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On a spanning supergraph of the complement of the sum annihilating ideal graph of a commutative ring

S. Visweswaran and Patat Sarman

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Abstract The rings considered in this article are commutative with identity and unless otherwise specified, they are not integral domains. Let R be a ring. Let $\mathbb{A}(R)$ denote the set of all annihilating ideals of R and let us denote $\mathbb{A}(R)\backslash\{(0)\}$ by $\mathbb{A}(R)^*$. With any ring R, Alilou and Amjadi, in [The sum-annihilating essential ideal graph of a commutative ring, Commun. comb. optim. **1** (2) (2016), 117-135] introduced and investigated an undirected graph, denoted by $\mathcal{AE}_R$, whose vertex set is $\mathbb{A}(R)^*$ and distinct vertices I, J are adjacent in $\mathcal{AE}_R$ if and only if $Ann_R(I) + Ann_R(J)$ is an essential ideal of R. In this article, with any ring R, we associate an undirected graph denoted by $\mathbb{G}(R)$, whose vertex set is $\mathbb{A}(R)^*$ and distinct vertices I, J are adjacent in $\mathbb{G}(R)$ if and only if $I + J$ is an essential ideal of R. The aim of this article is to investigate the interplay between the ring-theoretic properties of a ring R and the graph-theoretic properties of $\mathbb{G}(R)$.

1 Introduction

The rings considered in this article are commutative with identity and unless otherwise specified, they are not integral domains. Let R be a ring. Recall from [9] that an ideal I of R is said to be an *annihilating ideal* if there exists $r \in R\backslash\{0\}$ such that $Ir = (0)$. As in [9], we denote the set of all annihilating ideals of R by $\mathbb{A}(R)$ and $\mathbb{A}(R)\backslash\{(0)\}$ by $\mathbb{A}(R)^*$. We denote the set of all proper ideals of R by $\mathbb{I}(R)$ and $\mathbb{I}(R)\backslash\{(0)\}$ by $\mathbb{I}(R)^*$. Recall that an ideal I of R is said to be an *essential ideal* if $I \cap J \neq (0)$ for any non-zero ideal J of R. For the sake of convenience, we denote the set of all essential ideals of R by $\mathbb{E}(R)$. For an ideal I of R, the *annihilator of* I *in* R denoted by $Ann_R(I)$ is defined as $Ann_R(I) = \{r \in R \mid Ir = (0)\}$. This article is motivated by the research work done by Alilou and Amjadi on the sum-annihilating essential ideal graph of a commutative ring in [1] and on the essential ideal graph of a commutative ring by Amjadi in [2]. With each ring R, the authors of [1] introduced and investigated an undirected graph, denoted by $\mathcal{AE}_R$ whose vertex set is $\mathbb{A}(R)^*$ and distinct vertices I, J are joined by an edge in $\mathcal{AE}_R$ if and only if $Ann_R(I) + Ann_R(J)$ is an essential ideal of R. In [1], Alilou and Amjadi proved several interesting theorems on $\mathcal{AE}_R$ illustrating the interplay between the graph-theoretic properties of $\mathcal{AE}_R$ and the ring-theoretic properties of R. In [2] with each ring R, Amjadi introduced and investigated an undirected graph, denoted by $\mathcal{E}_R$ whose vertex set is $\mathbb{I}(R)^*$ and distinct vertices I, J are joined by an edge in $\mathcal{E}_R$ if and only if $I + J$ is an essential ideal of R. Let R be a ring such that $\mathbb{A}(R)^* \neq \emptyset$. In this article with R, we associate an undirected graph, denoted by $\mathbb{G}(R)$ whose vertex set is $\mathbb{A}(R)^*$ and distinct vertices I, J are joined by an edge in $\mathbb{G}(R)$ if and only if $I + J$ is an essential ideal of R and we study the interplay between the graph-theoretic properties of $\mathbb{G}(R)$ and the ring-theoretic properties of R. It is clear that $\mathbb{G}(R)$ is the subgraph of $\mathcal{E}_R$ induced by $\mathbb{A}(R)^*$. Thus for a ring R, if $\mathbb{I}(R)^* = \mathbb{A}(R)^*$, then $\mathcal{E}_R = \mathbb{G}(R)$. Let R be an Artinian ring. Then it is well-known that $\mathbb{I}(R)^* = \mathbb{A}(R)^*$. (One can prove it with the help of [6, Corollary 8.2 and Propositions 8.3, 8.4].) In the course of our investigation on the properties of $\mathbb{G}(R)$ to be presented in this article, we provide rings R such that $\mathbb{G}(R) \neq \mathcal{AE}_R$.

Let us first recall the following definitions and results from commutative ring theory. Let R be a ring. We denote the set of all zero-divisors of R by $Z(R)$ and $Z(R)\backslash\{0\}$ by $Z(R)^*$. We denote the set of all prime ideals of R by $Spec(R)$ and we denote the set of all maximal ideals of R by $Max(R)$. Let I be a proper ideal of R. Recall from [14] that $\mathfrak{p} \in Spec(R)$ is said to be a *maximal N-prime of* I if $\mathfrak{p}$ is maximal with respect to the property of being contained in $Z_R(\frac{R}{I}) = \{r \in R \mid rx \in I \text{ for some } x \in R\backslash I\}$. Hence, $\mathfrak{p} \in Spec(R)$ is a maximal N-prime of (0) if $\mathfrak{p}$ is maximal with respect to the property of being contained in $Z(R)$. For convenience, we denote the set of all maximal N-primes of (0) in R by $MNP(R)$. Let $S = R\backslash Z(R)$. Note that S is a multiplicatively closed subset of R. Let $x \in Z(R)$. Observe that $Rx \cap S = \emptyset$. Hence, we obtain from Zorn's lemma and [16, Theorem 1] that there exists $\mathfrak{p} \in MNP(R)$ such that $x \in \mathfrak{p}$. Therefore, if $MNP(R) = \{\mathfrak{p}_\alpha\}_{\alpha\in\Lambda}$, then it follows that $Z(R) = \bigcup_{\alpha\in\Lambda} \mathfrak{p}_\alpha$. We denote the cardinality of a set A by $|A|$. It is clear that $|MNP(R)| = 1$ if and only if $Z(R)$ is an ideal of R. Let I be a proper ideal of a ring R. Recall from [13] that $\mathfrak{p} \in Spec(R)$ is said to be an *associated prime of* I *in the sense of Bourbaki* if $\mathfrak{p} = (I :_R x)$ for some $x \in R$. In such a case, we say that $\mathfrak{p}$ is a B-prime of I. Let $\mathfrak{p} \in MNP(R)$. It is not hard to verify that $\mathfrak{p} \in \mathbb{A}(R)$ if and only if $\mathfrak{p}$ is a B-prime of (0) in R. We denote the nilradical of a ring R by

$nil(R)$. Recall that a ring R is said to be *reduced* if $nil(R) = (0)$. Let us denote the set of all minimal prime ideals of R by $Min(R)$. We know from [16, Theorem 10] that if $\mathfrak{p} \in Spec(R)$, then there exists $\mathfrak{p}' \in Min(R)$ such that $\mathfrak{p}' \subseteq \mathfrak{p}$. Hence, it follows from [6, Proposition 1.8] that $nil(R) = \bigcap_{\mathfrak{p} \in Min(R)} \mathfrak{p}$. Thus if R is a reduced ring, then $\bigcap_{\mathfrak{p} \in Min(R)} \mathfrak{p} = (0)$. For a ring R, the Krull dimension of R is referred to as the dimension of R and is denoted by $dim R$.

A ring R which admits a unique maximal ideal is referred to as a *quasilocal ring*. A ring R which admits only a finite number of maximal ideals is referred to as a *semiquasilocal ring*. A Noetherian quasilocal (respectively, semiquasilocal) ring is referred to as a *local* (respectively, *semilocal*) ring. The modules considered in this article are unitary. Let M be a module over a commutative ring R. Recall that a submodule N of M is said to be an *essential submodule of* M, if $N \cap K \neq (0)$ for each non-zero submodule K of M. A submodule N of M is said to be a *minimal submodule of* M if $N \neq (0)$ and the only submodules of N are (0) and N [17, page 51]. Recall from [17, page 59] that the *socle* of M, denoted by $Soc(M)$ is defined as the sum of all the minimal submodules of M. If M has no minimal submodule, then we define $Soc(M) = (0)$. Let R be a ring. Then the socle of R is the socle of R regarded as a module over R. Hence, $Soc(R)$ is the sum of all the minimal ideals of R and $Soc(R) = (0)$ if R has no minimal ideal. If $(R, \mathfrak{m})$ is a quasilocal ring, then it is easy to verify that $Soc(R) = Ann_R(\mathfrak{m})$ and hence, $Soc(R)$ has the structure of a vector space over the field $\frac{R}{\mathfrak{m}}$. Recall that a principal ideal ring R is said to be a *special principal ideal ring* (SPIR) if R has a unique prime ideal. If $\mathfrak{m}$ is the unique prime ideal of R, then $\mathfrak{m}$ is necessarily nilpotent and is principal. If R is an SPIR with $\mathfrak{m}$ as its only prime ideal, then we denote it by mentioning that $(R, \mathfrak{m})$ is an SPIR. Let $(R, \mathfrak{m})$ be a quasilocal ring such that $\mathfrak{m}$ is nilpotent. Let $n \geq 2$ be least with the property that $\mathfrak{m}^n = (0)$. If $\mathfrak{m}$ is principal with $\mathfrak{m} = Rm$, then it follows from the proof of $(iii) \Rightarrow (i)$ of [6, Proposition 8.8] that $\mathbb{I}(R)^* = \{\mathfrak{m}^i = Rm^i \mid i \in \{1, \ldots, n-1\}\}$ and hence, $(R, \mathfrak{m})$ is an SPIR. Whenever a set A is a subset of a set B and $A \neq B$, we denote it by either $A \subset B$ (or by $B \supset A$). For any $n \geq 2$, we denote the ring of integers modulo n by $\mathbb{Z}_n$. Let R be a ring with identity which is not necessarily commutative. Let M be a left module over R. With M, Matczuk and Majidinya in [18] introduced and investigated an undirected graph called the *sum-essential graph of* M denoted by $\mathcal{S}_R(M)$ whose vertex set is the set of all non-zero proper submodules of M and distinct vertices N_1, N_2 are joined by an edge in this graph if and only if $N_1 + N_2$ is an essential submodule of M. The authors of [18] also explored the subgraph $\mathcal{P}_R(M)$ of $\mathcal{S}_R(M)$ induced by the set of all non-essential submodules of M and in [18], they studied the interplay between the module properties of M and the graph properties of the graphs $\mathcal{S}_R(M)$ and $\mathcal{P}_R(M)$.

It is useful to recall the following definitions and results from graph theory before we give an account of the results that are proved in this article. The graphs considered in this article are undirected and simple. Let $G = (V, E)$ be a graph. Let $a, b \in V, a \neq b$. Suppose that there exists a path in G between a and b. Recall from [7] that the *distance* between a and b, denoted by $d(a, b)$ is defined as the length of a shortest path in G between a and b. We define $d(a, b) = \infty$ if there exists no path in G between a and b. We define $d(a, a) = 0$. A graph $G = (V, E)$ is said to be *connected* if for any distinct $a, b \in V$, there exists a path in G between a and b. Let $G = (V, E)$ be a connected graph. Recall from [7, Definition 4.2.1] that the *diameter* of G, denoted by $diam(G)$ is defined as $diam(G) = sup\{d(a, b) \mid a, b \in V\}$. For a graph G, we denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. Recall from [7] that a subgraph H of G is said to be a *spanning subgraph of* G if $V(H) = V(G)$. In such a case, we say that G is a *spanning supergraph of* H.

A simple graph $G = (V, E)$ is said to be *complete* if every pair of distinct vertices of G are adjacent in G. Let $n \in \mathbb{N}$. A complete graph on n vertices is denoted by K_n [7, Definition 1.1.11].

Let $G = (V, E)$ be a simple graph. Recall from [7, Definition 1.1.13] that the *complement of* G, denoted by G^c is a graph whose vertex set is V and distinct vertices u, v are joined by an edge in G^c if and only if there is no edge joining u and v in G.

Let R be a ring which is not an integral domain. Motivated by the research work done on the *annihilating-ideal graph* of a commutative ring by Behboodi and Rakeei in [9, 10] and the research work done on the *total graph* of a commutative ring by Anderson and Badawi in [3, 4, 5], we in [22] introduced and investigated an undirected graph, denoted by $\Omega(R)$ whose vertex set is $\mathbb{A}(R)^*$ and distinct vertices I, J are joined by an edge in $\Omega(R)$ if and only if $I + J \in \mathbb{A}(R)$. The graph $\Omega(R)$ was also investigated in [15] and the authors of [15] called $\Omega(R)$ as the *sum annihilating ideal graph of* R. In [23], we studied $(\Omega(R))^c$, the complement of $\Omega(R)$. It is useful to recall here that the vertex set of $(\Omega(R))^c$ is $\mathbb{A}(R)^*$ and distinct vertices I, J are joined by an edge in $(\Omega(R))^c$ if and only if $I + J \notin \mathbb{A}(R)$. This article consists of three sections including the introduction. Let us now give an account of the results that are proved in this article on $\mathbb{G}(R)$. Let R be a ring such that $\mathbb{A}(R)^* \neq \emptyset$. In Section 2 of this article, we prove some basic results on $\mathbb{G}(R)$. It is observed in Lemma 2.2 that $(\Omega(R))^c$ is a spanning subgraph of $\mathbb{G}(R)$ and hence, $\mathbb{G}(R)$ is a spanning supergraph of $(\Omega(R))^c$. In Proposition 2.4, it is shown that for a reduced ring R, $(\Omega(R))^c = \mathbb{G}(R)$. As $(\Omega(R))^c$ was already investigated in [23], and as $\mathbb{G}(R) = (\Omega(R))^c$ for a reduced ring R, we use the results that are proved on $(\Omega(R))^c$ in our study on $\mathbb{G}(R)$. Let R be a reduced ring such that $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in Min(R)$. It is proved in Proposition 2.5 that the following statements are equivalent: (1) $\mathbb{G}(R) = \mathcal{AE}_R$; and (2) $|Min(R)| = 2$. It is verified in Example 2.7 that the non-reduced ring $R = \mathbb{Z}_4 \times \mathbb{Z}_4$ is such that $|Min(R)| = 2$ but $\mathbb{G}(R) \neq \mathcal{AE}_R$. Let $n \geq 2$. Let $R = R_1 \times R_2 \times \cdots \times R_n$, where for given any $i \in \{1, 2, \ldots, n\}$, either R_i is a field or an SPIR. Then it is shown in Corollary 2.10 that $\mathbb{G}(R) \cong \mathcal{AE}_R$. It is proved in Proposition 2.13 that for a ring

R, $\mathbb{G}(R)$ is connected and $diam(\mathbb{G}(R)) \leq 3$ and if R is not reduced, then it is verified that $diam(\mathbb{G}(R)) \leq 2$. Let R be a ring such that $|\mathbb{A}(R)^*| \geq 2$. It is shown in Corollary 2.14 that the following statements are equivalent: (1) $(\Omega(R))^c = \mathbb{G}(R)$; and (2) R is reduced.

