & \text{if } 2 \in Z(R) \text{ or } u \in Z(R); \\ |Z(R)| & otherwise. \end{cases}$$

A regular graph is a graph where each vertex has the same number of neighbors; i.e. every vertex has the same degree. Theorem 2.7 shows that, $T(\Gamma(R))$ is regular if and only if $2 \in Z(R)$.

Theorem 2.8. [29] Let R be a finite ring, then

(i) $\text{Reg}(\Gamma(R))$ is a regular graph.

(ii) $Z(\Gamma(R))$ is regular graph if and only if R is a local ring.

If G has a walk that traverses each edge exactly once, goes through all vertices, and ends at the starting vertex, then G is called Eulerian. Equivalently, a nontrivial, connected graph G is Eulerian if and only if every vertex of G has even degree. An Eulerian trail of a graph G is an open trail containing every edge of G. A graph containing an Eulerian trail is said to be traversable. Or equivalently, a connected graph G is traversable if and only if exactly 2 vertices of G have odd degree. Furthermore, each Eulerian trail of G begins at one of these odd vertices and ends at the other. A graph is called Hamiltonian if there exists a cycle containing every vertex.

Theorem 2.9. [6] Let R be a finite commutative rings such that $Z(R)$ is not an ideal. Then the following hold

(i) $T(\Gamma(R)$ is a Hamiltonian graph.

(ii) $\text{Reg}(\Gamma(R))$ is a Hamiltonian graph if and only if R is isomorphic to none of the following rings: ${\mathbb{Z}_2}^{n+1}, {\mathbb{Z}_2}^n \times \mathbb{Z}_3, {\mathbb{Z}_2}^n \times \mathbb{Z}_4, {\mathbb{Z}_2}^n \times \mathbb{Z}_2[x]/(x^2)$.

Asir and Chelvam [14] relaxed the conditions so that $T(\Gamma(R)$ is a Hamiltonian graph in the following theorem

Theorem 2.10. [14] If R is a commutative ring and $\text{diam}(T(\Gamma(R))) = 2$, then $T(\Gamma(R)$ is Hamiltonian.

Corollary 2.11. [14] If R is finite ring, then $T(\Gamma(R))$ is Hamiltonian.

Corollary 2.12. [14] If R is an Artinian ring such that $Z(R)$ is not an ideal, then $T(\Gamma(R))$ is Hamiltonian.

Theorem 2.13. [29] Let R be a finite local ring, then

(i) $T(\Gamma(R))$ is non-Eulerian.

(ii) $\mathrm{Reg}(\Gamma(R))$ is Eulerian if and only if $R \cong \mathbb{Z}_2$.

(iii) $Z(\Gamma(R))$ is Eulerian if and only if $|R|$ is odd or R is a field.

Theorem 2.14. [14] If R is a commutative ring such that $Z(R)$ is not an ideal of R, then

(i) $T(\Gamma(R))$ is Eulerian if and only if $2 \in Z(R)$ and $|Z(R)|$ is odd.

(ii) $\overline{T(\Gamma(R))}$ is Eulerian if and only if $2 \in Z(R)$ and $|\mathrm{Reg}(R)|$ is even.

The next theorem, which is due to Shekarriz et al. [34], characterizes Eulerian $T(\Gamma(R))$ when R is a finite ring.

Theorem 2.15. [34] Let R be a finite ring, then the graph $T(\Gamma(R))$ is Eulerian if and only if R is a product of two or more fields of even orders.

Theorem 2.16. [29] Let R be a finite ring, then

(i) $\mathrm{Reg}(\Gamma(R))$ is Eulerian if and only if $R \cong \mathbb{Z}_2$ or R is a product of two or more fields of even orders.

(ii) $Z(\Gamma(R))$ is Eulerian if and only if R is a field or $|R|$ is odd.

Let R be a finite nontrivial ring. Since a traversal graph has exactly two vertices of odd degree, then $T(\Gamma(R))$ could not be regular and hence $|R|$ is odd. By Lemma 3.4 of [29], $|Z(R)|$ is odd and $|U(R)|$ is even. Thus we get the following theorem.

Theorem 2.17. Let R be a finite nontrivial ring Then

(i) $T(\Gamma(R))$ is traversable graph if and only if $|R|$ is odd and $|U(R)| = 2$

(ii) $\mathrm{Reg}(\Gamma(R))$ is never traversable.

A graph G is said to be locally connected if for all $v \in V(G)$, the neighborhood of v, $N(v)$, induces a connected graph in G.

Theorem 2.18. [29] Let R be a ring and $Z(R)$ be an ideal of R.

(i) $Z(\Gamma(R))$ is locally connected graph.

(ii) $\mathrm{Reg}(\Gamma(R))$ and $T(\Gamma(R))$ are locally connected graphs if and only if $2 \in Z(R)$, or R is an integral domain.

The next theorem considers the case when R is a product of two rings.

Theorem 2.19. [29] Let R_1 and R_2 be two rings, and $R = R_1 \times R_2$. Then $T(\Gamma(R))$ is locally connected if and only if either R_1 or R_2 is not an integral domain.

If R is a local ring, then $Z(R)$ is an ideal and hence $Z(\Gamma(R))$ is a complete graph which is obviously locally connected. When R is a product of two rings, we have the following theorem.

Theorem 2.20. [29] Let R_1 and R_2 be two rings, and $R = R_1 \times R_2$. Then $Z(\Gamma(R))$ is locally connected if and only if either R_1 or R_2 is not an integral domain.

Next we will investigate when $\mathrm{Reg}(\Gamma(R))$ is locally connected. If R is local ring, then $\mathrm{Reg}(\Gamma(R))$ is locally connected if R is an integral domain or $2 \in Z(R)$. If R is a product of two rings, then we have

Theorem 2.21. [29] Let $R = R_1 \times R_2$ and $2 \in \mathrm{Reg}(R)$. Then $\mathrm{Reg}(\Gamma(R))$ is locally connected.

Theorem 2.22. [29] Let R be a product of two local rings R_1 and R_2 such that $2 \in Z(R)$ and $|Reg(R_i)| \geq 2$ for $i = 1, 2$. Then Reg($\Gamma(R)$) is locally connected if and only if R_1 or R_2 is not an integral domain.

Theorem 2.23. [29] If $R = \Pi_{i=1}^{n} R_i, n \geq 3$, then Reg($\Gamma(R)$) is locally connected.

Theorem 2.24. [29] Let R be an Artinian ring, then

(i) $T(\Gamma(R))$ is not locally connected if and only if R is a local ring satisfying $2 \in Reg(R)$ and R is not an integral domain or $R = R_1 \times R_2$, where, R_1 and R_2 are both integral domains.

(ii $Z(\Gamma(R))$ is not locally connected if and only if R is a product of two integral domains.

(iii) Reg($\Gamma(R)$) is not locally connected if and only if R is a local ring satisfying $2 \in$ Reg(R) and R is not an integral domain or $R = R_1 \times R_2$, $2 \in Z(R)$, and $|$Reg$(R_i)| \geq 2$ and R_i is an integral domain for $i = 1, 2$.

A graph G is called locally H if for each vertex $v \in V(G)$, the subgraph induced by the set of neighbors of v, $N(v)$, is isomorphic to H.

Theorem 2.25. [29] Let R be a finite ring. Then

(i) $T(\Gamma(R))$ is locally H if and only if $|R|$ is even.

(ii) Reg($\Gamma(R)$) is locally H.

(iii) $Z(\Gamma(R))$ is locally H if and only if R is a local ring.

Theorem 2.26. [9] Let R be a commutative ring

(i) If $|R| \leq 3$, then $T_0(\Gamma(R))$ is connected,

(ii) If $|R| \geq 4$, then $T_0(\Gamma(R))$ is connected if and only if $T(\Gamma(R))$ is connected.

(iii) If R is a non-reduced commutative ring. Then $Z_0(\Gamma(R))$ is connected

(iv) If R is a reduced commutative ring with $|$Min(R)$| = 2$. Then $Z_0(\Gamma(R))$ is not connected.

(v) If R is a reduced commutative ring that is not an integral domain. Then $Z_0(\Gamma(R))$ is connected if and only if $|$Min(R)$| \geq 3$.

2.2 The Total Graph of a Module $T\Gamma(M)$

Theorem 2.27. [30] Let R be a commutative ring with identity, let M be an R-module. Then

(i) $T\Gamma(M)$ is complete if and only if $T(M) = M$.

(ii) $T\Gamma(M)$ is totally disconnected if and only if R has characteristic 2 and M is torsion-free.

(iii) If $T(M)$ is a proper submodule of M, then $T\Gamma(M)$ is disconnected.

