Palestine Journal of Mathematics

Vol. 12(2)(2023) , 116–124

On commutativity with derivations acting on prime ideals

L. Oukhtite, A. Mamouni, and H. El Mir

Communicated by Najib Mahdou

MSC 2010 Classifications: 16U80, 16W25, 16N60.

Keywords and phrases: Prime ring, commutativity, derivation.

Abstract The main purpose of the present paper is to study the commutativity of a quotient ring R/P by considering derivations of R satisfying algebraic identities involving the prime ideal P. This approach allows us to generalize some well known results characterizing commutativity of rings.

1 Introduction

Throughout this article, R will represent an associative ring with center $Z(R)$. Recall that an ideal P of R is said to be prime if $P \neq R$ and for all $x, y \in R$, $xRy \subseteq P$ implies that $x \in P$ or $y \in P$. Therefore, R is called a prime ring if and only if (0) is the only prime ideal of R. R is 2-torsion free if whenever $2x = 0$, with $x \in R$ implies $x = 0$. For any $x, y \in R$, the symbol $[x, y]$ will denote the commutator $xy - yx$; while the symbol $x \circ y$ will stand for the anti-commutator $xy + yx$. A map $d : R \longrightarrow R$ is a derivation of a ring R if d is additive and satisfies $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. An additive mapping $* : R \longrightarrow R$ is called an involution if $*$ is an anti-automorphism of order 2; that is $(x^*)^* = x$ for all $x \in R$. An element x in a ring with involution $(R, *)$ is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. The involution is said to be of the first kind if $*$ induces the identity map on the center $Z(R)$ of R, otherwise it is said to be of the second kind. In the latter case, it is worthwhile to see that $S(R) \cap Z(R) \neq \{0\}$.

Recently many authors have obtained commutativity of prime and semiprime rings admitting suitably constrained additive mappings, as automorphisms, derivations, skew derivations and generalized derivations acting on appropriate subsets of the rings. We first recall that for a subset S of R, a mapping $f : S \longrightarrow R$ is called centralizing if $[f(x), x] \in Z(R)$ for all $x \in S$, in the special case where $[f(x), x] = 0$ for all $x \in S$, the mapping f is said to be commuting on S. In [16], Posner proved that if a prime ring R admits a nonzero derivation d such that $[d(x), x] \in Z(R)$ for all $x \in R$, then R is commutative. Over the last few decades, several authors have subsequently refined and extended these result in various directions (see [1], [12], [13] and [14] where further references can be found).

Long ago Herstein [9] proved that if a prime ring R of characteristic different from two admits a derivation d such that $d(x)d(y) = d(y)d(x)$ holds for all $x, y \in R$, then R is commutative. Motivated by this result Bell and Daif [6] obtained the same result by considering the identity $d[x, y] = 0$ for all x, y in a nonzero ideal of R. Recall that a mapping $f : R \longrightarrow R$ preserves commutativity if $[f(x), f(y)] = 0$ whenever $[x, y] = 0$ for all $x, y \in R$. A mapping $f : R \longrightarrow R$ is said to be strong commutativity preserving (SCP) on a subset S of R if $[f(x), f(y)] = [x, y]$ for all $x, y \in S$. In [5], Bell and Daif investigated the commutativity of rings admitting a derivation that is SCP on a nonzero right ideal. Indeed, they proved that if a semiprime ring R admits a derivation d satisfying $[d(x), d(y)] = [x, y]$ for all x, y in a right ideal I of R, then $I \subseteq Z(R)$. In particular, R is commutative if $I = R$. Further, Ali and Huang [2] showed that if R is a 2-torsion free semiprime ring an d is a derivation of R satisfying $[d(x), d(y)] + [x, y] = 0$ for all x, y in a nonzero ideal I of R, then R contains a nonzero central ideal. Motivated by this results the authors in [11] introduce the notions of *-SCP, *-Skew SCP and give some commutativity criteria in case of prime ring with involution. Many related generalizations of these results can be found in the literature (see for instance [7] and [14]). In [8] Daif and Bell established commutativity of semiprime ring satisfying $d([x,y]) = [x,y]$ for all x, y in a nonzero ideal of R, and d a derivation of R. Further, in the year 1997 M. Hongan [10] proved that a 2-torsion free semiprime ring R must be commutative if it admits a derivation d satisfying $d([x,y]) + [x,y] \in Z(R)$ for all $x, y \in I$ or $d([x,y]) - [x,y] \in Z(R)$ for all $x, y \in I$, where I is an ideal of R. In [15] Oukhtite and all generalized these results for $*-$prime ring R satisfying any one of the properties: (i) $d[x,y] = 0$, (ii) $d([x,y]) - [x,y] \in Z(R)$, (iii) $d([x,y]) + [x,y] \in Z(R)$, (iv) $d(x \circ y) = 0$, (v) $d(x \circ y) - x \circ y \in Z(R)$, (vi) $d(x \circ y) + x \circ y \in Z(R)$ for all $x, y \in J$, where J is a nonzero Jordan ideal of R.