Let R be a ring such that $\mathbb{A}(R)^* \neq \emptyset$. In Section 3 of this article, we discuss some results on $diam(\mathbb{G}(R))$. It is proved in Proposition 3.1 that the following statements are equivalent: (1) $\mathbb{G}(R)$ is complete; and (2) If $I \in \mathbb{A}(R)^*$ is not a minimal ideal of R, then $I \in \mathbb{E}(R)$. If $Soc(R) \neq (0)$, then it is deduced in Corollary 3.2 that the following statements are equivalent: (1) $\mathbb{G}(R)$ is complete; and (2) If $I \in \mathbb{A}(R)^*$ is not a minimal ideal of R, then $Soc(R) \subseteq I$. Let R be such that $dimR = 0$. It is shown in Corollary 3.4 that if $\mathbb{G}(R)$ is complete, then $|Max(R)| \leq 2$. If $|Max(R)| = 2$, then it is deduced in Corollary 3.6 that the following statements are equivalent:(1) $\mathbb{G}(R)$ is complete; and (2) $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$. For a quasilocal ring $(R, \mathfrak{m})$, it is proved in Corollary 3.8 that if $\mathbb{G}(R)$ is complete, then $dim_{\frac{R}{\mathfrak{m}}}(Soc(R)) \leq 2$. If $(R, \mathfrak{m})$ is a local Artinian ring and if $dim_{\frac{R}{\mathfrak{m}}}(Soc(R)) = 1$, then it is verified in Proposition 3.10 that $\mathbb{G}(R)$ is complete. For a local Artinian ring $(R, \mathfrak{m})$ with $\mathfrak{m}^2 = (0)$, it is shown in Proposition 3.13 that the following statements are equivalent: (1) $\mathbb{G}(R)$ is complete; and (2) $dim_{\frac{R}{\mathfrak{m}}}(Soc(R)) \leq 2$. If $(R, \mathfrak{m})$ is an SPIR, then it is verified in Proposition 3.14 that $\mathbb{G}(R)$ is complete. Let R be a ring such that R is not reduced. It is proved in Proposition 3.15 that the following statements are equivalent: (1) $diam(\mathbb{G}(R)) = 2$; and (2) There exists $A \in \mathbb{A}(R)^*$ such that A is not a minimal ideal of R and $A \notin \mathbb{E}(R)$. Let R be a ring such that $|MNP(R)| = 1$ (that is, equivalently, $Z(R)$ is an ideal of R). If $Z(R) \in \mathbb{A}(R)^*$, then with the help of Proposition 2.13, it is deduced in Corollary 3.18 that $diam(\mathbb{G}(R)) \leq 2$. Such a ring R is given in Example 3.19(1) (respectively, in Example 3.19(2)) such that $diam(\mathbb{G}(R)) = 1$ (respectively, $diam(\mathbb{G}(R)) = 2$). Let R be a non-reduced ring with $Z(R)$ is an ideal of R and $Z(R) \notin \mathbb{A}(R)$. Such a ring R is provided in Example 3.20(1) (respectively, in Example 3.20(2)) such that $diam(\mathbb{G}(R)) = 1$ (respectively, $diam(\mathbb{G}(R)) = 2$). Let R be a non-reduced ring with $|MNP(R)| \geq 2$. Several sufficient conditions on $MNP(R)$ are determined in order that $diam(\mathbb{G}(R)) = 2$. If $\mathfrak{p} \notin \mathbb{E}(R)$ for some $\mathfrak{p} \in MNP(R)$, then it is proved in Proposition 3.24 that $diam(\mathbb{G}(R)) = 2$. If $|MNP(R) \cap \mathbb{A}(R)| \geq 2$ and if at least one member of $MNP(R) \cap \mathbb{A}(R)$ is not in $Max(R)$, then it is shown in Proposition 3.26 that $diam(\mathbb{G}(R)) = 2$. If $|MNP(R) \cap \mathbb{A}(R)| \geq 3$, then it is proved in Proposition 3.28 that $diam(\mathbb{G}(R)) = 2$. Motivated by [2, Theorem 2.7], for a ring R, it is proved in Theorem 3.31 that the following statements are equivalent: (1) $\mathbb{G}(R) \cong K_4 - e$, where e is an edge of K_4; (2) $\mathcal{E}_R \cong K_4 - e$, where e is an edge of K_4; (3) $R \cong F \times S$ as rings, where F is a field and $(S, \mathfrak{m})$ is an SPIR with $\mathfrak{m} \neq (0)$ but $\mathfrak{m}^2 = (0)$; and (4) $\mathcal{AE}_R \cong K_4 - e$, where e is an edge of K_4. Several examples are provided to illustrate the results proved in this section (see Examples 3.9, 3.11, 3.12, 3.19, 3.20, 3.23, 3.25, 3.27, and 3.30).

2 Some basic results on $\mathbb{G}(R)$

The rings considered in this article are commutative with identity and unless otherwise specified, they are not integral domains. Let R be a ring. The aim of this section is to prove some basic results on $\mathbb{G}(R)$.

Lemma 2.1. *Let R be a ring. Let I be any ideal of R. If $I \notin \mathbb{A}(R)$, then $I \in \mathbb{E}(R)$. If R is reduced, and if $J \in \mathbb{E}(R)$, then $J \notin \mathbb{A}(R)$.*

Proof. We are assuming that the ideal I of R is such that $I \notin \mathbb{A}(R)$. Let $r \in R\backslash\{0\}$. Then $Ir \neq (0)$ and so, $I \cap Rr \neq (0)$. This shows that $I \in \mathbb{E}(R)$.

Let R be reduced. Let A be any ideal of R. Let $r \in R\backslash\{0\}$. If $sr \in (A \cap Rr)\backslash\{0\}$ for some $s \in R$, then $sr^2 \in Ar$ and as R is reduced, $sr^2 \neq 0$. Let $J \in \mathbb{E}(R)$. Let $r \in R\backslash\{0\}$. Then $J \cap Rr \neq (0)$ and so, $Jr \neq (0)$. This proves that $J \notin \mathbb{A}(R)$. □

Lemma 2.2. *Let R be a ring. Then $(\Omega(R))^c$ is a spanning subgraph of $\mathbb{G}(R)$.*

Proof. Note that $V(\mathbb{G}(R)) = V((\Omega(R))^c) = \mathbb{A}(R)^*$. Let $I, J \in \mathbb{A}(R)^*$ be such that I and J are adjacent in $(\Omega(R))^c$. Hence, $I + J \notin \mathbb{A}(R)$ and so, we obtain from Lemma 2.1 that $I + J \in \mathbb{E}(R)$. Therefore, I and J are adjacent in $\mathbb{G}(R)$. This proves that $(\Omega(R))^c$ is a spanning subgraph of $\mathbb{G}(R)$. □

In Example 2.3, we provide a ring R such that $(\Omega(R))^c$ is not a subgraph of $\mathcal{AE}_R$.

Example 2.3. Let F_1, F_2, F_3 be fields and let $R = F_1 \times F_2 \times F_3$. Then $(\Omega(R))^c$ is not a subgraph of $\mathcal{AE}_R$.

Proof. Note that R is semilocal with $Max(R) = \{\mathfrak{m}_1 = (0) \times F_2 \times F_3, \mathfrak{m}_2 = F_1 \times (0) \times F_3, \mathfrak{m}_3 = F_1 \times F_2 \times (0)\}$. It is clear that $\mathbb{I}(R)^* = \mathbb{A}(R)^*$. Observe that $\mathfrak{m}_1 + \mathfrak{m}_2 = R \notin \mathbb{A}(R)$. Hence, $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are adjacent in $(\Omega(R))^c$. Note that $Ann_R(\mathfrak{m}_1) = \mathfrak{m}_2\mathfrak{m}_3$ and $Ann_R(\mathfrak{m}_2) = \mathfrak{m}_1\mathfrak{m}_3$. Therefore, $Ann_R(\mathfrak{m}_1) + Ann_R(\mathfrak{m}_2) = \mathfrak{m}_2\mathfrak{m}_3 + \mathfrak{m}_1\mathfrak{m}_3 = \mathfrak{m}_3$. As $\mathfrak{m}_3 \cap (\mathfrak{m}_1\mathfrak{m}_2) = (0) \times (0) \times (0)$, it follows that $\mathfrak{m}_3 \notin \mathbb{E}(R)$ and so, $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are not adjacent in $\mathcal{AE}_R$. This shows that $(\Omega(R))^c$ is not a subgraph of $\mathcal{AE}_R$. □

Let R be a ring. In Proposition 2.4, we provide a sufficient condition on R in order that $(\Omega(R))^c$ to be equal to $\mathbb{G}(R)$.

Proposition 2.4. *Let R be a reduced ring. Then $(\Omega(R))^c = \mathbb{G}(R)$.*

Proof. We know from Lemma 2.2 that for any ring R (R can possibly be non-reduced), $(\Omega(R))^c$ is a spanning subgraph of $\mathbb{G}(R)$. Let R be reduced. Let $I, J \in \mathbb{A}(R)^*$ be such that I and J are adjacent in $\mathbb{G}(R)$. Hence, $I + J \in \mathbb{E}(R)$. Since R is reduced, we obtain from Lemma 2.1 that $I + J \notin \mathbb{A}(R)$ and so, I and J are adjacent in $(\Omega(R))^c$. This proves that $\mathbb{G}(R)$ is a subgraph of $(\Omega(R))^c$ and so, we obtain that $(\Omega(R))^c = \mathbb{G}(R)$. $\square$

For a ring R with $|\mathbb{A}(R)^*| \geq 2$, we verify in Corollary 2.14 that $(\Omega(R))^c = \mathbb{G}(R)$ if and only if R is reduced.

Let R be as in Example 2.3. Note that R is a reduced ring with $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in Min(R)$ and it is verified in Example 2.3 that $(\Omega(R))^c \neq \mathcal{AE}_R$. Hence, $\mathbb{G}(R) \neq \mathcal{AE}_R$. Let R be a reduced ring such that $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in Min(R)$. We prove in Proposition 2.5 that $\mathbb{G}(R) = \mathcal{AE}_R$ if and only if $|Min(R)| = 2$.

Proposition 2.5. *Let R be a reduced ring such that $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in Min(R)$. The following statements are equivalent:*
(1) $\mathbb{G}(R) = \mathcal{AE}_R$.
(2) $|Min(R)| = 2$.

Proof. Since R is a reduced ring, as noted in Section 1, it follows that $\bigcap_{\mathfrak{p} \in Min(R)} \mathfrak{p} = (0)$. As R is not an integral domain, we get that $|Min(R)| \geq 2$.
(1) $\Rightarrow$ (2) We are assuming that $\mathbb{G}(R) = \mathcal{AE}_R$. As $|Min(R)| \geq 2$, it is possible to find distinct $\mathfrak{p}_1, \mathfrak{p}_2 \in Min(R)$. By hypothesis, $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in Min(R)$. Hence, $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathbb{A}(R)^*$. Since $\mathfrak{p}_1 + \mathfrak{p}_2 \not\subseteq \mathfrak{p}$ for any $\mathfrak{p} \in Min(R)$, we obtain from [23, Lemma 2.14] that $\mathfrak{p}_1 + \mathfrak{p}_2 \notin \mathbb{A}(R)$. Hence, we obtain from Lemma 2.1 that $\mathfrak{p}_1 + \mathfrak{p}_2 \in \mathbb{E}(R)$. Therefore, $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are adjacent in $\mathbb{G}(R)$. As $\mathbb{G}(R) = \mathcal{AE}_R$ by assumption, it follows that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are adjacent in $\mathcal{AE}_R$. Hence, $Ann_R(\mathfrak{p}_1) + Ann_R(\mathfrak{p}_2) \in \mathbb{E}(R)$ and so, we obtain from Lemma 2.1 that $Ann_R(\mathfrak{p}_1) + Ann_R(\mathfrak{p}_2) \notin \mathbb{A}(R)$. Let $\mathfrak{p} \in Min(R)$. By hypothesis, $\mathfrak{p} \in \mathbb{A}(R)$. Therefore, either $Ann_R(\mathfrak{p}_1) \not\subseteq \mathfrak{p}$ or $Ann_R(\mathfrak{p}_2) \not\subseteq \mathfrak{p}$. Without loss of generality, we can assume that $Ann_R(\mathfrak{p}_1) \not\subseteq \mathfrak{p}$. Note that $\mathfrak{p}_1 Ann_R(\mathfrak{p}_1) = (0) \subset \mathfrak{p}$. Therefore, we obtain that $\mathfrak{p}_1 \subseteq \mathfrak{p}$ and so, $\mathfrak{p} = \mathfrak{p}_1$. This proves that $Min(R) = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$. Therefore, $|Min(R)| = 2$.
(2) $\Rightarrow$ (1) We are assuming that R is a reduced ring with $|Min(R)| = 2$. Let $Min(R) = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$. Note that $\bigcap_{i=1}^{2} \mathfrak{p}_i = (0)$ and $Z(R) = \bigcup_{i=1}^{2} \mathfrak{p}_i$. It is clear that $\mathfrak{p}_i \in \mathbb{A}(R)^*$ for each $i \in \{1, 2\}$ and if $I \in \mathbb{A}(R)^*$, then $I \subseteq Z(R) = \bigcup_{i=1}^{2} \mathfrak{p}_i$. Hence, either $I \subseteq \mathfrak{p}_1$ or $I \subseteq \mathfrak{p}_2$ and from $\bigcap_{i=1}^{2} \mathfrak{p}_i = (0)$, it follows that I is contained in exactly one between $\mathfrak{p}_1$ and $\mathfrak{p}_2$. Therefore, $\mathbb{A}(R)^*$ is the set of all non-zero ideals I of R such that $I \subseteq \mathfrak{p}_i$ for some $i \in \{1, 2\}$. Let $I_1, I_2 \in \mathbb{A}(R)^*$ be such that $I_1 \neq I_2$. Suppose that I_1 and I_2 are adjacent in $\mathbb{G}(R)$. We know from Proposition 2.4 that $(\Omega(R))^c = \mathbb{G}(R)$. Hence, $I_1 + I_2 \notin \mathbb{A}(R)$. Without loss of generality, we can assume that $I_1 \subseteq \mathfrak{p}_1$ but $I_1 \not\subseteq \mathfrak{p}_2$. In such a case, it follows from $I_1 + I_2 \notin \mathbb{A}(R)$ that $I_2 \subseteq \mathfrak{p}_2$ but $I_2 \not\subseteq \mathfrak{p}_1$. From $I_1 Ann_R(I_1) = (0) \subset \mathfrak{p}_2$, we get that $Ann_R(I_1) \subseteq \mathfrak{p}_2$ and so, $Ann_R(I_1) \not\subseteq \mathfrak{p}_1$. From $I_2 Ann_R(I_2) = (0) \subset \mathfrak{p}_1$, it follows that $Ann_R(I_2) \subseteq \mathfrak{p}_1$ and so, $Ann_R(I_2) \not\subseteq \mathfrak{p}_2$. Hence, we obtain that $Ann_R(I_1) + Ann_R(I_2) \not\subseteq \mathfrak{p}_i$ for each $i \in \{1, 2\}$. This shows that $Ann_R(I_1) + Ann_R(I_2) \notin \mathbb{A}(R)$. Therefore, it follows from Lemma 2.1 that $Ann_R(I_1) + Ann_R(I_2) \in \mathbb{E}(R)$ and so, I_1 and I_2 are adjacent in $\mathcal{AE}_R$. This proves that $\mathbb{G}(R)$ is a subgraph of $\mathcal{AE}_R$. Let $I_1, I_2 \in \mathbb{A}(R)^*$ be such that I_1 and I_2 are adjacent in $\mathcal{AE}_R$. Then $Ann_R(I_1) + Ann_R(I_2) \in \mathbb{E}(R)$. As R is a reduced ring, we obtain from Lemma 2.1 that $Ann_R(I_1) + Ann_R(I_2) \notin \mathbb{A}(R)$. As $Ann_R(I_i) \in \mathbb{A}(R)^*$ for each $i \in \{1, 2\}$, it can be shown using the same reasoning as above that $I_1 + I_2 \notin \mathbb{A}(R)$. Therefore, I_1 and I_2 are adjacent in $(\Omega(R))^c = \mathbb{G}(R)$. This shows that $\mathcal{AE}_R$ is a subgraph of $\mathbb{G}(R)$ and therefore, $\mathbb{G}(R) = \mathcal{AE}_R$. $\square$

We provide Example 2.7 to illustrate that (2) $\Rightarrow$ (1) of Proposition 2.5 can fail to hold if we omit the hypothesis that R is reduced in the statement of Proposition 2.5. We use Lemma 2.6 in the verification of Example 2.7.