Theorem 2.28. [30] Let R be a commutative ring and M an R-module such that $T(M)$ is a proper submodule of M. $|T(M)| = \lambda$ and $|M/T(M)| = \mu$, then

$$T\Gamma(M) = \begin{cases} \bigcup\limits_{i=1}^{\mu} K_\lambda, & \text{if } 2 \in Z(R); \\ K_\lambda \cup \left(\bigcup\limits_{i=1}^{\frac{\mu-1}{2}} K_{\lambda,\lambda} \right), & \text{if } 2 \notin Z(R). \end{cases}$$

Theorem 2.29. [30] Let M be an R-module such that $T(M)$ is not a submodule. Then $T\Gamma(M)$ is connected if and only if M is generated by its torsion elements.

Theorem 2.30. [30] Let R be a commutative ring and M an R-module. If $T\Gamma(R)$ is connected, then $T\Gamma(M)$ is connected as well.

Theorem 2.31. [30] Let R be a commutative ring and M an R-module. Then $T\Gamma(M)$ is totally disconnected if and only if R has characteristic 2 and M is torsion-free.

2.3 The Total Graph $\Gamma_S(R)$

The following theorem is analogous to Theorem 2.1.

Theorem 2.32. [17] Suppose that S is an ideal of R with $|S| = \lambda$ and $|R/S| = \mu$, then

$$\Gamma_S(R)) = \begin{cases} \bigcup_{i=1}^{\mu} K_\lambda, & \text{if } 2 \in S; \\ K_\lambda \cup \left(\bigcup_{i=1}^{\frac{\mu-1}{2}} K_{\lambda,\lambda} \right), & \text{if } 2 \in R \setminus S. \end{cases}$$

Theorem 2.33. [17] The graph $\Gamma_S(R)$ is complete if and only if $S = R$ or $char(R) = 2$ and $S = R \setminus \{0\}$.

Theorem 2.34. [17] Let R be a finite ring such that $R \neq \mathbb{Z}_3$. Also, suppose that S is a saturated multiplicatively closed subset of R. Then $\Gamma_S(R)$ is a forest if and only if it is a complete matching.

Theorem 2.35. [17] Let $R = R_1 \times R_2 \times \cdots \times R_k$, where (R_i, m_i) is a finite local ring such that $R_i/m_i \cong \mathbb{Z}_2$, and let $S = S_1 \times S_2 \times \cdots \times S_k$ be a saturated multiplicatively closed subset of R. Then $\Gamma_S(R)$ is disconnected if and only if there exist $1 \leq i \neq j \leq n$ such that $S_i = U(R_i)$ and $S_j = U(R_j)$.

Theorem 2.36. [17] Let S be a multiplicatively closed subset of R such that $S = -S$. Then $\Gamma_S(R)$ is connected if and only if $(R, +)$ is generated by S.

Theorem 2.37. [17] For an arbitrary saturated multiplicatively closed subset S of R, in the graph $\Gamma_S(R)$, the following statements hold.

(i) If $x \in R \setminus S$, then $deg(x) = |S|$.

(ii) If $2 \notin S$, then $deg(x) = |S|$ for all $x \in R$.

2.4 The Total Graph $T\Gamma_I(R)$

The next two theorems illustrate the relation between $T(\Gamma_I(R))$ and $T(\Gamma(R/I))$.

Theorem 2.38. [2] Let R be a commutative ring with the proper ideal I, and let $x, y \in R$. Then

(i) If $x + I$ and $y + I$ are (distinct) adjacent vertices in $T(\Gamma(R/I))$, then x is adjacent to y in $T(\Gamma_I(R))$.

(ii) If x and y are (distinct) adjacent vertices in $T(\Gamma_I(R))$ and $x + I \neq y + I$ then $x + I$ is adjacent to $y + I$ in $T(\Gamma(R/I))$.

(iii) If x is adjacent to y in $T(\Gamma_I(R))$ and $x + I = y + I$, then $2x, 2y \in S(I)$ and all distinct elements of $x + I$ are adjacent in $T(\Gamma_I(R))$.

Corollary 2.39. [2] Let R be a commutative ring with the proper ideal I. Then $T(\Gamma_I(R))$ contains $|I|$ disjoint subgraphs isomorphic to $T(\Gamma(R/I))$.

Theorem 2.40. [2] Let R be a commutative ring with the proper ideal I. Then

(i) $S(\Gamma_I(R))$ is complete (connected) if and only if $Z(\Gamma(R/I))$ is complete (connected).

(ii) If $\bar{S}(\Gamma_I(R))$ is complete, then $Reg(\Gamma(R/I))$ is complete.

(iii) $\bar{S}(\Gamma_I(R))$ is connected if and only if $Reg(\Gamma(R/I))$ is connected.

Theorem 2.41. [2] Let R be a commutative ring with the proper ideal I such that $S(I)$ is an ideal of R. Then

(i) $S(\Gamma_I(R))$ is a complete (induced) subgraph $T(\Gamma_I(R))$ and is disjoint from $\bar{S}(\Gamma_I(R))$.

(ii) The (induced) subgraph $S(\Gamma_I(R))$ with vertices $\sqrt{I}$ is complete and each vertex of this subgraph is adjacent to each vertex of $S(\Gamma_I(R))$ and is disjoint from $\bar{S}(\Gamma_I(R))$.

Theorem 2.42. [2] Let R be a commutative ring with the proper ideal I such that $S(I)$ is an ideal of R. Then the following statements are equivalent.

(i) $\bar{S}(\Gamma_I(R))$ is connected.

(ii) Either $x+y \in S(I)$ or $x-y \in S(I)$ for all $x, y \in R \setminus S(I)$.

(iii) Either $x+y \in S(I)$ or $x+2y \in S(I)$ (but not both) for all $x, y \in R \setminus S(I)$. In particular, either $2x \in S(I)$ or $3x \in S(I)$ for all $x \in R \setminus S(I)$.

Theorem 2.43. [2] Let R be a commutative ring with the proper ideal I such that $S(I)$ is an ideal of R. let $|S(I)| = \lambda$ and $|R/S(I)| = \mu$ Then

$$\bar{S}(\Gamma_I(R)) = \begin{cases} \bigcup_{i=1}^{\mu-1} K_\lambda, & \text{if } 2 \in S(I); \\ \bigcup_{i=1}^{\frac{\mu-1}{2}} K_{\lambda,\lambda}, & \text{if } 2 \in R \setminus S(I). \end{cases}$$

Theorem 2.44. [2] Let R be a commutative ring with the proper ideal I such that $S(I)$ is an ideal of R. Then

(i) $\bar{S}(\Gamma_I(R))$ is complete if and only if either $R/S(I) \cong \mathbb{Z}_2$ or $R \cong \mathbb{Z}_3$

(ii) $\bar{S}(\Gamma_I(R))$ is connected if and only if either $R/S(I) \cong \mathbb{Z}_2$ or $R/S(R) \cong \mathbb{Z}_3$

(iii) $\bar{S}(\Gamma_I(R))$ (and hence $T(\Gamma_I(R))$ and $S(\Gamma_I(R))$ is totally disconnected if and only if $I = \{0\}$ and R is an integral domain with char$(R) = 2$.

Theorem 2.45. [2] Let R be a commutative ring with the proper ideal I such that $S(I)$ is not an ideal of R.

(i) $S(\Gamma_I(R))$ is connected.

(ii) Some vertex of $S(\Gamma_I(R))$ is adjacent to a vertex of $\bar{S}(\Gamma_I(R))$. In particular, the subgraphs $\bar{S}(\Gamma_I(R))$ and $S(\Gamma_I(R))$ are not disjoint.

(iii) If $\bar{S}(\Gamma_I(R))$ is connected, then $T(\Gamma_I(R))$ is connected.

2.5 The Generalized Total Graph $GT_H(R)$

Theorem 2.46. [10] Let H be a prime ideal of a commutative ring R. Then $GT_H(H)$ is a complete (induced) subgraph of $GT_H(R)$ and $GT_H(H)$ is disjoint from $GT_H(R \setminus H)$. In particular, $GT_H(H)$ is connected and $GT_H(R)$ is never connected.

The following theorem concerning the generalized total graph is analogous to Theorem 2.1

Theorem 2.47. [10] Let H be a prime ideal of a commutative ring R. Let $|H| = \lambda$, $|R/H| = \mu$. Then

$$GT_H(R) = \begin{cases} \bigcup_{i=1}^{\mu} K_\lambda, & \text{if } 2 \in H; \\ K_\lambda \cup \left(\bigcup_{i=1}^{\frac{\mu-1}{2}} K_{\lambda,\lambda} \right), & \text{if } 2 \in R \setminus H. \end{cases}$$

Theorem 2.48. [10] Let H be a prime ideal of a commutative ring R

(i) $G_H(R \setminus H)$ is complete if and only if either $R/H \cong \mathbb{Z}_2$ or $R \cong \mathbb{Z}_3$.