The present paper is motivated by the previous results and we here continue this line of investigation by considering a generalization to any ring rather then a prime ring. More precisely, we will establish a relationship between the structure of a quotient ring R/P and the behavior of derivations of R satisfying some identities involving the prime ideal P.

2 Main results

We will use frequently the following fact whose proof will be left to the reader. Indeed, this fact and the following lemmas are very crucial for developing the proofs of our main results.
Fact 1. Let R be a ring, I a non zero ideal of R, P a prime ideal such that $P \subsetneq I$ and $a, b \in R$. Then $aIb \subseteq P$ implies that $a \in P$ or $b \in P$.

In [[3], Theorem 2.2] it is proved that if P is a prime ideal of a ring R and d a derivation of R such that $\big[[x, d(x)], y\big] \in P$ for all $x, y \in R$, then $d(R) \subseteq P$ or R/P is a commutative ring. Using similar arguments with some modifications, we get the following lemma which plays a crucial role in developing the proofs of our main results.

Lemma 2.1. *Let R be a ring, I a nonzero ideal, P a prime ideal such that $P \subsetneq I$ and d a derivation of R. Then $[d(x), x] \in P$ for all $x \in I$ if and only if R/P is a commutative integral domain or $d(R) \subseteq P$.*

Lemma 2.2. *Let R be a ring, I a nonzero ideal, P a prime ideal such that $P \subsetneq I$ and R/P is 2-torsion free. If d is a derivation of R, then $\overline{[d(x), x]} \in Z(R/P)$ for all $x \in I$ if and only if R/P is a commutative integral domain or $d(R) \subseteq P$.*

Proof. We are given that

$$\overline{[d(x), x]} \in Z(R/P) \quad \text{for all } \; x \in I. \tag{2.1}$$

Writing x^2 instead of x we find that

$$\overline{x^2[d(x), x] + 2x[d(x), x]x + [d(x), x]x^2} \in Z(R/P) \quad \text{for all } \; x \in I. \tag{2.2}$$

By view of (2.1), equation (2.2) yields

$$4\overline{[d(x), x]x^2} \in Z(R/P) \quad \text{for all } \; x \in I. \tag{2.3}$$

Since R/P is 2-torsion free it follows that

$$\overline{[d(x), x]x^2} \in Z(R/P) \quad \text{for all } \; x \in I. \tag{2.4}$$

Commuting the last equation with $d(x)$ we arrive at

$$\overline{[d(x), x]^2 x} = \overline{0} \quad \text{for all } \; x \in I. \tag{2.5}$$

Using the hypothesis with equation (2.5) we get

$$\overline{[d(x), x]^2 d(x) x} = \overline{0} \quad \text{for all } \; x \in I. \tag{2.6}$$

Right multiplying (2.5) by $d(x)$ and combining with (2.6), we obtain

$$\overline{[d(x), x]^3} = \overline{0} \quad \text{for all } \; x \in I. \tag{2.7}$$

Since the center of a semiprime ring contains no nonzero nilpotent elements then $\overline{[d(x),x]} = \bar{0}$ for all $x \in I$. Hence by Lemma 2.1 we conclude that R/P is a commutative or $d(R) \subseteq P$. □

In [10], Hongan established commutativity of 2-torsion free semiprime ring R which admits a derivation d satisfying $d([x,y]) + [x,y] \in Z(R)$ for all $x, y \in I$ or $d([x,y]) - [x,y] \in Z(R)$ for all $x, y \in I$, where I is an ideal of R. Our next aim is to generalize this result in two directions. First of all, we will assume that the above algebraic identities belongs to $Z(R/P)$ where P is any prime ideal rather than the zero ideal. Secondly, we will treat more general differential identities involving two derivations.