Lemma 2.6 is well-known [19, Lemma 2.2]. For the sake of convenient reference, we mention [19, Lemma 2.2] here separately as Lemma 2.6.

Lemma 2.6. *Let R be a non-reduced ring. If I is a nilpotent ideal of R, then $Ann_R(I) \in \mathbb{E}(R)$.*

Example 2.7. Let $R = \mathbb{Z}_4 \times \mathbb{Z}_4$. Then $\mathbb{I}(R)^* = \mathbb{A}(R)^*$ and $|Min(R)| = 2$ but $\mathbb{G}(R) \neq \mathcal{AE}_R$.

Proof. It is clear that $|R| = 16$ and R is not reduced. It is well-known that in an Artinian ring T, $\mathbb{I}(T)^* = \mathbb{A}(T)^*$. Hence, $\mathbb{I}(R)^* = \mathbb{A}(R)^*$. Observe that $dim R = 0$ and so, $Spec(R) = Max(R) = Min(R)$. As $Max(\mathbb{Z}_4) = \{2\mathbb{Z}_4\}$, we get that $Min(R) = \{2\mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_4 \times 2\mathbb{Z}_4\}$. Therefore, $|Min(R)| = 2$. Let $I = \mathbb{Z}_4 \times (0)$ and let $J = 2\mathbb{Z}_4 \times (0)$. From $(\mathbb{Z}_4 \times (0)) \cap ((0) \times \mathbb{Z}_4) = (0) \times (0)$, we get that $I \notin \mathbb{E}(R)$. Note that $J \subset I$ and $I + J = I \notin \mathbb{E}(R)$. Hence, I and J are not adjacent in $\mathbb{G}(R)$. It is clear that $Ann_R(I) = (0) \times \mathbb{Z}_4$ and $Ann_R(J) = 2\mathbb{Z}_4 \times \mathbb{Z}_4$. Since J is a nilpotent ideal of R, we obtain from Lemma 2.6 that $Ann_R(J) \in \mathbb{E}(R)$. (It is not hard to verify directly that $Ann_R(J) \in \mathbb{E}(R)$.) Now, $Ann_R(I) \subset Ann_R(J)$ and so, $Ann_R(I) + Ann_R(J) = Ann_R(J) \in \mathbb{E}(R)$. Therefore, I and J are adjacent in $\mathcal{AE}_R$. This shows that $\mathbb{G}(R) \neq \mathcal{AE}_R$. $\square$

Recall from [11, page 14] that two graphs G and G' are said to be *isomorphic* (to each other) if there is a one-to-one correspondence between their vertices and between their edges such that the incidence relationship is preserved. In other words, suppose that the edge e is incident on vertices v_1 and v_2 in G, then the corresponding edge e' in G' must be incident on the vertices v_1' and v_2' that correspond to v_1 and v_2, respectively.

Let R be as in Example 2.7. We verify in Remark 2.11 that $\mathbb{G}(R) \cong \mathcal{AE}_R$.

Lemma 2.8. *Let R be a ring. Suppose that for each $I \in \mathbb{A}(R)$, there exists a non-zero ideal J of R such that $I = Ann_R(J)$. Then $\mathbb{G}(R) \cong \mathcal{AE}_R$.*

Proof. Observe that $V(\mathbb{G}(R)) = V(\mathcal{AE}_R) = \mathbb{A}(R)^*$. Define a mapping $\phi : \mathbb{A}(R)^* \to \mathbb{A}(R)^*$ by $\phi(I) = Ann_R(I)$. We claim that ϕ is a bijection and moreover, for any $I_1, I_2 \in \mathbb{A}(R)^*$, I_1 and I_2 are adjacent in $\mathbb{G}(R)$ if and only if $\phi(I_1)$ and $\phi(I_2)$ are adjacent in $\mathcal{AE}_R$. Let $I \in \mathbb{A}(R)^*$. By hypothesis, there exists a non-zero ideal J of R such that $I = Ann_R(J)$. Hence, $Ann_R(Ann_R(I)) = Ann_R(Ann_R(Ann_R(J))) = Ann_R(J) = I$. Using the above equality, it is easy to show that ϕ is a bijection. Let $I_1, I_2 \in \mathbb{A}(R)^*$ be distinct. Observe that I_1 and I_2 are adjacent in $\mathbb{G}(R)$ if and only if $I_1 + I_2 \in \mathbb{E}(R)$ if and only if $Ann_R(Ann_R(I_1)) + Ann_R(Ann_R(I_2)) \in \mathbb{E}(R)$ if and only if $Ann_R(\phi(I_1)) + Ann_R(\phi(I_2)) \in \mathbb{E}(R)$ if and only if $\phi(I_1)$ and $\phi(I_2)$ are adjacent in $\mathcal{AE}_R$. This shows that $\mathbb{G}(R) \cong \mathcal{AE}_R$. □

Lemma 2.9. *Let $n \geq 2$. Let $R_1, R_2, \ldots, R_n$ be rings. Suppose that for each $i \in \{1, 2, \ldots, n\}$, given any ideal I_i of R_i, there exists an ideal J_i of R_i such that $I_i = Ann_{R_i}(J_i)$. Let $R = R_1 \times R_2 \times \cdots \times R_n$. Then $\mathbb{G}(R) \cong \mathcal{AE}_R$.*

Proof. It is clear that $\mathbb{A}(R)^* \neq \emptyset$. Let $I \in \mathbb{I}(R)^*$. Note that $I = I_1 \times I_2 \times \cdots \times I_n$, where I_i is an ideal of R_i for each $i \in \{1, 2, \ldots, n\}$. Let $i \in \{1, 2, \ldots, n\}$. By hypothesis, there exists an ideal J_i of R_i such that $I_i = Ann_{R_i}(J_i)$. Let us denote the ideal $J_1 \times J_2 \times \cdots \times J_n$ of R by J. Observe that $I = Ann_R(J)$. It is clear that $J \neq (0) \times (0) \times \cdots \times (0)$. Now, it follows from Lemma 2.8 that $\mathbb{G}(R) \cong \mathcal{AE}_R$. □

Corollary 2.10. *Let $n \geq 2$. Let $R_1, R_2, \ldots, R_n$ be rings. For given any $i \in \{1, 2, \ldots, n\}$, suppose that either R_i is a field or an SPIR. Let $R = R_1 \times R_2 \times \cdots \times R_n$. Then $\mathbb{G}(R) \cong \mathcal{AE}_R$.*

Proof. Let F be a field. Then (0) and F are the only ideals of F, $(0) = Ann_F(F)$, and $F = Ann_F(0)$. Let $(S, \mathfrak{m})$ be an SPIR. Let $k \geq 2$ be least with the property that $\mathfrak{m}^k = (0)$. It follows from the proof of $(iii) \Rightarrow (i)$ of [6, Proposition 8.8] that $\mathbb{I}(S)^* = \{\mathfrak{m}^j \mid j \in \{1, \ldots, k-1\}\}$. Let $j \in \{1, \ldots, k-1\}$. It is clear that $\mathfrak{m}^j = Ann_S(\mathfrak{m}^{k-j}), (0) = Ann_S(S)$, and $S = Ann_S(0)$. Thus for each $i \in \{1, 2, \ldots, n\}$, given an ideal I_i of R_i, there exists an ideal J_i of R_i such that $I_i = Ann_{R_i}(J_i)$. Therefore, it follows from Lemma 2.9 that $\mathbb{G}(R) \cong \mathcal{AE}_R$. □

Remark 2.11. Let $R = \mathbb{Z}_4 \times \mathbb{Z}_4$. As $(\mathbb{Z}_4, 2\mathbb{Z}_4)$ is an SPIR, it follows from Corollary 2.10 that $\mathbb{G}(R) \cong \mathcal{AE}_R$.

For a ring R, we prove in Proposition 2.13 that $\mathbb{G}(R)$ is connected with $diam(\mathbb{G}(R)) \leq 3$ and moreover, if R is not reduced, then it is verified in Proposition 2.13 that $diam(\mathbb{G}(R)) \leq 2$. We use Lemma 2.12 in the proof of Proposition 2.13.

Lemma 2.12. *Let R be a ring. Let $A \in \mathbb{A}(R)^*$ be such that $A \in \mathbb{E}(R)$. Then for any $B \in \mathbb{A}(R)^*$ with $A \neq B$, then A and B are adjacent in $\mathbb{G}(R)$.*

Proof. Let $B \in \mathbb{A}(R)^*$ be such that $A \neq B$. Note that $A \subseteq A + B$ and as $A \in \mathbb{E}(R)$, we obtain that $A + B \in \mathbb{E}(R)$. Hence, A and B are adjacent in $\mathbb{G}(R)$. □

Proposition 2.13. *Let R be a ring. Then $\mathbb{G}(R)$ is connected and $diam(\mathbb{G}(R)) \leq 3$. If R is not reduced, then $diam(\mathbb{G}(R)) \leq 2$.*

Proof. We consider the following cases.

Case(1)**.** R is reduced.

We know from Proposition 2.4 that $\mathbb{G}(R) = (\Omega(R))^c$. Hence, we obtain from [23, Proposition 2.4] that $\mathbb{G}(R)$ is connected and $diam(\mathbb{G}(R)) \leq 3$.

Case(2)**.** R is not reduced.

Since R is not reduced, there exists $x \in R\backslash\{0\}$ such that $x^2 = 0$. Let $I = Rx$. Let $A, B \in \mathbb{A}(R)^*$ be such that $A \neq B$. We show that there exists a path of length at most two between A and B in $\mathbb{G}(R)$. We can assume that A and B are not adjacent in $\mathbb{G}(R)$. As $I^2 = (0)$ and $I \neq (0)$, it is clear that $Ann_R(I) \in \mathbb{A}(R)^*$. We know from Lemma 2.6 that $Ann_R(I) \in \mathbb{E}(R)$. Now, it follows from Lemma 2.12 that $A - Ann_R(I) - B$ is a path of length two between A and B in $\mathbb{G}(R)$.

This proves that $\mathbb{G}(R)$ is connected and $diam(\mathbb{G}(R)) \leq 3$. Moreover, in the case R is not reduced, it is shown in Case(2) that $diam(\mathbb{G}(R)) \leq 2$. □

For a ring R with $|\mathbb{A}(R)^*| \geq 2$, we verify in Corollary 2.14 that $(\Omega(R))^c = \mathbb{G}(R)$ if and only if R is reduced.

Corollary 2.14. *Let R be a ring such that $|\mathbb{A}(R)^*| \geq 2$. The following statements are equivalent:*
(1) $(\Omega(R))^c = \mathbb{G}(R)$.
(2) *R is reduced.*

Proof. (1) $\Rightarrow$ (2) By hypothesis, $|\mathbb{A}(R)^*| \geq 2$. We are assuming that $(\Omega(R))^c = \mathbb{G}(R)$. Hence, it follows from Proposition 2.13 that $(\Omega(R))^c$ is connected. Therefore, we obtain from [23, Lemma 2.1] that R is reduced.
(2) $\Rightarrow$ (1) As R is reduced by assumption, we obtain from Proposition 2.4 that $(\Omega(R))^c = \mathbb{G}(R)$. □

3 Some results on $diam(\mathbb{G}(R))$

As mentioned in Section 1, the rings considered in this article are commutative with identity and unless otherwise specified, they are not integral domains. Let R be a ring. It is shown in Proposition 2.13 that $\mathbb{G}(R)$ is connected and $diam(\mathbb{G}(R)) \leq 3$. If R is reduced, then we know from Proposition 2.4 that $(\Omega(R))^c = \mathbb{G}(R)$. We know from [23, Remark 2.19(i)] that $diam((\Omega(R))^c) = 1$ if and only if $R \cong K_1 \times K_2$ as rings, where K_i is a field for each $i \in \{1, 2\}$. It was remarked in [23, Remark 2.19(ii)] that $diam((\Omega(R))^c) = 2$ if and only if $|Min(R)| = 2$ and $R \ncong K_1 \times K_2$ as rings for any fields K_1, K_2 and it was noted in [23, Remark 2.19(iii)] that $diam((\Omega(R))^c) = 3$ if and only if $|Min(R)| \geq 3$. If the ring R is not reduced, then it is already verified in Proposition 2.13 that $diam(\mathbb{G}(R)) \leq 2$. In [1, Theorem 2] Artinian rings R were characterized in order that $\mathcal{AE}_R$ to be complete. Motivated by [1, Theorem 2], we first try to characterize ring R in order that $\mathbb{G}(R)$ to be complete. A non-zero ideal I of a ring R is said to be a *minimal ideal of* R if there exists no ideal J of R such that $(0) \subset J \subset I$.

In Proposition 3.1, we provide a necessary and sufficient condition in order that $\mathbb{G}(R)$ to be complete.