(ii) $GT_H(R \setminus H)$ is connected if and only if either $R/H \cong \mathbb{Z}_2$ or $R/H \cong \mathbb{Z}_3$

(iii) $GT_H(R \setminus H)$ is totally disconnected if and only if $H = \{0\}$ and $char(R) = 2$.

Theorem 2.49. [10] Let H be a prime ideal of a commutative ring R. Then the following statements are equivalent.

(i) $GT_H(R \setminus H)$ is connected.

(ii) Either $x + y \in H$ or $x - y \in H$ for every $x, y \in R \setminus H$.

(iii) Either $x + y \in H$ or $x + 2y \in H$ for every $x, y \in R \setminus H$. In particular, either $2x \in H$ or $3x \in H$ (but not both) for every $x \in R \setminus H$.

(iv) Either $R/H \cong \mathbb{Z}_2$ or $R/H \cong \mathbb{Z}_3$

Theorem 2.50. [10] Let R be a commutative ring and H be a multiplicative-prime subset of R. Then $GT_H(R)$ is connected if and only if $(H) = R$.

Theorem 2.51. [10] Let R be a commutative ring and H be a multiplicative-prime subset of R that is not an ideal of R.

(i) $GT_H(H)$ is connected.

(ii) Some vertex of $GT_H(H)$ is adjacent to a vertex of $GT_H(R \setminus H)$. In particular, the subgraphs $GT_H(H)$ and $GT_H(R \setminus H)$ of $GT_H(R)$ are not disjoint.

(iii) If $GT_H(R \setminus H)$ is connected, then $GT_H(R)$ is connected.

Theorem 2.52. [10] Let R be a commutative ring and H be a multiplicative-prime subset of R that is not an ideal of R. Then $GT_H(R)$ is connected if and only if $(H) = R$ (i.e. $R = (z_1, ..., z_n)$ for some $z_1, ..., z_n \in H$. In particular, if H is not an ideal of R and either $\dim(R) = 0$ (e.g. R is finite) or R is an integral domain with $\dim(R) = 1$, then $GT_H(R)$ is connected.

Corollary 2.53. [10] Let R be a commutative ring and H be a multiplicative-prime subset of R. Then $GT_H(R)$ is connected if and only if $(H) = R$.

3 Some Graph Invariants

For a connected graph G, the distance $d(u, v)$, between two vertices u and v is the minimum of the lengths of all $u - v$ paths of G. The eccentricity of a vertex v in G is the maximum distance from v to any vertex in G. The radius of G, $rad(G)$, is the minimum eccentricity among the vertices of G. The diameter of G, $diam(G)$, is the maximum eccentricity among the vertices of G. The open neighborhood of a vertex x in G is the set $N(x) = \{y : xy \in E(G)\}$ while the closed neighborhood of a vertex x in G is the set $N[x] = N(x) \cup \{x\}$. The girth of a graph is the length of a shortest cycle contained in the graph. If the graph does not contain any cycles (i.e. it's an acyclic graph), its girth is defined to be infinity. Let R be a commutative ring and $x, y \in R^*$ be distinct. We say that $x - a_1 - \cdots - a_n - y$ is a zero-divisor path from x to y if $a_1, \cdots, a_n \in Z(R^*)$ and $a_i + a_{i+1} \in Z(R)$ for every $0 \le i \le n$. let $a_0 = x$ and $a_{n+1} = y$. We define $d_Z(x, y)$ to be the length of a shortest zero-divisor path from x to y, $d_Z(x, x) = 0$ and $d_Z(x, y) = \infty$ if there is no such path. Let $\text{diam}_Z(R) = \sup\{d_Z(x, y) : x, y \in R^*\}$. Thus $d_T(x, y) = d_{T_0}(x, y) \le d_Z(x, y)$, for every $x, y \in R^*$, where $d_T(x, y)$ and $d_{T_0}(x, y)$ denote the distance between x and y in the graphs $T_0(\Gamma(R))$ and $T(\Gamma(R))$ respectively. We say that $x - a_1 - \cdots - a_n - y$ is a regular path from x to y if $a_1 \cdots a_n \in \text{Reg}(R)$ and $a_i + a_{i+1} \in Z(R)$ for every $0 \le i \le n$. Let $a_0 = x$ and $a_{n+1} = y$. We define $d_{\text{reg}}(x, y)$ to be the length of a shortest regular path from x to y, $d_{\text{reg}}(x, x) = 0$ and $d_{\text{reg}}(x, y) = \infty$ if there is no such path, and $\text{diam}_{\text{reg}}(R) = \sup\{d_{\text{reg}}(x, y) : x, y \in R^*\}$. Then $d_T(x, y) = d_{T_0}(x, y) \le d_{\text{reg}}(x, y)$ for every $x, y \in R^*$. For any commutative ring R, we have $\max\{\text{diam}(T_0(\Gamma(R))), \text{diam}(\text{Reg}(\Gamma(R)))\} \le \text{diam}_{\text{reg}}(R)$.

3.1 The Total Graph $T(\Gamma(R))$

If $Z(R)$ is an ideal, then the precise description for $T(\Gamma(R))$ of Theorm 2.1 makes it an easy task to characterize rings whose regular graphs, Reg$(\Gamma(R))$, is connected, complete or even totally disconnected. On the other hand, One may easily conclude the following

Theorem 3.1. [8] Let R be a commutative ring such that $Z(R)$ is an ideal of R. Then

(i) diam$(\text{Reg}(\Gamma(R))) = 0$ if and only if $R \cong \mathbb{Z}_2$.

(ii) diam$(\text{Reg}(\Gamma(R))) = 1$ if and only if either $R/Z(R) \cong \mathbb{Z}_2$ and $R \ncong \mathbb{Z}_2$ or $R \cong \mathbb{Z}_3$.

(iii) diam$(\text{Reg}(\Gamma(R))) = 2$ if and only if $R/Z(R) \cong \mathbb{Z}_3$ and $R \ncong \mathbb{Z}_3$.

(iv) Otherwise, diam$(\text{Reg}(\Gamma(R))) = \infty$.

Theorem 3.2. [8] Let R be a commutative ring such that $Z(R)$ is an ideal of R. Then

(i) Let G be an induced subgraph of Reg$(\Gamma(R))$, and let x and y be distinct vertices of G that are connected by a path in G. Then there is a path of length at most two between x and y in G. In particular, if Reg$(\Gamma(R))$ is connected, then diam$(\text{Reg}(\Gamma(R))) \leq 2$.

(ii) Let x and y be distinct regular elements of R that are connected by a path. If $x + y \notin Z(R)$, then $x - (-x) - y$ and $x - (-y) - y$ are paths of length two between x and y in Reg$(\Gamma(R))$.

Theorem 3.3. [8], Let R be a commutative ring such that $Z(R)$ is not an ideal of R and $T(\Gamma(R))$ is connected, if n is the least integer such that $R = (z_1, ..., z_n)$ for some $z_1, ..., z_n \in Z(R)$. then

(i) diam$(T(\Gamma(R))) = n$. In particular, if R is finite, then diam$(T(\Gamma(R))) = 2$.

(ii) If diam$(T(\Gamma(R))) = n$, then diam$(\text{Reg}(\Gamma(R))) \geq n - 2$.

(iii) diam$(T(\Gamma(R))) = d(0, 1)$.

Theorem 3.4. [31] Let R be a commutative ring such that $Z(R)$ is not an ideal of R and $T(\Gamma(R))$ is connected, if n is the least integer such that $R = (z_1, ..., z_n)$ for some $z_1, ..., z_n \in Z(R)$. then diam$(T(\Gamma(R[x]))) = n$. In particular, if R is finite, then diam$(T(\Gamma(R[x]))) = 2$.

Corollary 3.5. [8] Let $\{R_\alpha\}_{\alpha\in\Delta}$ be a family of commutative rings with $|\Delta| \geq 2$, and let $R = \Pi_{\alpha\in\Delta} R_\alpha$. Then $T(\Gamma(R))$ is connected with diam$(T(\Gamma(R))) = 2$.

Akbari et. [6] studied the relation between the diameter of the total graph and the diameter of the regular graph of a commutative Noetherian ring. for this purpose, they gave the following theorem.

Theorem 3.6. [6] Let R be a commutative Noetherian ring. For any to two regular elments a and b, the distance between a and b in Reg$(\Gamma(R))$ is equal to the distance between a and b in $T(\Gamma(R))$.

Corollary 3.7. [6] Let R be a commutative Noetherian ring such that $T(\Gamma(R))$ is connected and diam$(T(\Gamma(R))) = n$, then diam$(\text{Reg}(\Gamma(R))) \leq n$.