Theorem 2.3. *Let R be a ring, I a nonzero ideal of R, P a prime ideal such that $P \subsetneq I$ and R/P is 2-torsion free. If d is a derivation of R then the following assertions are equivalent:*
(1) $\overline{d[x,y]} - \overline{[x,y]} \in Z(R/P)$ for all $x, y \in I$;
(2) R/P is a commutative integral domain.
Moreover, if $\overline{d[x,y]} \in Z(R/P)$ for all $x, y \in I$, then $d(R) \subseteq P$ or R/P is a commutative integral domain.

Proof. We are given that

$$\overline{d[x,y]} - \overline{[x,y]} \in Z(R/P) \quad \text{for all } \ x, y \in I. \tag{2.8}$$

Replacing y by $[y,t]$, we get

$$\overline{[d(x),[y,t]]} + \overline{[x,d[y,t]]} - \overline{[x,[y,t]]} \in Z(R/P) \quad \text{for all } \ t, x, y \in I. \tag{2.9}$$

Combining (2.8) with (2.9), we may conclude that

$$\overline{[d(x),[y,t]]} \in Z(R/P) \quad \text{for all } \ t, x, y \in I. \tag{2.10}$$

Substituting ty for y, one can see that

$$\overline{[d(x),t[y,t]]} \in Z(R/P) \quad \text{for all } \ t, x, y \in I. \tag{2.11}$$

Therefore commuting with $d(x)$ we find that

$$\overline{[t[d(x),[y,t]] + [d(x),t][y,t], d(x)]} = \bar{0} \quad \text{for all } \ t, x, y \in I \tag{2.12}$$

and thus

$$2\overline{[d(x),t][[y,t],d(x)]} + \overline{[[d(x),t],d(x)][y,t]} = \bar{0} \quad \text{for all } \ t, x, y \in I. \tag{2.13}$$

Replacing y by yt and using the 2-torsion freeness of R/P, one can verify that

$$\overline{[d(x),t]^3} = \bar{0} \quad \text{for all } \ t, x \in I \tag{2.14}$$

in such a way that

$$\overline{[d(x),[t,y]]^3} = \bar{0} \quad \text{for all } \ t, x, y \in I. \tag{2.15}$$

By view of (2.10) equation (2.15) yields

$$\overline{[d(x),[t,y]]} = \bar{0} \quad \text{for all } \ t, x, y \in I. \tag{2.16}$$

Substituting ty for y we get

$$\overline{[d(x),t][t,y]} = \bar{0} \quad \text{for all } \ t, x, y \in I. \tag{2.17}$$

Writing ry instead of y, we arrive at

$$\overline{[d(x),t]r[t,y]} = \bar{0} \quad \text{for all } \ r, t, x, y \in I, \tag{2.18}$$

thereby obtaining

$$\overline{[d(x),t]}(I/P)\overline{[d(x),t]} = \bar{0} \quad \text{for all } \ t, x \in I. \tag{2.19}$$

Since P is a semiprime ideal then $[d(x), t] \in P$ for all $t, x \in I$. Using Lemma 2.1 we are forced to conclude that R/P is commutative or $d(R) \subseteq P$ in which case the hypothesis assures that $\overline{[x, y]} \in Z(R/P)$ for all $x, y \in I$ and thus I/P is commutative. Hence R/P is a commutative integral domain.