Proposition 3.1. *Let R be a ring. Then the following statements are equivalent:*
(1) *$\mathbb{G}(R)$ is complete.*
(2) *If $I \in \mathbb{A}(R)^*$ is not a minimal ideal of R, then $I \in \mathbb{E}(R)$.*

Proof. (1) $\Rightarrow$ (2) We are assuming that $\mathbb{G}(R)$ is complete. Let $I \in \mathbb{A}(R)^*$ be such that I is not a minimal ideal of R. Note that there exists a non-zero ideal J of R such that $J \subset I$. It is clear that $J \in \mathbb{A}(R)^*$. Since $\mathbb{G}(R)$ is complete, it follows that I and J are adjacent in $\mathbb{G}(R)$. Therefore, $I + J = I \in \mathbb{E}(R)$.
(2) $\Rightarrow$ (1) We are assuming that if $I \in \mathbb{A}(R)^*$ is such that I is not a minimal ideal of R, then $I \in \mathbb{E}(R)$. Let $A, B \in \mathbb{A}(R)^*$ be such that $A \neq B$. Then either $A \nsubseteq B$ or $B \nsubseteq A$. Without loss of generality, we can assume that $A \nsubseteq B$. Then $(0) \neq B \subset A + B$. Then $A + B$ is not a minimal ideal of R. If $A + B \in \mathbb{A}(R)$, then it follows from the assumption that $A + B \in \mathbb{E}(R)$. If $A + B \notin \mathbb{A}(R)$, then we know from Lemma 2.1 that $A + B \in \mathbb{E}(R)$. Therefore, A and B are adjacent in $\mathbb{G}(R)$. This shows that $\mathbb{G}(R)$ is complete. □

Let R be a ring such that $Soc(R) \neq (0)$. In Corollary 3.2, we prove that $\mathbb{G}(R)$ is complete if and only if $Soc(R) \subseteq I$ for each $I \in \mathbb{A}(R)^*$ such that I is not a minimal ideal of R.

Corollary 3.2. *Let R be a ring such that $Soc(R) \neq (0)$. The following statements are equivalent:*
(1) *$\mathbb{G}(R)$ is complete.*
(2) *If $I \in \mathbb{A}(R)^*$ is not a minimal ideal of R, then $Soc(R) \subseteq I$.*

Proof. (1) $\Rightarrow$ (2) We are assuming that $\mathbb{G}(R)$ is complete. Let $I \in \mathbb{A}(R)^*$ be such that I is not a minimal ideal of R. We know from (1) $\Rightarrow$ (2) of Proposition 3.1 that $I \in \mathbb{E}(R)$. Let J be any minimal ideal of R. Note that $I \cap J \neq (0)$ and so, $I \cap J = J$. Therefore, $J \subseteq I$. Since $Soc(R)$ is the sum of all the minimal ideals of R, we obtain that $Soc(R) \subseteq I$.
(2) $\Rightarrow$ (1) By hypothesis, $Soc(R) \neq (0)$. First, we verify that $Soc(R) \in \mathbb{E}(R)$. Let $A \in \mathbb{I}(R)^*$. If $A \notin \mathbb{A}(R)$, then $ASoc(R) \neq (0)$ and so, $Soc(R) \cap A \neq (0)$. Suppose that $A \in \mathbb{A}(R)^*$. If A is a minimal ideal of R, then it is clear that $A \subseteq Soc(R)$ and so, $Soc(R) \cap A = A \neq (0)$. If A is not a minimal ideal of R, then by assumption, $Soc(R) \subseteq A$ and hence, $Soc(R) \cap A = Soc(R) \neq (0)$. This shows that $Soc(R) \in \mathbb{E}(R)$. Let $I \in \mathbb{A}(R)^*$. If I is not a minimal ideal of R, then by assumption, $Soc(R) \subseteq I$. Since $Soc(R) \in \mathbb{E}(R)$, we get that $I \in \mathbb{E}(R)$. Therefore, we obtain from (2) $\Rightarrow$ (1) of Proposition 3.1 that $\mathbb{G}(R)$ is complete. □

Let R be a ring such that $dim R = 0$. In Corollary 3.4, we show that there is a restriction on $|Max(R)|$ in order that $\mathbb{G}(R)$ to be complete. We use Lemma 3.3 in the proof of Corollary 3.4.

Lemma 3.3. *Let R_1, R_2, R_3 be rings and let $R = R_1 \times R_2 \times R_3$. Then $\mathbb{G}(R)$ is not complete.*

Proof. Let $A = R_1 \times R_2 \times (0)$ and let $B = R_1 \times (0) \times (0)$. It is clear that $A, B \in \mathbb{A}(R)^*$ and $B \subset A$. Hence, A is not a minimal ideal of R. Let $C = (0) \times (0) \times R_3$. Observe that $C \in \mathbb{I}(R)^*$ and $A \cap C = (0) \times (0) \times (0)$. Therefore, $A \notin \mathbb{E}(R)$. It now follows from (1) $\Rightarrow$ (2) of Proposition 3.1 that $\mathbb{G}(R)$ is not complete. □

Corollary 3.4. *Let R be a ring such that $dim R = 0$. If $\mathbb{G}(R)$ is complete, then $|Max(R)| \leq 2$.*

Proof. We are assuming that $\mathbb{G}(R)$ is complete. Suppose that $|Max(R)| \geq 3$. Since $dim R = 0$ by hypothesis, we obtain from [24, Lemma 2.2] that there exist zero-dimensional rings R_1, R_2, R_3 such that $R \cong R_1 \times R_2 \times R_3$ as rings. Let us denote the ring $R_1 \times R_2 \times R_3$ by T. We know from Lemma 3.3 that $\mathbb{G}(T)$ is not complete. Since $R \cong T$ as rings, we get that $\mathbb{G}(R)$ is not complete. This is in contradiction to the assumption that $\mathbb{G}(R)$ is complete. Therefore, $|Max(R)| \leq 2$. □

For a ring R with $dim R = 0$ and $|Max(R)| = 2$, we prove in Corollary 3.6 that $\mathbb{G}(R)$ is complete if and only if $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$. We use Lemma 3.5 in the proof of Corollary 3.6.

Lemma 3.5. *Let R_1, R_2 be rings and let $R = R_1 \times R_2$. Then the following statements are equivalent:*
(1) *$\mathbb{G}(R)$ is complete.*
(2) *Both R_1 and R_2 are fields.*

Proof. (1) $\Rightarrow$ (2) We are assuming that $\mathbb{G}(R)$ is complete. Suppose that R_1 is not a field. Let $I_1 \in \mathbb{I}(R_1)^*$. Let $A = I_1 \times (0)$ and let $B = R_1 \times (0)$. Observe that $A, B \in \mathbb{A}(R)^*$, $A \subset B$ and so, B is not a minimal ideal of R. Note that $B \cap ((0) \times R_2) = (0) \times (0)$. Hence, $B \notin \mathbb{E}(R)$. Therefore, we obtain from (1) $\Rightarrow$ (2) of Proposition 3.1 that $\mathbb{G}(R)$ is not complete. This is in contradiction to the assumption that $\mathbb{G}(R)$ is complete. Hence, R_1 is a field. Similarly, it can be shown that R_2 is a field.
(2) $\Rightarrow$ (1) We are assuming that R_1 and R_2 are fields. Observe that $\mathbb{I}(R)^* = \mathbb{A}(R)^* = \{R_1 \times (0), (0) \times R_2\}$. As $(R_1 \times (0)) + ((0) \times R_2) = R \in \mathbb{E}(R)$, it follows that $\mathbb{G}(R)$ is complete. □

Corollary 3.6. *Let R be a ring such that $dim R = 0$ and $|Max(R)| = 2$. Then the following statements are equivalent:*
(1) *$\mathbb{G}(R)$ is complete.*
(2) *$R \cong F_1 \times F_2$ as rings, where F_1 and F_2 are fields.*

Proof. We are assuming that $dim R = 0$ and $|Max(R)| = 2$. We know from [24, Lemma 2.2] that there exist zero-dimensional rings R_1, R_2 such that $R \cong R_1 \times R_2$ as rings. Therefore, we obtain from Lemma 3.5 that $\mathbb{G}(R)$ is complete if and only if $R \cong F_1 \times F_2$ as rings, where F_1 and F_2 are fields. □

Remark 3.7. For any ring T with $\mathbb{A}(T)^* \neq \emptyset$, it is already observed in Section 1 that $\mathbb{G}(T)$ is the subgraph of $\mathcal{E}_T$ induced by $\mathbb{A}(T)^*$. As $V(\mathbb{G}(T)) = \mathbb{A}(T)^*$ and $V(\mathcal{E}_T) = \mathbb{I}(T)^*$, it follows that $\mathbb{G}(T) = \mathcal{E}_T$ if and only if $\mathbb{I}(T)^* = \mathbb{A}(T)^*$. It is already noted in Section 1 that for any Artinian ring R, $\mathbb{G}(R) = \mathcal{E}_R$. Let F_1, F_2 be fields and let $T = F_1 \times F_2$. Observe that $\mathbb{I}(T)^* = \mathbb{A}(T)^* = \{\mathfrak{m}_1 = (0) \times F_2, \mathfrak{m}_2 = F_1 \times (0)\}$. It is clear that $\mathfrak{m}_1 + \mathfrak{m}_2 = T \notin \mathbb{A}(T)$ and so, $\mathfrak{m}_1, \mathfrak{m}_2$ are adjacent in $(\Omega(T))^c$ and hence, they are adjacent in $\mathbb{G}(T)$. From $Ann_T(\mathfrak{m}_1) = \mathfrak{m}_2$ and $Ann_T(\mathfrak{m}_2) = \mathfrak{m}_1$, it follows that $Ann_T(\mathfrak{m}_1) + Ann_T(\mathfrak{m}_2) = T \in \mathbb{E}(T)$ and so, $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are adjacent in $\mathcal{AE}_T$. Therefore, we get that $(\Omega(T))^c = \mathbb{G}(T) = \mathcal{E}_T = \mathcal{AE}_T$. Let R_1, R_2 be rings and let $R = R_1 \times R_2$. If $\mathbb{G}(R)$ is complete, then it follows from (1) $\Rightarrow$ (2) of Lemma 3.5 and the above given arguments that $(\Omega(R))^c = \mathbb{G}(R) = \mathcal{E}_R = \mathcal{AE}_R$.

Let $(R, \mathfrak{m})$ be a quasilocal ring. It is already noted in Section 1 that $Soc(R) = Ann_R(\mathfrak{m})$. Hence, $Soc(R)$ has the structure of a vector space over $\frac{R}{\mathfrak{m}}$ by defining $(r + \mathfrak{m})s = rs$ for any $r + \mathfrak{m} \in \frac{R}{\mathfrak{m}}$ and $s \in Soc(R)$. In Corollary 3.8, we provide a necessary condition on $dim_{\frac{R}{\mathfrak{m}}}(Soc(R))$ in order that $\mathbb{G}(R)$ to be complete.

Corollary 3.8. *Let $(R, \mathfrak{m})$ be a quasilocal ring. If $\mathbb{G}(R)$ is complete, then $dim_{\frac{R}{\mathfrak{m}}}(Soc(R)) \leq 2$.*

Proof. We are assuming that $\mathbb{G}(R)$ is complete. Suppose that $dim_{\frac{R}{\mathfrak{m}}}(Soc(R)) \geq 3$. Let $x, y, z \in Soc(R)$ be such that $\{x, y, z\}$ is linearly independent over $\frac{R}{\mathfrak{m}}$. Let $I = Rx + Ry$. Note that $I\mathfrak{m} = (0)$ and so, $I \in \mathbb{A}(R)^*$. It is clear that I is not a minimal ideal of R. Since $\mathbb{G}(R)$ is complete, we obtain from (1) $\Rightarrow$ (2) of Corollary 3.2 that $Soc(R) \subseteq I$. Hence, $Rz \subseteq I$. This is impossible since $\{x, y, z\}$ is linearly independent over $\frac{R}{\mathfrak{m}}$. Therefore, $dim_{\frac{R}{\mathfrak{m}}}(Soc(R)) \leq 2$. □

In Example 3.9, we provide a local Artinian ring $(R, \mathfrak{m})$ such that $dim_{\frac{R}{\mathfrak{m}}}(Soc(R)) = 2$ but $\mathbb{G}(R)$ is not complete thereby illustrating that the converse of Corollary 3.8 can fail to hold.

Example 3.9. Let $T = \mathbb{Z}_2[X, Y, Z]$ be the polynomial ring in three variables X, Y, Z over $\mathbb{Z}_2$. Let I be the ideal of T generated by $\{X^2, Y^2, Z^2, XZ, YZ\}$. Let $R = \frac{T}{I}$. Let $\mathfrak{m} = \frac{TX+TY+TZ}{I}$. Then $(R, \mathfrak{m})$ is a local Artinian ring with $dim_{\frac{R}{\mathfrak{m}}}(Soc(R)) = 2$ but $\mathbb{G}(R)$ is not complete.

Proof. As T is Noetherian by [6, Corollary 7.6], it follows that $R = \frac{T}{I}$ is Noetherian. It is convenient to denote $X + I$ by x, $Y + I$ by y, and $Z + I$ by z. Note that $TX + TY + TZ \in Max(T)$ and so, $\mathfrak{m} = \frac{TX+TY+TZ}{I} \in Max(R)$. It is easy to verify that $\mathfrak{m}^3 = (0 + I)$. Hence, it follows that $Spec(R) = Max(R) = \{\mathfrak{m}\}$. Therefore, $dim R = 0$. It now follows from [6, Theorem 8.5] that R is Artinian. Hence, $(R, \mathfrak{m})$ is a local Artinian ring. It is not hard to verify that $Soc(R) = Rxy + Rz$ and $\{xy, z\}$ is linearly independent over $\frac{R}{\mathfrak{m}}$. Therefore, $dim_{\frac{R}{\mathfrak{m}}}(Soc(R)) = 2$. We next verify that $\mathbb{G}(R)$ is not complete. Observe that $A = Rx + Ry \in \mathbb{I}(R)^* = \mathbb{A}(R)^*$ and as $(0 + I) \neq Rx \subset A$, we obtain that A is not a minimal ideal of R. Since $Rz \in \mathbb{I}(R)^*$ and $A \cap Rz = (0 + I)$, we get that $A \notin \mathbb{E}(R)$. Hence, we obtain from (1) $\Rightarrow$ (2) of Proposition 3.1 that $\mathbb{G}(R)$ is not complete. □

Let $(R, \mathfrak{m})$ be the local Artinian ring mentioned in Example 3.9. It is shown in Example 3.9 that $\mathbb{G}(R)$ is not complete. We know from [1, Lemma 2] that $\mathcal{AE}_R$ is complete. Therefore, $\mathbb{G}(R) \not\cong \mathcal{AE}_R$.

Let $(R, \mathfrak{m})$ be a local Artinian ring. In Proposition 3.10, we provide a sufficient condition on $dim_{\frac{R}{\mathfrak{m}}}(Soc(R))$ in order that $\mathbb{G}(R)$ to be complete.

Proposition 3.10. *Let $(R, \mathfrak{m})$ be a local Artinian ring. If $dim_{\frac{R}{\mathfrak{m}}}(Soc(R)) = 1$, then $\mathbb{G}(R)$ is complete.*

Proof. We are assuming that $(R, \mathfrak{m})$ is a local Artinian ring with $dim_{\frac{R}{\mathfrak{m}}}(Soc(R)) = 1$. Let $m \in Soc(R)$ be such that $Soc(R) = Rm$. Observe that R has Rm as its unique minimal ideal. Let $I \in \mathbb{I}(R)^* = \mathbb{A}(R)^*$ be such that I is not a minimal ideal of R. As in an Artinian ring, each non-zero ideal contains a minimal ideal, we get that $Soc(R) = Rm \subset I$. Hence, we obtain from (2) $\Rightarrow$ (1) of Corollary 3.2 that $\mathbb{G}(R)$ is complete. □

We provide Example 3.11 to illustrate Proposition 3.10.