Combining part (ii) of Theorem 3.2 and Cor 3.7 , we get

Theorem 3.8. Let R be a commutative Noetherian ring such that $T(\Gamma(R))$ is connected and diam$(T(\Gamma(R))) = n$ then $n - 2 \leq$ diam$(\text{Reg}(\Gamma(R))) \leq n$.

Theorem 3.9. [31] Let R be a commutative ring such that $Z(R)$ is not an ideal of R and $T(\Gamma(R))$ is connected. Then if n is the least integer such that $R = (z_1, ..., z_n)$ for some $z_1, ..., z_n \in Z(R)$. Then $r(T(\Gamma(R))) = n$. In particular, if R is finite, then $r(T(\Gamma(R))) = 2$.

The above results show that for a connected $T(\Gamma(R))$, if n is the least integer such that $R = (z_1, ..., z_n)$ for some $z_1, ..., z_n \in Z(R)$, we have

(i) diam$(T(\Gamma(R))) = r(T(\Gamma(R))) = n$.

(ii) diam$(T(\Gamma(R[x]))) = r(T(\Gamma(R[x]))) = n$.

The center of G is the set of all vertices of G with minimum eccentricity. A graph G is said to be self centered if center$(G) = G$.

Theorem 3.10. [22] Let R be a finite commutative ring. Then $T(\Gamma(R))$ is self centered if and only if $Z(R)$ is not an ideal of R.

Theorem 3.11. [9] Let R be a commutative ring with $|R| \geq 2$. Then $\text{diam}(T_0(\Gamma(R))) = \text{diam}(T(\Gamma(R)))$

Theorem 3.12. [9] Let R be a finite commutative ring. Then $\text{diam}(Z_0(\Gamma(R))) \in \{0, 1, 2, \infty\}$. Moreover,

(i) $Z_0(\Gamma(R))$ is the empty graph if and only if R is a field,

(ii) $\text{diam}(Z_0(\Gamma(R))) = 0$ if and only if R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$,

(iii) $\text{diam}(Z_0(\Gamma(R))) = 1$ if and only if R is a local ring with maximal ideal M and $|M| \geq 3$,

(iv) $\text{diam}(Z_0(\Gamma(R))) = 2$ if and only if either $|Max(R)| \geq 3$ or R is not reduced with $|Max(R)| = 2$, and

(v) $\text{diam}(Z_0(\Gamma(R))) = \infty$ if and only if R is reduced with $|\text{Max}(R)| = 2$.

Theorem 3.13. [9] Let R be a commutative ring that is not an integral domain, Let $\text{Min}(R)$ be the set of all minimal ideals of R. Then there is a zero-divisor path from x to y for every $x, y \in R^*$ if and only if one of the following two statements holds.

(i) R is reduced, $|\text{Min}(R)| \geq 3$, and $R = (z_1, z_2)$ for some $z_1, z_2 \in Z(R)^*$.

(ii) R is not reduced and $R = (z_1, z_2)$ for some $z_1, z_2 \in Z(R)^*$.

Moreover, if there is a zero-divisor path from x to y for every $x, y \in R^*$, then R is not quasilocal and $\text{diam}_Z(R) \in \{2, 3\}$.

Theorem 3.14. [9]

(i) Let $R = R_1 \times R_2$ for commutative quasilocal rings R_1, R_2 with maximal ideals M_1, M_2, respectively. If there are $a_1 \in U(R_1)$ and $a_2 \in U(R_2)$ with $(2a_1, 2a_2) \in U(R)$ and $(3a_1, 3a_2) \notin Z(R)$, then $\text{diam}_Z(R) \in \{3, \infty\}$. Moreover, $\text{diam}_Z(R) = 3$ if either R_1 or R_2 is not reduced.

(ii) Let $R = R_1 \times R_2 \times R_n$ for commutative rings $R_1, ..., R_n$ with $n \geq 3$. Then $\text{diam}_Z(R) = 2$.

Theorem 3.15. [9] Let R be a commutative ring with $\text{diam}(T_0(\Gamma(R))) = n < \infty$.

(i) Let $u \in U(R), s \in R^*$, and P be a shortest path from s to u of length $n - 1$ in $T_0(\Gamma(R))$. Then P is a regular path from s to u.

(ii) Let $u \in U(R), s \in R^*$, and $P : s - a_1 - \cdots a_n = u$ be a shortest path from s to u of length n in $T_0(\Gamma(R))$. Then either P is a regular path from s to u, or $a_1 \in Z(R)^*$ and $a_1 - \cdots - a_n = u$ is a regular path from a_1 to u of length $n - 1 = d_{T_0}(a_1, u)$.

Theorem 3.16. [9] Let R be a commutative ring

(i) If $s \in \text{Reg}(R)$ and $w \in \text{Nil}(R)^*$, then there is no regular path from s to w.

(ii) If R is reduced and quasilocal, then there is no regular path from any unit to any nonzero nonunit in R.

Theorem 3.17. [9] Let R be a commutative ring. Then there is a regular path from x to y for every $x, y \in R^*$ if and only if R is reduced, $\text{Reg}(\Gamma(R)$ is connected, and for every $z \in Z(R)^*$ there is a $w \in Z(R)^*$ such that $d_Z(z, w) > 1$ (possibly with $d_Z(z, w) = \infty$).

A commutative ring R is a p.p. ring if and only if every if every principal ideal is projective.

Theorem 3.18. [9] Let R be a commutative p.p. ring that is not an integral domain. Then there is a regular path from x to y for every $x, y \in R^*$. Moreover, $\text{diam}_{\text{reg}}(R) = \text{diam}(T_0(\Gamma(R))) = diam(T(\Gamma(R))) = 2$.

Theorem 3.19. [8] Let R be a commutative ring such that $Z(R)$ is an ideal of R. Then

(i) (a) gr(Reg($\Gamma(R)$)) $= 3$ if and only if $2 \in Z(R)$ and $|Z(R)| \geq 3$.
(b) gr(Reg($\Gamma(R)$)) $= 4$ if and only if $2 \notin Z(R)$ and $|Z(R)| \geq 2$.
(c) Otherwise, gr(Reg($\Gamma(R)$)) $= \infty$.

(ii) (a) gr($T(\Gamma(R))$) $= 3$ if and only if $|Z(R)| \geq 3$.
(b) gr($T(\Gamma(R))$) $= 4$ if and only if $2 \notin Z(R)$ and $|Z(R)| = 2$.
(c) Otherwise, gr($T(\Gamma(R))$) $= \infty$.

Theorem 3.20. [8] Let R be a commutative ring such that $Z(R)$ is not an ideal of R. Then

(i) gr($Z(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2))$) $= \infty$, otherwise gr($Z(\Gamma(R))$) $= 3$.

(ii) gr($T(\Gamma(R))$) $= 3$ if and only if $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(iii) gr($T(\Gamma(R))$) $= 4$ if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Theorem 3.21. [9] Let R be a commutative ring which is not an integral domain. Then gr($Z_0(\Gamma(R))$) $=$ ∞ if and only if R is isomorphic to $\mathbb{Z}_4, \mathbb{Z}_2[X]/(X^2), \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_6, \mathbb{Z}_9, \mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_3[X]/(X^2)$. Otherwise, gr($Z_0(\Gamma(R))$) $= 3$.

Theorem 3.22. [9] Let R be a finite commutative ring. Then gr($T_0(\Gamma(R))$) $\in \{3, 4, \infty\}$. Moreover,

(i) gr($T_0(\Gamma(R))$) $= \infty$ if and only if R is an integral domain or is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(ii) gr($T_0(\Gamma(R))$) $= 4$ if and only if R is isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3[X]/(X^2)$ and

(iii) gr($T_0(\Gamma(R))$) $= \infty$ otherwise.

Theorem 3.23. [22] Let R be a commutative ring. Then gr($\overline{T(\Gamma(R))}$) $\in \{3, 4, 6, \infty\}$. In particular

(i) gr($\overline{T(\Gamma(R))}$) $= \infty$ if and only if $R \cong \mathbb{Z}_3$ or $R \cong \mathbb{Z}_2^r$ for any natural number r.

(ii) gr($\overline{T(\Gamma(R))}$) $= 6$ if and only if $R \cong \mathbb{Z}_2^r \times \mathbb{Z}_3$ for any natural number r.

(iii) gr($\overline{T(\Gamma(R))}$) $= 4$ if and only if Reg(R) $+$ Reg(R) $\subseteq Z(R)$ and for any $z_1, z_2, z_3 \in Z(R)$, $z_i + z_j \in Z(R)$ for some $i \neq j, 1 \leq i, j \leq 3$.