Example 3.11. Let $T = \mathbb{Z}_2[X, Y]$ be the polynomial ring in two variables X, Y over $\mathbb{Z}_2$. Let $I = TX^2 + TY^2$ and let $R = \frac{T}{I}$. Then $\mathbb{G}(R)$ is complete.

Proof. Note that R is local with $\mathfrak{m} = \frac{TX+TY}{I}$ as its unique maximal ideal. It is clear that $XY \notin I$ and hence, $\mathfrak{m}^2 \neq (0 + I)$. Observe that $\mathfrak{m}^3 = (0 + I)$, $dim_{\frac{R}{\mathfrak{m}}}(\mathfrak{m}^2) = 1$, and $dim_{\frac{R}{\mathfrak{m}}}(\frac{\mathfrak{m}}{\mathfrak{m}^2}) = 2$. Let us denote $X + I$ by x and $Y + I$ by y. Note that $\mathfrak{m} = Rx + Ry$ and $\mathfrak{m}^2 = Rxy$. From $\mathfrak{m}^3 = (0 + I)$ and $\mathfrak{m} \in Max(R)$, it follows that $Spec(R) = Max(R) = \{\mathfrak{m}\}$ and so, $dim R = 0$. Hence, we obtain from [6, Theorem 8.5] that R is Artinian. Thus $(R, \mathfrak{m})$ is a local Artinian ring. It is clear that $\mathfrak{m}^2 \subseteq Soc(R)$. Let $t \in TX + TY$ be such that $t + I \in Soc(R)$. Let $t = t_1X + t_2Y$ for some $t_1, t_2 \in T$. As $t(TX + TY) \subseteq I$, it can be shown that $t \in TX^2 + TXY + TY^2$ and so, $t + I \in Rxy = \mathfrak{m}^2$. This shows that $Soc(R) \subseteq \mathfrak{m}^2$ and so, $Soc(R) = \mathfrak{m}^2 = Rxy$. Therefore, $dim_{\frac{R}{\mathfrak{m}}}(Soc(R)) = 1$. Hence, we obtain from Proposition 3.10 that $\mathbb{G}(R)$ is complete. □

We provide Example 3.12 to illustrate that the conclusion of Proposition 3.10 can fail to hold if the hypothesis that R is Artinian is omitted in the statement of Proposition 3.10.

Example 3.12. Let $T = K[[X, Y]]$ be the power series ring in two variables X, Y over a field K. Let $I = TX^2 + TXY$. Let $R = \frac{T}{I}$. Then $\mathbb{G}(R)$ is not complete.

Proof. It follows from [6, Exercise 5(iv), page 11] that $Max(T) = \{TX + TY\}$ and we know from [16, Theorem 71] that T is Noetherian. Therefore, $R = \frac{T}{I}$ is Noetherian and $\mathfrak{m} = \frac{TX+TY}{I}$ is its unique maximal ideal. Observe that $I = TX \cap (TX + TY)^2$ is an irredundant primary decomposition of I in T with $TX \in Spec(T)$ and $(TX + TY)^2$ is a $TX + TY$-primary ideal of T. It is convenient to denote $X + I$ by x and $Y + I$ by y. Since $TX \in Spec(T)$ with $TX \supset I$, it follows that $Rx \in Spec(R)$. As $X \notin TY$, $TY \in Spec(T)$, and $TX + TY \neq T$, it follows that $X \notin I$ and so, $x \neq 0 + I$. We claim that $Soc(R) = Rx$. From $\mathfrak{m} = Rx + Ry$, we get that $x(rx + sy) = 0 + I$ for any $r, s \in R$. Therefore, $Rx \subseteq Ann_R(\mathfrak{m}) = Soc(R)$. Let $t \in TX + TY$ be such that $t + I \in Soc(R)$. Hence, $(t + I)(Y + I) = 0 + I$ and this implies that $tY \in I \subset TX$. From $Y \notin TX$, we get that $t \in TX$. Therefore, $t + I \in Rx$. This shows that $Soc(R) \subseteq Rx$ and so, $Soc(R) = Rx$. Therefore, $dim_{\frac{R}{\mathfrak{m}}}(Soc(R)) = 1$. Note that if $A \in \mathbb{I}(R)^*$, then $Ax = (0 + I)$ and so, $A \in \mathbb{A}(R)^*$. Therefore, $\mathbb{I}(R)^* = \mathbb{A}(R)^*$. Let $A = Ry$ and let $B = Ry^2$. If $A = B$, then $Y - tY^2 \in TX^2 + TXY$ for some $t \in T$. This implies that $1 - tY \in TX$. This is impossible. Therefore, $A \neq B$. Observe that $A + B = A$. Note that $Rx \in \mathbb{I}(R)^*$ and $A \cap Rx = \frac{TX^2+TY}{I} \cap \frac{TX}{I}$ equals $(0 + I)$. This shows that $A \notin \mathbb{E}(R)$. Therefore, A and B are not adjacent in $\mathbb{G}(R)$ and so, $\mathbb{G}(R)$ is not complete. □

Let $(R, \mathfrak{m})$ be a quasilocal ring with $\mathfrak{m}^2 = (0)$. Note that $Soc(R) = \mathfrak{m}$ and $Spec(R) = Max(R) = \{\mathfrak{m}\}$. Thus if $dim_{\frac{R}{\mathfrak{m}}}(Soc(R)) \leq 2$, then for any $I \in \mathbb{I}(R)^*$, $dim_{\frac{R}{\mathfrak{m}}}(I) \leq 2$ and so, I is generated as an ideal of R by at most two elements. Therefore, R is Noetherian and as $dim R = 0$, we get that R is Artinian. For a local Artinian ring $(R, \mathfrak{m})$ with $\mathfrak{m}^2 = (0)$, we verify in Proposition 3.13 that $\mathbb{G}(R)$ is complete if and only if $dim_{\frac{R}{\mathfrak{m}}}(Soc(R)) \leq 2$..

Proposition 3.13. *Let $(R, \mathfrak{m})$ be a local Artinian ring with $\mathfrak{m}^2 = (0)$. The following statements are equivalent:*
(1) $\mathbb{G}(R)$ *is complete.*
(2) $dim_{\frac{R}{\mathfrak{m}}}(Soc(R)) \leq 2$.

Proof. (1) $\Rightarrow$ (2) This follows from Corollary 3.8. (For this part of the proof, we do not need the hypothesis that $\mathfrak{m}^2 = (0)$.)
(2) $\Rightarrow$ (1) We are assuming that $dim_{\frac{R}{\mathfrak{m}}}(Soc(R)) \leq 2$. From $\mathfrak{m}^2 = (0)$ by hypothesis, we get that $Soc(R) = \mathfrak{m}$. Therefore, $dim_{\frac{R}{\mathfrak{m}}}(\mathfrak{m}) \in \{1, 2\}$. If $dim_{\frac{R}{\mathfrak{m}}}(\mathfrak{m}) = 1$, then $\mathbb{I}(R)^* = \mathbb{A}(R)^* = \{\mathfrak{m}\}$ and so, $\mathbb{G}(R)$ is a graph on a single vertex and so, it is complete. Suppose that $dim_{\frac{R}{\mathfrak{m}}}(\mathfrak{m}) = 2$. Let $I \in \mathbb{A}(R)^*$ be such that I is not a minimal ideal of R. Then $dim_{\frac{R}{\mathfrak{m}}}(I) = 2$ and so, $I = \mathfrak{m} = Soc(R)$. Therefore, we obtain from (2) $\Rightarrow$ (1) of Corollary 3.2 that $\mathbb{G}(R)$ is complete. □

Let $T = K[X, Y]$ be the polynomial ring in two variables X, Y over a field K. Let $I = TX^2 + TXY + TY^2$ and let $R = \frac{T}{I}$. Observe that $TX + TY \in Max(T)$ and $I = (TX + TY)^2$. It is clear that $(R, \mathfrak{m} = \frac{TX+TY}{I})$ is a local Artinian ring with $\mathfrak{m}^2 = (0 + I)$, $Soc(R) = \mathfrak{m}$, and $dim_{\frac{R}{\mathfrak{m}}}(Soc(R)) = 2$. Hence, it follows from (2) $\Rightarrow$ (1) of Proposition 3.13 that $\mathbb{G}(R)$ is complete.

Example 3.9 of this article illustrates that (2) $\Rightarrow$ (1) of Proposition 3.13 can fail to hold if the hypothesis that $\mathfrak{m}^2 = (0)$ is omitted in the statement of Proposition 3.13.

Recall from [12, Exercise 8, page 184] that a ring R is said to be a *chained ring* if the set of ideals of R is linearly ordered by inclusion.

Proposition 3.14. *Let R be a chained ring. Then $\mathbb{G}(R)$ is complete. In particular, if $(R, \mathfrak{m})$ is an SPIR, then $\mathbb{G}(R)$ is complete.*

Proof. Let $I \in \mathbb{I}(R)^*$. Then for any $J \in \mathbb{I}(R)^*$, $I \cap J$ is either I or J and hence, $I \cap J \neq (0)$. This shows that $I \in \mathbb{E}(R)$ for any $I \in \mathbb{I}(R)^*$. It is now clear that $\mathbb{G}(R)$ is complete.

Suppose that $(R, \mathfrak{m})$ is an SPIR and let $n \geq 2$ be least with the property that $\mathfrak{m}^n = (0)$. In such a case, it is remarked in Section 1 that $\mathbb{I}(R)^* = \{\mathfrak{m}^i \mid i \in \{1, \ldots, n-1\}\}$. Hence, $(R, \mathfrak{m})$ is a chained ring and so, $\mathbb{G}(R)$ is complete. □

Proposition 3.15. *Let R be a ring such that R is not reduced. Then the following statements are equivalent:*
(1) $diam(\mathbb{G}(R)) = 2$.
(2) *There exists at least one $A \in \mathbb{A}(R)^*$ such that A is not a minimal ideal of R and $A \notin \mathbb{E}(R)$.*

Proof. (1) $\Rightarrow$ (2) We are assuming that $diam(\mathbb{G}(R)) = 2$. Hence, there exist $I_1, I_2 \in \mathbb{A}(R)^*$ such that $d(I_1, I_2) = 2$ in $\mathbb{G}(R)$. Therefore, I_1, I_2 are not adjacent in $\mathbb{G}(R)$ and so, $I_1 + I_2 \notin \mathbb{E}(R)$. As $I_1 \neq I_2$, we obtain that either $I_1 \not\subseteq I_2$ or $I_2 \not\subseteq I_1$. Without loss of generality, we can assume that $I_1 \not\subseteq I_2$. Then $I_2 \subset I_1 + I_2$. Let us denote $I_1 + I_2$ by A. As $A \notin \mathbb{E}(R)$, we obtain from Lemma 2.1 that $A \in \mathbb{A}(R)$. From $(0) \neq I_2 \subset A$, it follows that A is not a minimal ideal of R. Therefore, there exists $A \in \mathbb{A}(R)^*$ such that A is not a minimal ideal of R and $A \notin \mathbb{E}(R)$.
(2) $\Rightarrow$ (1) We are assuming that there exists $A \in \mathbb{A}(R)^*$ such that A is not a minimal ideal and $A \notin \mathbb{E}(R)$. It follows from (1) $\Rightarrow$ (2) of Proposition 3.1 that $diam(\mathbb{G}(R)) \geq 2$. As R is not reduced by hypothesis, we know from Proposition 2.13 that $diam(\mathbb{G}(R)) \leq 2$ and so, $diam(\mathbb{G}(R)) = 2$. □

Let R be as given in Example 3.12. In the notation of Example 3.12, $x \neq 0 + I$ is such that $x^2 = 0 + I$ and so, R is not reduced. It is already verified in the proof of Example 3.12 that $A = Ry \in \mathbb{A}(R)^*$ is such that A is not a minimal ideal of R and $A \notin \mathbb{E}(R)$. Hence, we obtain from (2) $\Rightarrow$ (1) of Proposition 3.15 that $diam(\mathbb{G}(R)) = 2$.

Corollary 3.16. *Let R be a ring such that R is not reduced. If there exist $I_1, I_2, I_3 \in \mathbb{A}(R)^*$ such that $I_1 \supset I_2 \supset I_3$, with $a^2 \neq 0$ for each $a \in I_2 \backslash \{0\}$, then $diam(\mathbb{G}(R)) = 2$.*

Proof. As $I_3 \in \mathbb{I}(R)^*$ with $I_3 \subset I_2$, it follows that I_2 is not a minimal ideal of R. As $a^2 \neq 0$ for each $a \in I_2 \backslash \{0\}$, it follows that $I_2 \cap Ann_R(I_2) = (0)$. From $I_1 \in \mathbb{A}(R)^*$, it follows that $Ann_R(I_1) \neq (0)$. Observe that from $I_1 \supset I_2$, we get that $Ann_R(I_1) \subseteq Ann_R(I_2)$. Hence, $I_2 \cap Ann_R(I_1) \subseteq I_2 \cap Ann_R(I_2) = (0)$. This implies that $I_2 \cap Ann_R(I_1) = (0)$. Therefore, $I_2 \notin \mathbb{E}(R)$. Thus the non-reduced ring R admits $I_2 \in \mathbb{A}(R)^*$ such that I_2 is not a minimal ideal of R and $I_2 \notin \mathbb{E}(R)$. Therefore, we obtain from (2) $\Rightarrow$ (1) of Proposition 3.15 that $diam(\mathbb{G}(R)) = 2$. □

We provide an example of a non-reduced ring R which satisfies the hypotheses of Corollary 3.16. Let R be as in Example 3.12. Let $I_1 = Ry, I_2 = Ry^2$, and $I_3 = Ry^3$, where y is as in the proof of Example 3.12. Observe that $I_i \in \mathbb{A}(R)^*$ for each $i \in \{1, 2, 3\}$ and $I_1 \supset I_2 \supset I_3$. It is not hard to verify that if $r \in I_2 \backslash \{0 + I\}$, then $r^2 \neq 0 + I$. From $x \neq 0 + I$ but $x^2 = 0 + I$, it follows that R is not reduced. It now follows from Corollary 3.16 that $diam(\mathbb{G}(R)) = 2$.

An element e of a ring R is said to be *idempotent* if $e = e^2$. An idempotent element e of R is said to be *non-trivial* if $e \notin \{0, 1\}$.

Remark 3.17. Let R be a ring such that R is not reduced. Suppose that R has a non-trivial idempotent element e. Let $R_1 = Re$ and let $R_2 = R(1-e)$. Observe that the mapping $f : R \to R_1 \times R_2$ defined by $f(r) = (re, r(1-e))$ is an isomorphism of rings. Since R is not reduced, it follows that either R_1 or R_2 is not a field. Hence, it follows from (1) $\Rightarrow$ (2) of Lemma 3.5 that $diam(\mathbb{G}(R_1 \times R_2)) \geq 2$ and so, $diam(\mathbb{G}(R)) \geq 2$. Since R is not reduced, we obtain from Proposition 2.13 that $diam(\mathbb{G}(R)) \leq 2$ and so, $diam(\mathbb{G}(R)) = 2$. Suppose that R does not admit any non-trivial idempotent. It is clear that $|MNP(R)| = 1$ (equivalently, $Z(R)$ is an ideal of R) is a sufficient condition in order that R to have no non-trivial idempotent.

Let R be a ring such that $|MNP(R)| = 1$. Next, we discuss some results on $diam(\mathbb{G}(R))$.

Corollary 3.18. *Let R be a ring such that $MNP(R) = \{\mathfrak{p}\}$. If $\mathfrak{p} \in \mathbb{A}(R)^*$, then $diam(\mathbb{G}(R)) \leq 2$.*

Proof. Note that $Z(R) = \mathfrak{p}$ and there exists $x \in R\backslash\{0\}$ such that $\mathfrak{p}x = (0)$. It is clear that $x \in Z(R)$ and so, $x^2 = 0$. Hence, we obtain that R is not reduced. It now follows from Proposition 2.13 that $diam(\mathbb{G}(R)) \leq 2$. □

Example 3.19. (1) Let R be as in Example 3.11. Then $Z(R) \in \mathbb{A}(R)^*$ and $diam(\mathbb{G}(R)) = 1$.
(2) Let R be as in Example 3.12. Then $Z(R) \in \mathbb{A}(R)^*$ and $diam(\mathbb{G}(R)) = 2$.

Proof. In the notation of Example 3.11, $(R, \mathfrak{m})$ is a local Artinian ring with $\mathfrak{m}^3 = (0+I)$ but $\mathfrak{m}^2 \neq (0+I)$, and so, $Z(R) = \mathfrak{m} \in \mathbb{A}(R)^*$. It is verified in the proof of Example 3.11 that $diam(\mathbb{G}(R)) = 1$.
(2) In the notation of Example 3.12, R is Noetherian, $Max(R) = \{\mathfrak{m} = Rx + Ry\}$, $x \neq 0 + I$, and $\mathfrak{m}x = (0+I)$. Therefore, $Z(R) = \mathfrak{m} \in \mathbb{A}(R)^*$. It is already verified in the paragraph which appears just preceding the statement of Corollary 3.16 that $diam(\mathbb{G}(R)) = 2$. □

Let R be a ring such that $MNP(R) = \{\mathfrak{p}\}$. It is clear that $Z(R) = \mathfrak{p}$. Since $((0) :_R x) \subseteq \mathfrak{p}$ for any $x \in R\backslash\{0\}$, it follows that $\mathfrak{p} \notin \mathbb{A}(R)$ if and only if $\mathfrak{p}$ is not a B-prime of (0) in R. We provide in Example 3.20(1) (respectively, in Example 3.20(2)) a ring R such that $Z(R)$ is an ideal of R with $Z(R) \notin \mathbb{A}(R)$ and $diam(\mathbb{G}(R)) = 1$ (respectively, $diam(\mathbb{G}(R)) = 2$).

Let M be a module over a ring R. Then $R \times M = \{(r, m) \mid r \in R, m \in M\}$ can be made into a ring by defining addition and multiplication as follows: for any $(r_1, m_1)(r_2, m_2) \in R \times M$, $(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2)$ and $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$. The ring obtained in this way is called the ring obtained by using *Nagata's principle of idealization* and is denoted by $R(+)M$. Example 3.20(2) given below is from [16, Exercises 6 and 7, pages 62-63].

Example 3.20. (1) Let $(V, \mathfrak{m})$ be a rank one valuation domain which is not discrete. Let $m \in \mathfrak{m}\backslash\{0\}$. Let $R = \frac{V}{mV}$. Then $Z(R)$ is an ideal of R, $Z(R) \notin \mathbb{A}(R)$, and $diam(\mathbb{G}(R)) = 1$.
(2) Let $S = K[X, Y]$ be the polynomial ring in two variables X, Y over a field K. Let $\mathfrak{m} = SX + SY$ and let $T = S_{\mathfrak{m}}$. Let $\mathcal{P}$ denote the set of all pairwise non-associate prime elements of T. Let $W = \bigoplus_{p \in \mathcal{P}} \frac{T}{Tp}$ be the direct sum of the T-modules $\frac{T}{Tp}$, where p varies over $\mathcal{P}$. Let $R = T(+)W$ be the ring obtained by using Nagata's principle of idealization. Then $Z(R)$ is an ideal of R, $Z(R) \notin \mathbb{A}(R)$, and $diam(\mathbb{G}(R)) = 2$.

Proof. (1) Let us denote the ideal $\frac{\mathfrak{m}}{mV}$ by $\mathfrak{p}$. It was verified in [22, Example 2.7] that $Z(R) = \mathfrak{p}$. Therefore, $Z(R)$ is an ideal of R. Moreover, we know from the proof of [20, Example 3.1(ii)] that $\mathfrak{p} \notin \mathbb{A}(R)$. As V is a chained ring, it follows that R is a chained ring and hence, we obtain from Proposition 3.14 that $diam(\mathbb{G}(R)) = 1$.

(2) Note that S is a unique factorization domain (UFD). Hence, we obtain from [16, Theorem 5] and [6, Proposition 3.11(iv)] that $T = S_{\mathfrak{m}}$ is a UFD. As S is Noetherian by [6, Corollary 7.6], we obtain from [6, Proposition 7.3] that T is Noetherian. Observe that T is local and $Max(T) = \{\mathfrak{m}T\}$. Since $height(\mathfrak{m}T) = 2$, it follows from [16, Theorem 144] that T has an infinite number of height one prime ideals. Hence, the set of pairwise non-associate prime elements of T is infinite. It was verified in [22, Example 2.8] that $Z(R) = \mathfrak{m}T(+)W$. Let us denote the ideal $\mathfrak{m}T(+)W$ by $\mathfrak{p}$. Therefore, $Z(R)$ is an ideal of R. Moreover, it was shown in [22, Example 2.8] that $\mathfrak{p} \notin \mathbb{A}(R)$. Let us denote the zero element of W by 0_W. It is clear that $((0)(+)W)^2 = (0)(+)(0_W)$. Therefore, R is not reduced. We assert that $diam(\mathbb{G}(R)) = 2$. For any element $w \in W$ and for $p \in \mathcal{P}$, let us denote by w_p, the component of w corresponding to p. Fix two distinct elements $p, q \in \mathcal{P}$. Let $w \in W$ be such that $w_p = 1 + Tp$ and $w_{p'} = 0 + Tp'$ for any $p' \in \mathcal{P}\backslash\{p\}$. Let N_1 be the submodule of W given by $N_1 = Tw$. Let $w' \in W$ be such that $w'_q = 1 + Tq$ and $w'_{p'} = 0 + Tp'$ for any $p' \in \mathcal{P}\backslash\{q\}$. Let N_2 be the submodule of W given by $N_2 = Tw'$. Let $I_i = (0)(+)N_i$ for each $i \in \{1, 2\}$. Note that $I_i \in \mathbb{A}(R)^*$ for each $i \in \{1, 2\}$, $I_1 \neq I_2$, and $I_1 + I_2 = (0)(+)N$, where $N = N_1 + N_2$. Let $I = Tp(+)(0_W)$. It is clear that I is a non-zero ideal of R and $(I_1 + I_2) \cap I = (0)(+)(0_W)$. Hence, $I_1 + I_2 \notin \mathbb{E}(R)$. Observe that $(0)(+)(0_W) \neq I_1 \subset I_1 + I_2$. Hence, $I_1 + I_2 \in \mathbb{A}(R)^*$ is not a minimal ideal of R and $I_1 + I_2 \notin \mathbb{E}(R)$. Since R is not reduced, it follows from (2) $\Rightarrow$ (1) of Proposition 3.15 that $diam(\mathbb{G}(R)) = 2$. □

Proposition 3.21. *Let R be a ring. If $Z(R)$ is an ideal of R, then $Z(R) \in \mathbb{E}(R)$.*

Proof. By hypothesis, $Z(R)$ is an ideal of R. Hence, $|MNP(R)| = 1$. Let us denote $Z(R)$ by $\mathfrak{p}$. Observe that $MNP(R) = \{\mathfrak{p}\}$. Let $a \in R\backslash\{0\}$. If $a \notin Z(R)$, then for any $x \in \mathfrak{p}\backslash\{0\}$, $ax \neq 0$ and $ax \in \mathfrak{p} \cap Ra$. If $a \in \mathfrak{p}$, then $Ra \subseteq \mathfrak{p}$ and so, $\mathfrak{p} \cap Ra = Ra \neq (0)$. This shows that $Z(R) \in \mathbb{E}(R)$. □

Let $n \in \mathbb{N}$ be such that $n \geq 2$. Let R be a ring such that $|MNP(R)| = n$. In Proposition 3.22, we provide a necessary and sufficient condition in order that at least one member of $MNP(R)$ to be a member of $\mathbb{E}(R)$.

Proposition 3.22. *Let $n \in \mathbb{N}$ be such that $n \geq 2$. Let R be a ring such that $|MNP(R)| = n$. Let $MNP(R) = \{\mathfrak{p}_i \mid i \in \{1, 2, \ldots, n\}\}$. Then the following statements are equivalent:*
(1) $\mathfrak{p}_i \in \mathbb{E}(R)$ *for at least one* $i \in \{1, 2, \ldots, n\}$.
(2) $\bigcap_{k=1}^{n} \mathfrak{p}_k \neq (0)$.

Proof. (1) $\Rightarrow$ (2) We are assuming that $\mathfrak{p}_i \in \mathbb{E}(R)$ for some $i \in \{1, 2, \ldots, n\}$. Since distinct members of $MNP(R)$ are not comparable under the inclusion relation, $|MNP(R)| = n$, it follows from [6, Proposition 1.11(ii)] that $\bigcap_{j \in \{1,2,\ldots,n\}\setminus\{i\}} \mathfrak{p}_j \neq (0)$. As $\mathfrak{p}_i \in \mathbb{E}(R)$, we get that $\mathfrak{p}_i \cap (\bigcap_{j \in \{1,2,\ldots,n\}\setminus\{i\}} \mathfrak{p}_j) \neq (0)$ and this shows that $\bigcap_{k=1}^{n} \mathfrak{p}_k \neq (0)$.
(2) $\Rightarrow$ (1) We are assuming that $\bigcap_{k=1}^{n} \mathfrak{p}_k \neq (0)$. Suppose that $\mathfrak{p}_i \notin \mathbb{E}(R)$ for each $i \in \{1, 2, \ldots, n\}$. Then for each $i \in \{1, 2, \ldots, n\}$, there exists $A_i \in \mathbb{I}(R)^*$ such that $\mathfrak{p}_i \cap A_i = (0)$. Let $i \in \{1, 2, \ldots, n\}$. From $\mathfrak{p}_i A_i = (0) \subseteq \mathfrak{p}_j$ for each $j \in \{1, 2, \ldots, n\}\setminus\{i\}$, and $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$, it follows that $A_i \subseteq \mathfrak{p}_j$. We obtain from $\mathfrak{p}_i \cap A_i = (0)$ that $A_i \not\subseteq \mathfrak{p}_i$. Thus $A_i \subseteq \mathfrak{p}_j$ for all $j \in \{1, 2, \ldots, n\}\setminus\{i\}$ but $A_i \not\subseteq \mathfrak{p}_i$. Since $MNP(R) = \{\mathfrak{p}_i \mid i \in \{1, 2, \ldots, n\}\}$, it follows that $Z(R) = \bigcup_{i=1}^{n} \mathfrak{p}_i$. Observe that it follows from [6, Proposition 1.11(i)] that $\sum_{i=1}^{n} A_i \not\subseteq \bigcup_{i=1}^{n} \mathfrak{p}_i = Z(R)$. Hence, there exist $a_1, a_2, \ldots, a_n$ such that $a_i \in A_i$ for each $i \in \{1, 2, \ldots, n\}$ and $\sum_{i=1}^{n} a_i \notin Z(R)$. We are assuming that $\bigcap_{k=1}^{n} \mathfrak{p}_k \neq (0)$. Let $x \in \bigcap_{k=1}^{n} \mathfrak{p}_k, x \neq 0$. It follows from $\mathfrak{p}_i A_i = (0)$ for each $i \in \{1, 2, \ldots, n\}$ that $(\sum_{i=1}^{n} a_i)x = 0$. This is impossible, since $\sum_{i=1}^{n} a_i \notin Z(R)$ and $x \neq 0$. Therefore, $\mathfrak{p}_i \in \mathbb{E}(R)$ for at least one $i \in \{1, 2, \ldots, n\}$. □

For any ring R, we denote the group of units of R by $U(R)$ and the set of all non-units of R by $NU(R)$. Recall from [12, Exercise 16, page 111] that a ring R is said to be *von Neumann regular* if given $a \in R$, there exists $b \in R$ such that $a = a^2 b$. We know from $(a) \Leftrightarrow (d)$ of [12, Exercise 16, page 111] that a ring R is von Neumann regular if and only if $dim R = 0$ and R is reduced. Let $a \in NU(R)$, where R is von Neumann regular. From $a = a^2 b$ for some $b \in R$, it follows that $a(1 - ab) = 0$. As $1 - ab \neq 0$, we get that $a \in Z(R)$. Thus in a von Neumann regular ring R, $Z(R) = NU(R)$. As $dim R = 0$ and $Z(R) = NU(R)$, we get that $Spec(R) = Max(R) = Min(R) = MNP(R)$. Let $a \in R$. Then we know from $(1) \Rightarrow (3)$ of [12, Exercise 29, page 113] that $a = ue$ for some $u \in U(R)$ and $e \in R$ is idempotent. Let $\mathfrak{p} \in MNP(R) = Max(R)$. Let $\mathfrak{p} \in \mathbb{A}(R)$. Then there exists $a \in R\setminus\{0\}$ such that $\mathfrak{p}a = (0)$. Hence, $\mathfrak{p} \subseteq ((0) :_R a)$. Since R is reduced, $a^2 \neq 0$ and so, $a \notin \mathfrak{p}$. Therefore, $((0) :_R a) \subseteq \mathfrak{p}$ and hence, $\mathfrak{p} = ((0) :_R a)$. As $a = ue$ for some $u \in U(R)$ and $e \in R$ is idempotent, we obtain that $\mathfrak{p} = ((0) :_R ue) = ((0) :_R e) = R(1 - e)$ is principal. We provide Example 3.23 to illustrate that $(1) \Rightarrow (2)$ of Proposition 3.22 can fail to hold if the hypothesis that $|MNP(R)| < \infty$ is omitted in the statement of Proposition 3.22.

Example 3.23. Let L be the field of algebraic numbers (that is, L is the algebraic closure of $\mathbb{Q}$). Let A be the ring of all algebraic integers. Let $R = \frac{A}{\sqrt{2A}}$. Then $\mathfrak{p} \in \mathbb{E}(R)$ for each $\mathfrak{p} \in MNP(R)$ but $\bigcap_{\mathfrak{p} \in MNP(R)} \mathfrak{p} = (0 + \sqrt{2A})$.