The genus of a graph is the minimal integer n such that the graph can be drawn without crossing itself on a sphere with n handles (i.e. an oriented surface of genus n). Thus, a planar graph has genus 0, because it can be drawn on a sphere without self-crossing.

Theorem 3.24. [28] Let R be a finite ring such that $T(\Gamma(R))$ is planar. Then the following hold:

(i) If R is local ring, then R is a field or R is isomorphic to the one of the 9 following rings: $\mathbb{Z}_4, \mathbb{Z}_2[X]/(X^2), \mathbb{Z}_2[X]/(X^3), \mathbb{Z}_2[X,Y]/(X,Y)^2, \mathbb{Z}_4[X]/(2X, X^2), \mathbb{Z}_4[X]/(2X, X^2{-}2), \mathbb{Z}_8$, $\mathbb{F}_4[X]/(X^2), \mathbb{Z}_4[X]/(X^2+X+1)$, where $\mathbb{F}_4$ is a field with exactly four elements.

(ii) If R is not local ring, then R is an infinite integral domain or R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_6$.

Theorem 3.25. [28] Let R be a finite ring such that $T(\Gamma(R))$ is toroidal. Then the following statements hold:

(i) If R is local ring, then R is isomorphic to $\mathbb{Z}_9$, or $\mathbb{Z}_3[X]/(X^2)$.

(ii) If R is not local ring, then R is isomorphic to one of the following rings: $\mathbb{Z}_2 \times \mathbb{F}_4, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Theorem 3.26. [28] For any positive integer g, There are finitely many finite rings R whose total graph has genus g.

Theorem 3.27. [21] Let R be a finite commutative ring with identity, I be an ideal contained in $Z(R)$, $|I| = \lambda \geq 3$, $|R/I| = \mu$. then

(i) If $2 \in I$, then $g(T(\Gamma(R))) \geq \mu\lceil \frac{(\lambda-3)(\lambda-4)}{12} \rceil$;

(ii) If $2 \notin I$, then $g(T(\Gamma(R))) \geq \lceil \frac{(\lambda-3)(\lambda-4)}{12} \rceil + (\frac{\mu-1}{2})\lceil \frac{(\lambda-2)^2}{4} \rceil$

Theorem 3.28. [21] Let R be a finite commutative ring with identity, I be a annihilator ideal with maximal cardinality among the proper annihilator ideals in R, $|I| = \lambda$, $|R/I| = \mu$ and $2 \leq \lambda \not\equiv 5, 9(mod 12)$. If $Z(R)$ is not an ideal of R and $2 \in I$, Then

$$g(T(\Gamma(R))) \geq \begin{cases} \frac{\mu}{2}\lceil \frac{(\lambda-2)(\lambda-3)}{6} \rceil, & \text{if } \mu \; is \; even; \\ (\frac{\mu-1}{2})\lceil \frac{(\lambda-2)(\lambda-3)}{6} \rceil + \lceil \frac{(\lambda-3)(\lambda-4)}{12} \rceil & \text{if } \mu \; is \; odd. \end{cases}$$

Theorem 3.29. [21] Let R be a finite commutative ring and $Z(R) = \cup_{i=1}^{m} P_i$ where $P_i's$ are ideals of R. Let $|P_i| = \alpha_i$ and $|R_i/P_i| = \beta_i$ for $i = 1, 2 \ldots m$. Suppose that $2 \in P_t$ for all $1 \leq t \leq j$, and $2 \notin P_t$, for all $j+1 \leq t \leq m$, where $j = 0$ if $2 \notin Z(R)$. Then

$$g(T(\Gamma(R))) \leq (m-1)(|R|-1) + \sum_{t=1}^{j} \beta_t \left\lceil \frac{(\alpha_t-3)(\alpha_t-4)}{12} \right\rceil + \sum_{t=j+1}^{m} \left\{ \left(\frac{\beta_t-1}{2}\right) \left\lceil \frac{(\alpha_t-2)^2}{4} \right\rceil + \left\lceil \frac{(\alpha_t-3)(\alpha_t-4)}{12} \right\rceil \right\}$$

(If $\alpha_i = 2$ for some i, then we take $\left\lceil \frac{(\alpha_i-3)(\alpha_i-4)}{12} \right\rceil = 0$).

Theorem 3.30. [21] Let R be a finite commutative ring, the genus of $T(\Gamma(R))$ is 2 if and only if R is isomorphic to $\mathbb{Z}_{10}$ or $\mathbb{Z}_3 \times \mathbb{F}_4$.

Let $G = (V, E)$ be a graph. A subset S of $V(G)$ is called a dominating set if every vertex in $V \setminus S$ is adjacent to at least one vertex in S. The domination number of G is defined to be minimum cardinality of a dominating set in G and is denoted by $\gamma(G)$. A set of edges M of $E(G)$ is called an edge dominating set if every edge of $E(G) \setminus M$ is adjacent to an element of M. The number of elements of a minimum edge dominating set is the edge domination number and is denoted by $\acute{\gamma}(G)$. The domination number of $T(\Gamma(R))$ is determined by [20] and [34] independently.

Theorem 3.31. [20] [34] Let R be a finite ring, I be a maximum annihilator ideal of R and $|R/I| = \mu$ then

(i) if R is a field of odd order then $T(\Gamma(R)) = \frac{\mu-1}{2} + 1$ and

(ii) $T(\Gamma(R)) = \mu$, otherwise.

Conjecture 1. [20] Let R be a commutative ring with identity which is not an Artin ring, $Z(R)$ be not an ideal of R and $I_i's$ are maximal annihilator ideals of R. If $|R/I_i|$ is finite for some i, then $\gamma(T(\Gamma(R))) = \min\{|R/I_i|\}$, I_i is a maximal annihilator ideal of R, where the minimum is taken over all I_i for which $|R/I_i|$ is finite.

Theorem 3.32. [29] Let R be a finite. Then

(i) the domination number of $Reg(\Gamma(R))$ is $\gamma(Reg(\Gamma(R)) = \mu - 1$

(ii) the domination number of $Z(\Gamma(R))$ is $\gamma(Z(\Gamma(R))) = 1$.

Theorem 3.33. [29] Let R be a finite ring with unity. Then

(i) If $|R| = 2r$ for some odd integer r, then the edge domination number of $T(\Gamma(R))$ is $\acute{\gamma}(T(\Gamma(R))) = r - 1$.

(ii) If $|R| \neq 2r$ for some odd integer r, then the edge domination number of $T(\Gamma(R))$ is $\acute{\gamma}(T(\Gamma(R))) = \lfloor \frac{|R|}{2} \rfloor$.

(iii) the edge domination number of $Reg(\Gamma(R))$ is $\acute{\gamma}(Reg(\Gamma(R))) = \lfloor \frac{|Reg(R)|}{2} \rfloor$.

(iv) the edge domination number of $Z(\Gamma(R))$ is $\acute{\gamma}(Z(\Gamma(R))) = \lfloor \frac{|Z(R)|}{2} \rfloor$.

The chromatic number of a graph G, $\chi(G)$, is the minimum k such that G is k-colorable (i.e can be colored using k different colors such that no two adjacent vertices have the same color). The clique number, $\omega(G)$, of a graph G is the maximum order among the complete subgraphs of G. It is easy to see that $\chi(G) \geq \omega(G)$, because every vertex of a clique should get a different colour. A graph G is called weakly perfect provided $\chi(G) = \omega(G)$.

For a noncommutative ring R the graphs $T(\Gamma(R))$ and $Reg(\Gamma(R))$ are defined the same way as for the commutative case. Let $M_n(R), GL_n(R)$, and $T_n(R)$ denote the set of $n \times n$ matrices over R, the set of $n \times n$ invertible matrices over R, and the set of $n \times n$ upper triangular matrices over R, respectively. The chromatic number and the clique number for the total graph and the regular graphs of such rings are studied in [5], where the authors obtained the following results

Theorem 3.34. [5] Let F be a field with char$(F) \neq 2$, and n be a positive integer. Then $\omega(\text{Reg}(\Gamma(M_n(F)))) \leq \infty$. Moreover, $\omega(\text{Reg}(\Gamma(M_n(F)))) \leq \sum_{k=0}^{n} k![\binom{n}{k}]^2$

The authors in [5], conjectured that the above result is also true if we replace field by division ring.

Conjecture 2. [5] Let D be a division ring, char$(D) \neq 2$ and n be a natural number. Then $\omega(\text{Reg}(\Gamma(M_n(D)))) \leq \infty$

Theorem 3.35. [5] Let F be a field with char$(F) \neq 2$. Then $\omega(\text{Reg}(\Gamma(M_2(F)))) = 5$

Theorem 3.36. [3] Let F be a field with char$(F) \neq 2$, and n be a positive integer. Then Reg$(\Gamma(T_n(F)))$ is weakly perfect and $\chi(\text{Reg}(\Gamma(T_n(F)))) = \omega(\text{Reg}(\Gamma(T_n(F)))) = 2^n$

Aalipour and Akbari [1] considered the the chromatic number and the the clique number for the total graph and the regular graph of any commutative ring.

Theorem 3.37. [1] Let R be a finite ring such that one of the following conditions holds:

(i) The residue field of R of minimum size has even characteristic,

(ii) Every residue field of R has odd characteristic and $\frac{R}{J(R)}$ has no summand isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. Then both $T(\Gamma(R))$ and $Z(\Gamma(R)$ are weakly perfect. Moreoverer, we have: $\chi(T(\Gamma(R))) = \omega(T(\Gamma(R))) = \chi(Z(\Gamma(R))) = \omega(Z(\Gamma(R))) = \max\{|m| : m \in \text{Max}(R)\}$

Theorem 3.38. [4] Let R be a ring and $2 \notin Z(R)$. If $Z(R) = \cup_{i=1}^{n} P_i$ where $P_1, P_2, \ldots, P_n$ are prime ideals of R and $Z(R) \neq \cup_{i\neq j} P_i$, for $j = 1, 2, \ldots, n$, then Reg$(\Gamma(R))$ is weakly perfect. Moreoverer, we have:
$\chi(\text{Reg}(\Gamma(R))) = \omega(\text{Reg}(\Gamma(R))) = 2^n$

Corollary 3.39. [1] Let R be a finite ring. If every residue field of R has odd characteristic, then Reg$(\Gamma(R))$ is weakly perfect. Moreoverer, we have:
$\chi(\text{Reg}(\Gamma(R))) = \omega(\text{Reg}(\Gamma(R))) = 2^{|\text{Max}(R)|}$

Theorem 3.40. [1] Let R be a finite ring and m be a maximal ideal of R of maximum size. If $\frac{R}{m}$ has characteristic 2, then
$\chi(\text{Reg}(\Gamma(R))) = \omega(\text{Reg}(\Gamma(R))) = \frac{|\text{Reg}(R)|}{\frac{R}{m}-1}$

The results of Aalipour and Akbari motivated the following conjecture.

Conjecture 3. [1] Let R be a finite ring. The total graph of R, $T(\Gamma(R))$, is weakly perfect. Moreover

$$\chi(T(\Gamma(R))) = \omega(T(\Gamma(R))) = \begin{cases} 4, & \text{if } R \approx \mathbb{Z}_3 \times \mathbb{Z}_3; \\ \max\{|m| : m \in \text{Max}(R)\}, & \text{otherwise.} \end{cases}$$

Theorem 3.41. [27] If R is a finite commuatative ring, then the unit graph $G(R)$ is weakly perfect.

An independent set of vertices (also called a coclique) in a graph is a set of pairwise non-adjacent vertices. The independence number of a graph G, $\alpha(G)$, is the greatest integer n such that $\overline{K_n}$ is a subgraph of G.

From Theorem 2.2 of [8], we have the following.

Theorem 3.42. Let R be a commutative ring with unity such that $Z(R)$ is an ideal of R and $|Z(R)| = t$ and $|R/Z(R)| = \beta$. Then

(i) If $2 \in Z(R)$, Then $\alpha(T(\Gamma(R))) = \beta$ and $\alpha(\text{Reg}(\Gamma(R))) = \beta - 1$.

(ii) If $2 \in \text{Reg}(R)$, Then $\alpha(T(\Gamma(R))) = t(\frac{\beta-1}{2}) + 1$ and $\alpha(\text{Reg}(\Gamma(R))) = t(\frac{\beta-1}{2})$.

Observe that for any graph G, $\alpha(G) = \omega(\overline{G})$, is the clique number of the complement graph of the graph G. Combining this to the results in [27] we get

Theorem 3.43. Let $R = \prod_{i=1}^{n} R_i$ where every R_i is local ring with maximal ideal m_i. If $2 \in Z(R)$, assume that $R_1, R_2, ..., R_l$ all have characterstic equal to 2 with $|R_1|/|m_1| \leq |R_2|/|m_2| \leq ... \leq |R_l|/|m_l|$, and $R_{l+1}, R_{l+2}, ..., R_n$ all have characterstic not equal to 2. If $U(R)$ is the set of all unit elements in R, then

$$\alpha(T(\Gamma(R))) = \begin{cases} \frac{1}{2^n}\prod_{i=1}^{n}(|R_i| - |m_i|) + n, & \text{if } 2 \in U(R); \\ |R_1|/|m_1| & \text{if } 2 \notin U(R). \end{cases}$$

For a graph G, a map $f : E(G) \longrightarrow \mathbb{Z}$ is called a flow. A zero-sum flow of an unoriented graph G is a flow of G such that for every vertex $v \in V(G)$ the sum of the values of all edges incident with v is 0. For a natural number k, a zero-sum k-flow is a zero-sum flow with values from the set $\{\pm 1, \pm 2, ... \pm (k-1)\}$. A minimum zero-sum k-flow is the smallest natural number k such that a graph G admits a zero-sum k-flow, but G does not admit a zero-sum $(k-1)$-flow. Minimum flows in $T(\Gamma(R))$ for a finite commutative ring R are studied in [32].

Theorem 3.44. [32] Let R be a finite ring. Then

(i) If $|R|$ is even and $|Z(R)| > 2$, then $T(\Gamma(R))$ has a zero-sum 3-flow, but no zero-sum 2-flow.

(ii) If $|R|$ is odd and $Z(R)$ is an ideal with $|Z(R)| > 2$, then $T(\Gamma(R))$ has a zero-sum 3-flow, but no zero-sum 2-flow.

Based on their results, Sander and Nazzal gave the following conjecture

Conjecture 4. [32] Let R be a finite ring, such that $|Z(R)| > 2$, then $T(\Gamma(R))$ has a zero-sum 3-flow, but no zero-sum 2-flow.

3.2 The Graph $T\Gamma(M)$

Theorem 3.45. [30] If every element of a module M is a sum of at most n torsion elements, then $\text{diam}(T\Gamma(M)) \leq n$. If n is the smallest such number, then $\text{diam}(T\Gamma(M)) = n$

Corollary 3.46. [30] Let R be a commutative ring such that $Z(R)$ is not an ideal of R and $(Z(R)) = R$. Let M be an R-module. If $\text{diam}(T\Gamma(R)) = n$, then $\text{diam}(T\Gamma(M)) \leq n$. In particular, if R is finite, then $\text{diam}(T\Gamma(M)) \leq 2$

3.3 The Graph $\Gamma_S(R)$

Theorem 3.47. [17] Let $R = R_1 \times R_2 \times \cdots \times R_k$, where (R_i, m_i) is a finite local ring such that $R_i/m_i \cong \mathbb{Z}_2$, and let $S = S_1 \times S_2 \times \cdots \times S_k$ be a saturated multiplicatively closed subset of R. Then $\text{diam}(\Gamma_S(R)) \in \{1, 2, \infty\}$

Theorem 3.48. [17] Let R be a finite ring. For a saturated multiplicatively closed subset S of R, we have that $\text{diam}(\Gamma_S(R)) \in \{1, 2, 3, \infty\}$.

Theorem 3.49. [17] Let R be finite and S be a saturated multiplicatively closed subset of R. Then $\text{gr}(\Gamma_S(R)) \in \{3, 4, 6, \infty\}$.

Theorem 3.50. [17] Let R be finite and S be a saturated multiplicatively closed subset of R. Then $\mathrm{gr}(\Gamma_S(R)) = \infty$ if and only if one of the following statements holds:

(i) $R \cong \mathbb{Z}_3$.

(ii) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cdots \mathbb{Z}_2$ and $|S| = 1$.

3.4 The Graph $T(\Gamma_I(R))$

Theorem 3.51. [2] Let R be a commutative ring with the proper ideal I such that $S(I)$ is an ideal of R.

(i) Assume that Γ is an induced subgraph of $\overline{S}(\Gamma_I(R))$ and let x and y be distinct vertices of Γ that are connected by a path in Γ. Then there exists a path in Γ of length 2 between x and y. In particular, if $\overline{S}(\Gamma_I(R))$ is connected,then $\mathrm{diam}(\overline{S}(\Gamma_I(R))) \leq 2$.

(ii) Suppose x and y are distinct elements of $\overline{S}(\Gamma_I(R))$ that are connected by a path. If $x+y \notin S(I)$, then $x-(-x)-y$ and $x-(-y)-y$ are paths of length 2 between x and y in $\overline{S}(\Gamma_I(R))$.

Theorem 3.52. [2] Let R be a commutative ring with the proper ideal I such that $S(I)$ is an ideal of R.