Proof. It was already verified in the proof of [21, Example 2.20(3)] that R is von Neumann regular. Hence, $Spec(R) = Max(R) = Min(R) = MNP(R)$. Since R is reduced, it follows from [6, Proposition 1.8] that $\bigcap_{\mathfrak{p} \in Spec(R) = MNP(R)} \mathfrak{p} = (0 + \sqrt{2A})$. Let $\mathfrak{p} \in MNP(R) = Max(R)$. With the help of [12, Proposition 42.8], it was already shown in the proof of [21, Example 2.20(3)] that $\mathfrak{p}$ is not principal and hence, it follows from the arguments given in the paragraph which appears just preceding the statement of Example 3.23 that $\mathfrak{p} \notin \mathbb{A}(R)$. Hence, we obtain from Lemma 2.1 that $\mathfrak{p} \in \mathbb{E}(R)$. □

Let R be a ring such that $|MNP(R)| \geq 2$. In Proposition 3.24, we verify that $diam(\mathbb{G}(R)) = 2$ if R is not reduced and $\mathfrak{p} \notin \mathbb{E}(R)$ for some $\mathfrak{p} \in MNP(R)$.

Proposition 3.24. *Let R be a ring such that $|MNP(R)| \geq 2$. If R is not reduced and $\mathfrak{p} \notin \mathbb{E}(R)$ for some $\mathfrak{p} \in MNP(R)$, then $diam(\mathbb{G}(R)) = 2$.*

Proof. By assumption, $\mathfrak{p} \notin \mathbb{E}(R)$ for some $\mathfrak{p} \in MNP(R)$. Hence, it follows from Lemma 2.1 that $\mathfrak{p} \in \mathbb{A}(R)$. As $|MNP(R)| \geq 2$ by assumption, it is possible to find $\mathfrak{p}' \in MNP(R)\setminus\{\mathfrak{p}\}$. Since R is not reduced by assumption, we get that $\mathfrak{p} \cap \mathfrak{p}' \neq (0)$. From $\mathfrak{p} \cap \mathfrak{p}' \subset \mathfrak{p}$, it follows that $\mathfrak{p}$ is not a minimal ideal of R. Thus $\mathfrak{p} \in \mathbb{A}(R)$ is such that $\mathfrak{p}$ is not a minimal ideal of R and $\mathfrak{p} \notin \mathbb{E}(R)$. Therefore, we obtain from $(2) \Rightarrow (1)$ of Proposition 3.15 that $diam(\mathbb{G}(R)) = 2$. □

We provide Example 3.25 to illustrate Proposition 3.24.

Example 3.25. Let $T = K[X, Y]$ be the polynomial ring in two variables X, Y over a field K. Let $I = TX^2Y$ and let $R = \frac{T}{I}$. Then $diam(\mathbb{G}(R)) = 2$.

Proof. It is clear that $I = TX^2 \cap TY$ is an irredundant primary decomposition of I in T with TX^2 is a TX-primary ideal of T and $TY \in Spec(T)$. It is convenient to denote $X + I$ by x and $Y + I$ by y. Observe that $(0 + I) = Rx^2 \cap Ry$ is an irredundant primary decomposition of the zero ideal in R with Rx^2 is a Rx-primary ideal of R and $Ry \in Spec(R)$. Hence, we obtain from [6, Proposition 4.7] that $Z(R) = Rx \cup Ry$. Let us denote Rx by $\mathfrak{p}_1$ and Ry by $\mathfrak{p}_2$. As $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are not comparable under the inclusion relation, it follows from $Z(R) = \bigcup_{i=1}^{2} \mathfrak{p}_i$ that $MNP(R) = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$. Thus $|MNP(R)| = 2$. Note that $Rx^2 \in \mathbb{I}(R)^*$ and from $(0 + I) = Rx^2 \cap Ry$, it follows that $\mathfrak{p}_2 = Ry \notin \mathbb{E}(R)$. As $xy \neq 0 + I$ but $x^2y^2 = 0 + I$, we get that R is not reduced. It now follows from Proposition 3.24 that $diam(\mathbb{G}(R)) = 2$. □

Let R be a ring which is not reduced. If $|MNP(R) \cap \mathbb{A}(R)| \geq 2$ and if at least one member of $MNP(R) \cap \mathbb{A}(R)$ is not in $Max(R)$, then we verify in Proposition 3.26 that $diam(\mathbb{G}(R)) = 2$.

Proposition 3.26. *Let R be a ring such that R is not reduced. If $|MNP(R) \cap \mathbb{A}(R)| \geq 2$ and if at least one member of $MNP(R) \cap \mathbb{A}(R)$ is not in $Max(R)$, then $diam(\mathbb{G}(R)) = 2$.*

Proof. Let $\{\mathfrak{p}_i \mid i \in \{1, 2\}\} \subseteq MNP(R) \cap \mathbb{A}(R)$ be such that $\mathfrak{p}_i \notin Max(R)$ for some $i \in \{1, 2\}$. Let $i \in \{1, 2\}$. As $\mathfrak{p}_i \in \mathbb{A}(R)$, there exists $u_i \in R\backslash\{0\}$ such that $\mathfrak{p}_i = ((0) :_R u_i)$. It is clear that $Ru_i \in \mathbb{A}(R)^*$ for each $i \in \{1, 2\}$. If $\sum_{i=1}^{2} \mathfrak{p}_i \subseteq Z(R)$, then it follows from Zorn's lemma and [16, Theorem 1] that there exists $\mathfrak{p} \in MNP(R)$ such that $\sum_{i=1}^{2} \mathfrak{p}_i \subseteq \mathfrak{p}$. This implies that $\mathfrak{p}_i = \mathfrak{p}$ for each $i \in \{1, 2\}$. This is in contradiction to the assumption that $\mathfrak{p}_1 \neq \mathfrak{p}_2$. Therefore, $\sum_{i=1}^{2} \mathfrak{p}_i \not\subseteq Z(R)$. Hence, there exists $a_i \in \mathfrak{p}_i$ for each $i \in \{1, 2\}$ such that $\sum_{i=1}^{2} a_i \notin Z(R)$. We claim that $Ru_1 \cap Ru_2 = (0)$. Let $x \in Ru_1 \cap Ru_2$. Then $x = ru_1 = su_2$ for some $r, s \in R$. From $a_iu_i = 0$ for each $i \in \{1, 2\}$, we get that $(\sum_{i=1}^{2} a_i)x = 0$. As $\sum_{i=1}^{2} a_i \notin Z(R)$, it follows that $x = 0$ and so, $Ru_1 \cap Ru_2 = (0)$. Hence, $Ru_i \notin \mathbb{E}(R)$ for each $i \in \{1, 2\}$. By assumption, $\mathfrak{p}_i \notin Max(R)$ for at least one $i \in \{1, 2\}$. Without loss of generality, we can assume that $\mathfrak{p}_1 \notin Max(R)$. Note that the mapping $f : R \to Ru_1$ given by $f(r) = ru_1$ is an onto homomorphism of R-modules with $ker(f) = \mathfrak{p}_1$. Hence, it follows from the fundamental theorem of homomorphism of modules that $\frac{R}{\mathfrak{p}_1} \cong Ru_1$ as R-modules. As $\mathfrak{p}_1 \notin Max(R)$, we obtain that Ru_1 is not a minimal ideal of R. Thus $Ru_1 \in \mathbb{A}(R)^*$ is not a minimal ideal of R and $Ru_1 \notin \mathbb{E}(R)$. Since R is not reduced by hypothesis, we obtain from $(2) \Rightarrow (1)$ of Proposition 3.15 that $diam(\mathbb{G}(R)) = 2$. □

We provide Example 3.27 to illustrate Proposition 3.26.

Example 3.27. Let $T = K[X_1, X_2]$ be the polynomial ring in two variables X_1, X_2 over a field K. Let $I = T(\prod_{i=1}^{2} X_i^2)$. Let $R = \frac{T}{I}$. Then R is not reduced, $|MNP(R)| = 2$, each member of $MNP(R)$ belongs to $(\mathbb{A}(R) \cap \mathbb{E}(R))\backslash Max(R)$, and $diam(\mathbb{G}(R)) = 2$.

Proof. Note that $I = \bigcap_{i=1}^{2} TX_i^2$ is an irredundant primary decomposition of I in T with TX_i^2 is a TX_i-primary ideal of T for each $i \in \{1, 2\}$. It is convenient to denote $X_i + I$ by x_i for each $i \in \{1, 2\}$. Note that $x_1x_2 \neq 0 + I$ but $x_1^2x_2^2 = 0 + I$. Therefore, R is not reduced. Observe that $(0 + I) = \bigcap_{i=1}^{2} Rx_i^2$ is an irredundant primary decomposition of the zero ideal in R with Rx_i^2 is a Rx_i-primary ideal of R for each $i \in \{1, 2\}$. Let us denote Rx_i by $\mathfrak{p}_i$ for each $i \in \{1, 2\}$. It follows from [6, Proposition 4.7] that $Z(R) = \bigcup_{i=1}^{2} \mathfrak{p}_i$. Since $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are not comparable under the inclusion relation, it follows from $Z(R) = \bigcup_{i=1}^{2} \mathfrak{p}_i$ that $MNP(R) = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$. Observe that $\mathfrak{p}_1 = ((0 + I) :_R x_1x_2^2)$ and $\mathfrak{p}_2 = ((0 + I) :_R x_1^2x_2)$. Hence, $\mathfrak{p}_i \in \mathbb{A}(R)$ for each $i \in \{1, 2\}$. Let $i \in \{1, 2\}$. As $Rx_1 + Rx_2 \neq R$ and $Rx_i \subset Rx_1 + Rx_2$, we get that $\mathfrak{p}_i \notin Max(R)$. Since R is not reduced, it follows from Proposition 3.26 that $diam(\mathbb{G}(R)) = 2$. We next verify that $\mathfrak{p}_i \in \mathbb{E}(R)$ for each $i \in \{1, 2\}$. Let $A \in \mathbb{I}(R)^*$. If $A \subseteq \mathfrak{p}_i$, then $\mathfrak{p}_i \cap A = A \neq (0 + I)$. Suppose that $A \not\subseteq \mathfrak{p}_i$. As Rx_i^2 is a Rx_i-primary ideal of R with $Rx_i \neq Rx_i^2$, it follows that $\mathfrak{p}_iA \not\subseteq Rx_i^2$ and so, $\mathfrak{p}_iA \neq (0 + I)$. Hence, $\mathfrak{p}_i \cap A \neq (0 + I)$. This shows that $\mathfrak{p}_i \in \mathbb{E}(R)$ for each $i \in \{1, 2\}$. □

Let R be a ring such that R is not reduced. If $|MNP(R) \cap \mathbb{A}(R)| \geq 3$, then we verify in Proposition 3.28 that $diam(\mathbb{G}(R)) = 2$.

Proposition 3.28. *Let R be ring such that R is not reduced. If $|MNP(R) \cap \mathbb{A}(R)| \geq 3$, then $diam(\mathbb{G}(R)) = 2$.*

Proof. Let $\{\mathfrak{p}_i \mid i \in \{1, 2, 3\}\} \subseteq MNP(R) \cap \mathbb{A}(R)$. Note that for each $i \in \{1, 2, 3\}$, there exists $u_i \in R\backslash\{0\}$ such that $\mathfrak{p}_i = ((0) :_R u_i)$. It follows from [8, Lemma 3.6] that $u_1u_2 = u_2u_3 = u_1u_3 = 0$. Since distinct members of $MNP(R)$ are not comparable under the inclusion relation, it follows that Ru_i and Ru_j are not comparable under the inclusion relation for all distinct $i, j \in \{1, 2, 3\}$. Let $I = Ru_1 + Ru_2$. As $Iu_3 = (0)$, we get that $I \in \mathbb{A}(R)^*$. Let $J = Ru_1$. Note that $J \neq (0)$ and $J \subset I$. Therefore, I is not a minimal ideal of R. We claim that $Ru_3 \cap I = (0)$. Let $a \in Ru_3 \cap I$. Then $a = ru_3 = su_1 + tu_2$ for some $r, s, t \in R$. It follows from [6, Proposition 1.11(ii)] that $\mathfrak{p}_3 \not\supseteq \bigcap_{k=1}^{2} \mathfrak{p}_k$. Let $b \in (\bigcap_{k=1}^{2} \mathfrak{p}_k)\backslash\mathfrak{p}_3$. Then $bu_1 = bu_2 = 0$ and so, from $a = ru_3 = su_1 + tu_2$, we get that $bru_3 = 0$. Hence, $br \in \mathfrak{p}_3$. From $b \notin \mathfrak{p}_3$, it follows that $r \in \mathfrak{p}_3$ and so, $a = ru_3 = 0$. This shows that $Ru_3 \cap I = (0)$. Therefore, $I \notin \mathbb{E}(R)$. Thus the ideal $I \in \mathbb{A}(R)^*$ is such that I is not a minimal ideal of R and $I \notin \mathbb{E}(R)$. As R is not reduced by hypothesis, we obtain from $(2) \Rightarrow (1)$ of Proposition 3.15 that $diam(\mathbb{G}(R)) = 2$. □

Remark 3.29. Let R be a ring such that (0) admits a strong primary decomposition (for example, we can take R to be any strongly Laskerian ring, in particular R to be any Noetherian ring). Let $(0) = \bigcap_{i=1}^{n} \mathfrak{q}_i$ be an irredundant strong primary decomposition of (0) in R with $n \geq 3$, where $\mathfrak{q}_i$ is a strongly primary ideal of R for each $i \in \{1, 2, 3, \ldots, n\}$. Let $\sqrt{\mathfrak{q}_i} = \mathfrak{p}_i$ for each $i \in \{1, 2, 3, \ldots .n\}$. It can be shown as in the proof of [6, Proposition 7.14] that for each $i \in \{1, 2, 3, \ldots, n\}$ that there exists $r_i \in R\backslash\{0\}$ such that $\mathfrak{p}_i = ((0) :_R r_i)$. Thus $\mathfrak{p}_i \in \mathbb{A}(R)^*$ for each $i \in \{1, 2, 3, \ldots, n\}$. We know from [6, Proposition 4.7] that $Z(R) = \bigcup_{i=1}^{n} \mathfrak{p}_i$. Let us denote the set $\{\mathfrak{p}_i \mid i \in \{1, 2, 3, \ldots, n\}\}$ by $\sum$. It follows from [6, Proposition 1.11(i)]that $MNP(R) = \{\mathfrak{p}_j \mid \mathfrak{p}_j$ is a maximal member of $\sum\}$. If $|MNP(R)| = k \geq 3$ and if R is not reduced, then it follows from Proposition 3.28 that $diam(\mathbb{G}(R)) = 2$.

We provide Example 3.30 to illustrate Proposition 3.28.

Example 3.30. Let $T = \mathbb{Q}[X_1, X_2, X_3]$ be the polynomial ring in three variables X_1, X_2, X_3 over the field $\mathbb{Q}$. Let $i \in \{1, 2, 3\}$ and let $\mathfrak{Q}_i = T(X_1 - i) + TX_2^2 + TX_3$. Let $I = \bigcap_{i=1}^{3} \mathfrak{Q}_i$. Let $R = \frac{T}{I}$. Then R is not reduced, $|MNP(R)| = 3$, each member of $MNP(R)$ belongs to $\mathbb{A}(R) \cap \mathbb{E}(R) \cap Max(R)$, and $diam(\mathbb{G}(R)) = 2$.