(i) $\mathrm{diam}(\overline{S}(\Gamma_I(R))) = 0$ if and only if $R \cong \mathbb{Z}_2$.

(ii) $\mathrm{diam}(\overline{S}(\Gamma_I(R))) = 1$ if and only if $R/S(I) \cong \mathbb{Z}_2$ and $|S(I)| \geq 2$ or $R \cong \mathbb{Z}_3$

(iii) $\mathrm{diam}(\overline{S}(\Gamma_I(R))) = 2$ if and only if $R/S(I) \cong \mathbb{Z}_3$ and $|S(I)| \geq 2$

(iv) Otherwise, $\mathrm{diam}(\overline{S}(\Gamma_I(R))) = \infty$.

Corollary 3.53. [2] Let R be a commutative ring with the proper ideal I such that $S(I)$ is an ideal of R, $I \neq \{0\}$.Then

(i) If $\mathrm{diam}(\mathrm{Reg}(\Gamma(R/I))) = 0$, then $\mathrm{diam}(\overline{S}(\Gamma_I(R))) = 1$ and $I = S(I)$.

(ii) Let $\mathrm{diam}(\mathrm{Reg}(\Gamma(R/I))) = 1$. Then $\mathrm{diam}(\overline{S}(\Gamma_I(R))) = 1$ if $I \subsetneq S(I)$ and $\mathrm{diam}(\overline{S}(\Gamma_I(R))) = 2$ if $I = S(I)$.

(iii) If $\mathrm{diam}(\mathrm{Reg}(\Gamma(R/I))) = 2$, then $\mathrm{diam}(\overline{S}(\Gamma_I(R))) = 2$.

(iv) $\mathrm{diam}(\overline{S}(\Gamma_I(R))) = \infty$ if and only if $\mathrm{diam}(\mathrm{Reg}(\Gamma(R/I))) = \infty$.

Theorem 3.54. [2] Let R be a commutative ring with the proper ideal I such that $S(I)$ is not an ideal of R and $R = (S(I))$. Let $n \geq 2$ be the least integer such that $R = (x_1, \cdots, x_n)$ for some $x_1, \cdots, x_n \in S(I)$. Then $\mathrm{diam}(T(\Gamma_I(R))) = n$. In particular, if R/I is a fnite ring and $I \subseteq Jac(R)$, then $\mathrm{diam}(T(\Gamma_I(R))) = 2$.

Theorem 3.55. [2] Let R be a commutative ring with the proper ideal I such that $S(I)$ is an not ideal of R. If $T(\Gamma_I(R))$ is connected, then

(i) $\mathrm{diam}(T(\Gamma_I(R))) = d(0, 1)$.

(i) If $\mathrm{diam}(T(\Gamma_I(R))) = n$ then $\mathrm{diam}(\overline{S}(\Gamma_I(R))) \geq n-2$.

Theorem 3.56. [2] Let R be a commutative ring with the proper ideal I such that $S(I)$ is not an ideal of R. then $\mathrm{diam}(S(\Gamma_I(R))) = 2$.

Theorem 3.57. [2] Let R be a commutative ring with the proper ideal I. Then

(i) $\mathrm{gr}(T(\Gamma_I(R))) \leq \mathrm{gr}(T(\Gamma(R/I)))$. If $T(\Gamma(R/I))$ contains a cycle, then so does $T(\Gamma_I(R))$, and therefore $\mathrm{gr}(T(\Gamma_I(R))) \leq \mathrm{gr}(T(\Gamma(R/I)) \leq 4$.

(ii) If $S(I)$ is an ideal of R and $\{0\} \neq \sqrt{I} \subset S(I)$, then $\mathrm{gr}(S(\Gamma_I(R))) = 3$

(iii) If $S(I)$ is an ideal of R then $\mathrm{gr}(\overline{S}(\Gamma_I(R))) = 3, 4$ or ∞. In particular, $\mathrm{gr}(\overline{S}(\Gamma_I(R))) \leq 4$ if $\overline{S}(\Gamma_I(R))$ contains a cycle.

Theorem 3.58. [2] Let R be a commutative ring. Suppose that $S(I)$ is an ideal of R. Then

(i) (a) $\text{gr}(\overline{S}(\Gamma_I(R))) = 3$ if and only if $2 \in S(I)$ and $|S(I)| \geq 3$.
(b) $\text{gr}(\overline{S}(\Gamma_I(R))) = 4$ if and only if $2 \notin S(I)$ and $|S(I)| = 2$.
(c) Otherwise, $\text{gr}(\overline{S}(\Gamma_I(R))) = \infty$.

(ii) (a) $\text{gr}(T(\Gamma_I(R))) = 3$ if and only if $|S(I)| \geq 3$.
(b) $\text{gr}(T(\Gamma_I(R))) = 4$ if and only if $2 \notin S(I)$ and $|S(I)| = 2$.
(c) Otherwise, $\text{gr}(T(\Gamma_I(R))) = \infty$.

Theorem 3.59. [2] Let R be a commutative ringwith a proper ideal I such that $S(I)$ is not an ideal of R. Then

(i) If $I \neq \{0\}$ then $\text{gr}(S(\Gamma_I(R))) = 3$.

(ii) $\text{gr}(T(\Gamma_I(R))) = 3$ if and only if $\text{gr}(S(\Gamma_I(R))) = 3$.

(iii) $\text{gr}(S(\Gamma_I(R))) = 3$ when $\left|\sqrt{I}\right| \geq 3$.

(iv) If $\text{gr}(T(\Gamma_I(R))) = 4$ then $\text{gr}(S(\Gamma_I(R))) = \infty$.

(v) If $2 \in I$, then $\text{gr}(\overline{S}(\Gamma_I(R))) = 3$ or ∞.

(vi) If $2 \notin I$, then $\text{gr}(\overline{S}(\Gamma_I(R))) = 3, 4$ or ∞.

3.5 The Generalized Total Graph $GT_H(R)$

Theorem 3.60. [10] Let H be a prime ideal of a commutative ring R.

(i) Let G be an induced subgraph of $GT_H(R \setminus H)$, and let x and y be distinct vertices of G that are connected by a path in G. Then there is a path of length at most two between x and y in G. In particular, if $GT_H(R \setminus H)$ is connected, then $\text{diam}(GT_H(R \setminus H)) \leq 2$.

(ii) Let x and y be distinct elements of $R \setminus H$ that are connected by a path in $GT_H(R \setminus H)$. If $x + y \notin H$, then $x - (-x) - y$ and $x - (-y) - y$ are paths of length two tween x and y in $GT_H(R \setminus H)$.

Theorem 3.61. [10] Let R be a commutative ring and H a multiplicative-prime subset of R that is not an ideal of R such that $(H) = R$ (i.e. $GT_H(R)$ is connected). Let $n \geq 2$ be the least integer such that $R = (z_1, ..., z_n)$ for some $z_1, ..., z_n \in H$. Then $\text{diam}(GT_H(R)) = n$. In particular, if H is not an ideal of R and either $\dim(R) = 0$ (e.g. R is finite) or R is an integral domain with $\dim(R) = 1$, then $\text{diam}(GT_H(R)) = 2$.

Theorem 3.62. [10] Let H be a prime ideal of a commutative ring R.

(i) $\text{diam}(GT_H(R \setminus H)) = 0$ if and only if $R \cong \mathbb{Z}_2$.

(ii) $\text{diam}(GT_H(R \setminus H)) = 1$ if and only if $R/H \cong \mathbb{Z}_2$ and $R \neq \mathbb{Z}_2$.

(iii) $\text{diam}(GT_H(R \setminus H)) = 2$ if and only if $R/H \cong \mathbb{Z}_3$ and $R \ncong \mathbb{Z}_3$.

(iv) Otherwise, $\text{diam}(GT_H(R \setminus H)) = \infty$.

Theorem 3.63. [10] Let R be a commutative ring and H be a multiplicative-prime subset of R that is not an ideal of R such that $GT_H(R)$ is connected.

(i) $\text{diam}(GT_H(H)) = 2$

(ii) $\text{diam}(GT_H(R)) = d(0, 1)$.

(iii) If $\text{diam}(GT_H(R)) = n$, then $\text{diam}(GT_H(R \setminus H)) \geq n - 2$.

Theorem 3.64. [10] Let R be a commutative ring and H be a multiplicative-prime subset of R that contains two co-maximal ideals of R. Then $GT_H(R)$ is connected with $\text{diam}(GT_H(R)) = 2$. In particular, this holds if H is not an ideal of R and either $\dim(R) = 0$ or R is an integral domain with $\dim(R) = 1$.