Proof. Let $i \in \{1, 2, 3\}$. Note that $\mathfrak{M}_i = T(X_1 - i) + TX_2 + TX_3 \in Max(T)$. Since $\sqrt{\mathfrak{Q}_i} = \mathfrak{M}_i \in Max(T)$, we obtain from [6, Proposition 4.2] that $\mathfrak{Q}_i$ is a $\mathfrak{M}_i$-primary ideal of T and it is clear that $\mathfrak{M}_i^2 \subseteq \mathfrak{Q}_i$. Thus $I = \bigcap_{i=1}^{3} \mathfrak{Q}_i$ is an irredundant strong primary decomposition of I in T. It is convenient to denote $X_i + I$ by x_i for each $i \in \{1, 2, 3\}$. As $x_2 \neq 0 + I$ but $x_2^2 = 0 + I$, we get that R is not reduced. Since $\mathbb{Q} \cap I = (0)$, for any $\alpha \in \mathbb{Q}$, we identify $\alpha + I$ with α. For each $i \in \{1, 2, 3\}$, let us denote $R(x_1 - i) + Rx_2 + Rx_3$ by $\mathfrak{m}_i$ and $R(x_1 - i) + Rx_3$ by $\mathfrak{q}_i$. It is clear that $\mathfrak{m}_i \in Max(R)$ and $\mathfrak{q}_i$ is a $\mathfrak{m}_i$-primary ideal of R and $\mathfrak{m}_i^2 \subseteq \mathfrak{q}_i$. Note that $(0 + I) = \bigcap_{i=1}^{3} \mathfrak{q}_i$ is an irredundant strong primary decomposition of the zero ideal in R. As distinct maximal ideals of a ring are not comparable under the inclusion relation, it follows from the arguments given in Remark 3.29 that $MNP(R) = \{\mathfrak{m}_i \mid i \in \{1, 2, 3\}\}$. It now follows from Remark 3.29 that $diam(\mathbb{G}(R)) = 2$. Thus $|MNP(R)| = 3$ and each member of $MNP(R)$ is a member of $\mathbb{A}(R) \cap Max(R)$. Let $i \in \{1, 2, 3\}$. We verify that $\mathfrak{m}_i \in \mathbb{E}(R)$. Let $A \in \mathbb{I}(R)^*$. If $A \subseteq \mathfrak{m}_i$, then $\mathfrak{m}_i \cap A = A \neq (0 + I)$. Suppose that $A \nsubseteq \mathfrak{m}_i$. As $x_2 \in \mathfrak{m}_i \backslash \mathfrak{q}_i$, it follows that $\mathfrak{m}_i \neq \mathfrak{q}_i$ and so, $\mathfrak{m}_i \nsubseteq \mathfrak{q}_i$. Since $\mathfrak{q}_i$ is a $\mathfrak{m}_i$-primary ideal of R, we get that $A\mathfrak{m}_i \nsubseteq \mathfrak{q}_i$ and so, $A\mathfrak{m}_i \neq (0 + I)$. As $A \nsubseteq \mathfrak{m}_i$ by assumption, it follows that $A + \mathfrak{m}_i = R$ and so, $\mathfrak{m}_i \cap A = \mathfrak{m}_i A \neq (0 + I)$. This shows that $\mathfrak{m}_i \in \mathbb{E}(R)$ for each $i \in \{1, 2, 3\}$. □

Let $G = (V, E)$ be a graph. Let $e \in E$. Then $G - e$ is the subgraph of G obtained by deleting e from G. It is useful to mention here that $V(G - e) = V$ and $E(G - e) = E\backslash\{e\}$. Theorem 3.31 is motivated by [2, Theorem 2.7]. Let R be a ring. In Theorem 3.31, necessary and sufficient conditions are determined in order that $\mathbb{G}(R)$ to be isomorphic to $K_4 - e$, where e is an edge of K_4. For any ring T, we denote the Jacobson radical of T by $J(T)$.

Theorem 3.31. *Let R be a ring. The following statements are equivalent:*
(1) $\mathbb{G}(R) \cong K_4 - e$*, where e is an edge of K_4.*
(2) $\mathcal{E}_R \cong K_4 - e$*, where e is an edge of K_4.*
(3) $R \cong F \times S$ *as rings, where F is a field and $(S, \mathfrak{m})$ is an SPIR with* $\mathfrak{m} \neq (0)$ *but* $\mathfrak{m}^2 = (0)$.
(4) $\mathcal{AE}_R \cong K_4 - e$*, where e is an edge of K_4.*

Proof. (1) $\Rightarrow$ (2) We are assuming that $\mathbb{G}(R) \cong K_4 - e$, where e is an edge of K_4. This implies that $|V(\mathbb{G}(R))| = 4$. As $V(\mathbb{G}(R)) = \mathbb{A}(R)^*$, we get that $|\mathbb{A}(R)^*| = 4$. Therefore, R satisfies d.c.c. on annihilating ideals of R. Hence, we obtain from [9, Theorem 1.1] that R is Artinian. In such a case, it is already noted in Section 1 that $\mathbb{I}(R)^* = \mathbb{A}(R)^*$ and $\mathcal{E}_R = \mathbb{G}(R)$. Therefore, $\mathcal{E}_R \cong K_4 - e$, where e is an edge of K_4

(2) $\Rightarrow$ (3) We are assuming that $\mathcal{E}_R \cong K_4 - e$, where e is an edge of K_4. As $V(\mathcal{E}_R) = \mathbb{I}(R)^*$, it follows that $|\mathbb{I}(R)^*| = 4$. Hence, R is Artinian and so, $\mathbb{I}(R)^* = \mathbb{A}(R)^*$ and therefore, $\mathcal{E}_R = \mathbb{G}(R)$. We claim that $|Max(R)| = 2$. If R has more than two maximal ideals, then $|\mathbb{I}(R)^*| \geq 6$. This is impossible. Hence, $|Max(R)| \leq 2$. Suppose that R is local. Let $Max(R) = \{\mathfrak{m}\}$. If $\mathfrak{m}$ is principal, then as $\mathfrak{m}$ is nilpotent, it follows as is remarked in Section 1 that $(R, \mathfrak{m})$ is an SPIR. In such a case, we know from Proposition 3.14 that $\mathbb{G}(R)$ is complete. This is a contradiction, since $\mathbb{G}(R)$ is not complete. Hence, $\mathfrak{m}$ cannot be principal. It is clear that $\mathfrak{m}$ is finitely generated. Therefore, it follows from [6, Proposition 2.8] that there exist $x, y \in \mathfrak{m}$ such that $x + \mathfrak{m}^2, y + \mathfrak{m}^2$ are linearly independent over $\frac{R}{\mathfrak{m}}$. As $Rx, Ry, R(x + y), Rx + Ry$ are distinct members of $\mathbb{I}(R)^*$ and $|\mathbb{I}(R)^*| = 4$, we get that $\mathbb{I}(R)^* = \{Rx, Ry, R(x + y), Rx + Ry\}$. As $\mathfrak{m}^2 \in \mathbb{I}(R)$ and $\mathfrak{m}^2 \notin \{Rx, Ry, R(x + y), Rx + Ry\}$, it follows that $\mathfrak{m}^2 = (0)$. Moreover, $Soc(R) = \mathfrak{m}$ and $dim_{\frac{R}{\mathfrak{m}}}(\mathfrak{m}) = 2$. In such a case, we know from (2) $\Rightarrow$ (1) of Proposition 3.13 that $\mathbb{G}(R)$ is complete. This is a contradiction. Therefore, $|Max(R)| = 2$. Let $Max(R) = \{\mathfrak{m}_i \mid i \in \{1, 2\}\}$. As $|\mathbb{I}(F_1 \times F_2)^*| = 2$, where F_i is a field for each $i \in \{1, 2\}$ and $|\mathbb{I}(R)^*| = 4$, we obtain that R cannot be isomorphic to the direct product of two fields. If $\bigcap_{i=1}^{2} \mathfrak{m}_i = (0)$, then from $\mathfrak{m}_1 + \mathfrak{m}_2 = R$, it follows from [6, Proposition 1.10(ii) and (iii)] that $R \cong \frac{R}{\mathfrak{m}_1} \times \frac{R}{\mathfrak{m}_2}$ as rings. This is impossible, since $\frac{R}{\mathfrak{m}_i}$ is a field for each $i \in \{1, 2\}$. Hence, $\bigcap_{i=1}^{2} \mathfrak{m}_i \neq (0)$. If $\mathfrak{m}_i^2 \neq \mathfrak{m}_i$ for each $i \in \{1, 2\}$, then $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_2, \mathfrak{m}_2^2, \mathfrak{m}_1 \cap \mathfrak{m}_2$ are distinct members of $\mathbb{I}(R)^*$. This implies that $|\mathbb{I}(R)^*| \geq 5$. This is impossible. Therefore, $\mathfrak{m}_i = \mathfrak{m}_i^2$ for some $i \in \{1, 2\}$. Without loss of generality, we can assume that $\mathfrak{m}_1 = \mathfrak{m}_1^2$. Observe that $\mathfrak{m}_2 \neq \mathfrak{m}_2^2$. For if $\mathfrak{m}_2 = \mathfrak{m}_2^2$, then we obtain that $J(R) = \mathfrak{m}_1 \cap \mathfrak{m}_2 = \mathfrak{m}_1\mathfrak{m}_2 = \mathfrak{m}_1^2\mathfrak{m}_2^2 = (J(R))^2$. This

implies by Nakayama's lemma [6, Proposition 2.6] that $J(R) = (0)$. This is impossible. Therefore, $\mathfrak{m}_2 \neq \mathfrak{m}_2^2$. Since $|\mathbb{I}(R)^*| = 4$, we get that $\mathbb{I}(R)^* = \{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_2^2, \mathfrak{m}_1 \cap \mathfrak{m}_2\}$. Hence, $(J(R))^2 = (0)$. Therefore, $\mathfrak{m}_1 \cap \mathfrak{m}_2^2 = \mathfrak{m}_1^2 \cap \mathfrak{m}_2^2 = (0)$. As $\mathfrak{m}_1 + \mathfrak{m}_2^2 = R$, we obtain from [6, Proposition 1.10(ii) and (iii)] that the mapping $f : R \to \frac{R}{\mathfrak{m}_1} \times \frac{R}{\mathfrak{m}_2^2}$ defined by $f(r) = (r + \mathfrak{m}_1, r + \mathfrak{m}_2^2)$ is an isomorphism of rings. Let us denote $\frac{R}{\mathfrak{m}_1}$ by F and $\frac{R}{\mathfrak{m}_2^2}$ by S. Note that for any $y \in \mathfrak{m}_2 \backslash (\mathfrak{m}_1 \cup \mathfrak{m}_2^2)$, $\mathfrak{m}_2 = Ry$. Let us denote $\frac{\mathfrak{m}_2}{\mathfrak{m}_2^2}$ by $\mathfrak{m}$. It is clear that F is a field and $(S, \mathfrak{m})$ is an SPIR with $\mathfrak{m} \neq (0 + \mathfrak{m}_2^2)$ but $\mathfrak{m}^2 = (0 + \mathfrak{m}_2^2)$ and $R \cong F \times S$ as rings.

$(3) \Rightarrow (1)$ Let us denote the ring $F \times S$ by T, where F is a field and $(S, \mathfrak{m})$ is an SPIR with $\mathfrak{m} \neq (0)$ but $\mathfrak{m}^2 = (0)$. Observe that $\mathbb{I}(T)^* = \mathbb{A}(T)^* = \{F \times (0), (0) \times \mathfrak{m}, F \times \mathfrak{m}, (0) \times S\}$. Note that $\mathbb{G}(T)$ is the union of the cycle $F \times (0) - (0) \times \mathfrak{m} - F \times \mathfrak{m} - (0) \times S - F \times (0)$ and the edge $F \times (0) - F \times \mathfrak{m}$. Hence, $\mathbb{G}(T) \cong K_4 - e$, where e is an edge of K_4. Since $R \cong T$ as rings, we obtain that $\mathbb{G}(R) \cong K_4 - e$, where e is an edge of K_4.

$(3) \Rightarrow (4)$ We are assuming that $R \cong F \times S$ as rings, where F is a field and $(S, \mathfrak{m})$ is an SPIR with $\mathfrak{m} \neq (0)$ but $\mathfrak{m}^2 = (0)$. Now, it follows from Corollary 2.10 and $(3) \Rightarrow (1)$ of this theorem that $\mathcal{AE}_R \cong K_4 - e$, where e is an edge of K_4.

$(4) \Rightarrow (3)$ We are assuming that $\mathcal{AE}_R \cong K_4 - e$, where e is an edge of K_4. As $V(\mathcal{AE}_R) = \mathbb{A}(R)^*$, it follows that $|\mathbb{A}(R)^*| = 4$. It follows as in the proof of $(1) \Rightarrow (2)$ this theorem that R is Artinian. It follows as in the proof of $(2) \Rightarrow (3)$ of this theorem that $|Max(R)| \leq 2$. If R is local, then we know from [1, Lemma 2] that $\mathcal{AE}_R$ is complete. This is impossible, since $\mathcal{AE}_R$ is not complete. Therefore, $|Max(R)| = 2$. Since $\mathbb{I}(R)^* = \mathbb{A}(R)^*$ and $|\mathbb{I}(R)^*| = 4$, proceeding as in the proof of $(2) \Rightarrow (3)$ of this theorem, it can be shown that $R \cong F \times S$ as rings, where F is a field and $(S, \mathfrak{m})$ is an SPIR with $\mathfrak{m} \neq (0)$ but $\mathfrak{m}^2 = (0)$. □

Remark 3.32. Let $R = F \times S$, where F is a field and $(S, \mathfrak{m})$ is a SPIR with $\mathfrak{m} \neq (0)$ but $\mathfrak{m}^2 = (0)$. We know from Corollary 2.10 that $\mathbb{G}(R) \cong \mathcal{AE}_R$. We verify in this remark that $\mathbb{G}(R) \neq \mathcal{AE}_R$. Let us denote the ideal $F \times (0)$ of R by I and the ideal $F \times \mathfrak{m}$ of R by J. Observe that $I + J = F \times \mathfrak{m} \in \mathbb{E}(R)$. Hence, I and J are adjacent in $\mathbb{G}(R)$. However, $Ann_R(I) = (0) \times S$ and $Ann_R(J) = (0) \times \mathfrak{m}$ and so, $Ann_R(I) + Ann_R(J) = (0) \times S \notin \mathbb{E}(R)$. Hence, I and J are not adjacent in $\mathcal{AE}_R$. This shows that $\mathbb{G}(R) \neq \mathcal{AE}_R$.

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Author information

S. Visweswaran, Department of Mathematics, Saurashtra University, Rajkot, 360005, India.
E-mail: `s_visweswaran2006@yahoo.co.in`

Patat Sarman, Government Polytechnic, Bilkha Road, To: Khadiya Junagadh, 362263, India.
E-mail: `patat.sarman03@gmail.com`

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