Theorem 3.65. [10] Let R be a commutative ring and H be a multiplicative-prime subset of R that is not an ideal of R. Let $H = \cup_\alpha P_\alpha$ for prime ideals P_α of R. Suppose that $a - b - c$ is a path of length two in $GT_H(R \setminus H)$ for distinct vertices $a, b, c \in R \setminus H$.

(i) If $2k \in H$ for some $k \in \{a, b, c\}$ and $\cap_\alpha P_\alpha \neq \{0\}$, then gr$(GT_H(R \setminus H)) = 3$.

(ii) If $2k = 0$ for some $k \in \{a, b, c\}$ and char$(R) \neq 2$, gr$(GT_H(R \setminus H)) = 3$.

(iii) If $2k \notin H$ for every $k \in \{a, b, c\}$, then gr$(GT_H(R \setminus H)) \leq 4$.

Theorem 3.66. [10] Let H be a prime ideal of a commutative ring R.

(i) (a) gr$(GT_H(R \setminus H)) = 3$ if and only if $2 \in H$ and $|H| \geq 3$.
(b) gr$(GT_H(R \setminus H)) = 4$ if and only if $2 \notin H$ and $|H| \geq 2$.
(c) Otherwise, gr$(GT_H(R \setminus H)) = \infty$.

(iii) (a) gr$(GT_H(R)) = 3$ if and only if $|H| \geq 3$.
(b) gr$(GT_H(R)) = 4$ if and only if $2 \notin H$ and $|H| = 2$.
(c) Otherwise, gr$(GT_H(R)) = \infty$.

Theorem 3.67. [10] Let R be a commutative ring and H be a multiplicative-prime subset of R that is not an ideal of R.

(i) Either gr$(GT_H(H)) = 3$ or gr$(GT_H(H)) = \infty$. Moreover, if gr$(GT_H(H)) = \infty$ then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $H = Z(R)$.

(ii) gr$(GT_H(R)) = 3$ if and only if gr$(GT_H(H)) = 3$.

(iii) gr$(GT_H(R)) = 4$ if and only if gr$(GT_H(H)) = \infty$

(iv) If char$(R) = 2$, then gr$(GT_H(R \setminus H)) = 3$ or ∞.

(v) gr$(GT_H(R \setminus H)) = 3, 4$, or ∞.

4 Conclusion

The graph $T(\Gamma(R))$ motivated the definition of the total graphs $T\Gamma(M)$, $\Gamma_S(R)$, $T(\Gamma_I(R))$ and $GT_H(R)$. Most results obtained concerning these total graphs are extensions to the results obtained by Anderson and Badwai in [8]. However, the graph $T(\Gamma(R))$ is extensively studied in [1], [3], [5], [6], [9], [14], [20], [21], [22], [28], [29], [31], [32], and [34]. It is natural then to ask " can we extend the results in these papers to other total graphs?" the similarities in the structures of such graphs suggest analogous results. This presents tens of open problems to be answered.

References

[1] G. Aalipoura, S. Akbaria, Application of some combinatorial arrays in coloring of total graph of a commutative ring, (submitted) arXiv:1305.4315v1.

[2] A. Abbasi, S. Habibi, The total graph of a commutative ring with respect to a proper ideal, J. Korean Math. Soc. 49 (2012) 85-98, doi: 10.4134/JKMS.2012.49.1.085.

[3] S. Akbari, M. Aryapoor, M. Jamaali, Chromatic number and clique number of subgraphs of regular graph of matrix algebras. Linear Algebra Appl. 436 (2012) 2419-2424.

[4] Akbari, S., Haydari, F. The regular graph of a commutative ring, Period. Math. Hungar. 67 (2) (2013), 211-220.

[5] S. Akbari, M. Jamaali, S.A. Seyed Fakhari, The clique numbers of regular graphs of matrix algebras are finite, Linear Algebra Appl. 431 (2009) 1715-1718.

[6] S. Akbari, D. Kiani, F. Mohammadi, S. Moradi, The total graph and regular graph of a commutative ring, J. Pure Appl. Algebra 213 (2009) 2224-2228.

[7] D. F. Anderson, M. C. Axtell, J. A. Stickles, Zero-divisor graphs in commutative rings, in Commutative Algebra: Noetherian and Non-Noetherian Perspectives, ed. by M. Fontana, S.E. Kabbaj, B. Olberding, I. Swanson (Springer, New York, 2010), 23-45.

[8] D. F. Anderson, A. Badawi, The total graph of a commutative ring, J. Algebra 320 (2008) 2706-2719.

[9] D. F. Anderson, A. Badawi, The total graph of a commutative ring without the zero element, J. Algebra Appl. 11 (2012), doi: 10.1142/S0219498812500740.

[10] D. F. Anderson, A. Badawi, The gereralized total graph of a commutative ring, J. Algebra App. 12 (2013), doi: 10.1142/S021949881250212X.

[11] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, J. Algebra, 217 (1999) 434-447.

[12] C. Arthur (1878). "Desiderata and suggestions: No. 2. The Theory of groups: graphical representation". Amer. J. Math. 1(2): 174-176. JSTOR 2369306. In his Collected Mathematical Papers 10: 403-405.

[13] N. Ashrafi, H. R. Maimani, M. R. Pournaki, and S. Yassemi, Unit graph associated with rings, Comm. Algebra, 38 (2010) 2851-2871, doi: 10.1080/00927870903095574.

[14] T. Asir, T. T. Chelvam, On the total graph and its complement of a commutative ring, Comm. Algebra, 41 (2013) 3820-3835, doi: 10.1080/00927872.2012.678956

[15] R. E. Atani, The L-total graph of an L-module, ISRN Dis. Math., (2011) ID 491936, 10 pages, doi: 10.5402/2011/491936.

[16] S. E. Atani, S. Habibi, The total torsion element graph of a module M over a commutative ring R. An. St. Univ. Ovidius Constanta, 19 (2011) 23-34.

[17] Z. Barati, K. Khashyarmanesh, F. Mohammadi, Kh. Nafar, on the associated graphs to a commutative ring. J. Algebra Appl. 11 (2012) (17 pages) doi: 10.1142/S0219498811005610.

[18] I. Beck, Coloring of commutative rings, J. Algebra, 116 (1988) 208-226.

[19] L. W. Beineke, R. J. Wilson, (2004). Topics in Algebraic Graph Theory. Cambridge: Cambridge University Press.

[20] T. T. Chelvam, T. Asir, Domination in the total graph of a commutative ring, J. Comb. Math. Comb. Comput., 87 (2013) 147-158.

[21] T. T. Chelvam, T. Asir, On the genus of the total graph of a commutative ring, Comm. Algebra, 41 (2013) 142-153.

[22] T. T. Chelvam, T. Asir, On the total graph and its complement of a commutative ring, Comm. Algebra, 41 (2013) 3820-3835, doi: 10.1080/00927872.2012.678956.

[23] J. Coykendall, S. Sather-Wagstaff, L. Sheppardson, S. SpiroffJim, On zero divisor graphs, in Progress in Commutative Algebra 2, 241-299, De Gruyter 2012, Editor C. Francisco, L Klinger, S. Sather-wagstaff, J. C. Vassilev.

[24] L. Fuchs, On primal ideals, Proc. Amer. Math. Soc. 1 (1950) 1-6.

[25] Harary, F. Graph Theory. Publishing Co., Reading Mass. (1972).

[26] I. Kaplansky, Commutative Rings. (University of Chicago Press, Chicago, 1974).

[27] H.R. Maimani, M. R. Pournaki, S. Yassemi, Weakly perfect graphs arising from rings, Glasgow Math. J., 52 (2010) 417-425.

[28] H. R. Maimani, C. Wickham, S. Yassemi, Rings whose total graphs have genus at most one, Rocky Mt. J. Math. 42 (2010) 1551–1560, doi:10.1216/RMJ-2012-42-5-1551.

[29] Kh. Nazzal, M. Ghanem, Some properties of the total graph of a commutative ring, (submitted).

[30] Z. Pucanović, The total graph of a module, Matematicki Vesnik 63 (2011) 305-312.

[31] Z. Pucanović and Z. Petrović, On the radius and the relation between the total graph of a commutative ring and its extensions. Pub. Ins. Math, 89 (2011) 1-9.

[32] T. Sander , K. Nazzal, Minimum flows in the total graph of a commutative ring, Trans. Comb. 3 (2014) 11-20.

[33] F. E. K. Saraei, H. H. Astaneh, R. Navidinia, The total graph of a module with respect to multiplicative-prime subsets, Rom. J. Math. Comp Sc., 4 (2014) 151-166.

[34] M. H. Shekarriz, M. H. Shirdareh Haghighi, H. Sharif, On the total graph of a finite commutative ring, Comm. Algebra 40 (2012) 2798-2807.

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