Now, if $\overline{d[x, y]} \in Z(R/P)$ for all $x, y \in I$; replacing y by $[y, t]$ we obtain

$$\overline{[d(x), [y, t]]} + \overline{[x, d[y, t]]} \in Z(R/P) \quad \text{for all } \ t, x, y \in I \tag{2.20}$$

whence

$$\overline{[d(x), [y, t]]} \in Z(R/P) \quad \text{for all } \ t, x, y \in I. \tag{2.21}$$

That is just equation (2.10), so we may argue as before that $d(R) \subseteq P$ or R/P is a commutative integral domain. $\square$

Remark 2.4. Using the same technics as in the preceding proof, it is obvious to see that $\overline{d[x, y]} + \overline{[x, y]} \in Z(R/P)$ for all $x, y \in I$ implies that R/P is a commutative integral domain.

In ([10],Theorem 1) it is proved that if R is a 2-torsion free semiprime ring, I a nonzero ideal of R, and d a derivation of R. Then the following assertions are equivalent:
(1) $d[x, y] - [x, y] \in Z(R)$ for all $x, y \in I$;
(2) $d[x, y] + [x, y] \in Z(R)$ for all $x, y \in I$;
(3) $I \subseteq Z(R)$.

As an application of our Theorem 1, the following proposition gives an improved version of ([10],Theorem 1).

Corollary 2.5. *Let R be a 2-torsion free semiprime ring, I a nonzero ideal of R, and d a derivation of R. Then the following assertions are equivalent :*
(1) $d[x, y] - [x, y] \in Z(R)$ *for all* $x, y \in I$;
(2) $d[x, y] + [x, y] \in Z(R)$ *for all* $x, y \in I$;
(3) R *is commutative.*

Proof. Assume that $d[x, y] - [x, y] \in Z(R)$ for all $x, y \in I$. By view of the semiprimeness of the ring R, there exists a family Γ of prime ideals such that $\bigcap_{P\in\Gamma} P = \{0\}$, thereby obtaining $[d[x, y] - [x, y], r] \in P$ for all $P \in \Gamma$. Invoking Theorem 2.3 we conclude that R/P is a commutative integral domain for all $P \in \Gamma$ which, because of $\bigcap_{P\in\Gamma} P = \{0\}$, assures that R is commutative. Similarly, if $d[x, y] + [x, y] \in Z(R)$ for all $x, y \in I$, then the same reasoning proves that R is commutative. $\square$

As mentioned before, our present objective is to prove the following theorem which generalizes [[10], Theorem 1].

Theorem 2.6. *Let R be a ring, I a nonzero ideal of R, P a prime ideal such that $P \subsetneq I$ and R/P is 2-torsion free, d_1 and d_2 two derivations of R. The following assertions are equivalent:*
(1) $\overline{[d_1(x), y]} + \overline{[x, d_2(y)]} - \overline{[x, y]} \in Z(R/P)$ *for all* $x, y \in I$;
(2) R/P *is a commutative integral domain.*
Moreover, if $\overline{[d_1(x), y]} + \overline{[x, d_2(y)]} \in Z(R/P)$ for all $x, y \in I$, then R/P is a commutative integral domain or $\big(d_1(R) \subseteq P$ and $d_2(R) \subseteq P\big)$.

Proof. We are given that

$$\overline{[d_1(x), y]} + \overline{[x, d_2(y)]} - \overline{[x, y]} \in Z(R/P) \quad \text{for all } \ x, y \in I. \tag{2.22}$$

Substituting $[y, t]$ for y, we obtain

$$\overline{[d_1(x), [y, t]]} + \overline{[x, [d_2(y), t]]} + \overline{[x, [y, d_2(t)]]} - \overline{[x, [y, t]]} \in Z(R/P) \tag{2.23}$$

for all $t, x, y \in I$. Combining (2.22) with (2.23), we may write

$$\overline{[d_1(x), [y,t]]} + \overline{[x, [d_2(y), t]]} - \overline{[x, [d_1(y), t]]} \in Z(R/P) \quad \text{for all} \ \ t, x, y \in I. \tag{2.24}$$

Taking $y = t$ in the last equation one can see that

$$\overline{[x, [d(t), t]]} \in Z(R/P) \quad \text{for all} \ \ t, x \in I, \tag{2.25}$$

where $d = d_1$ - d_2, and thus for $x = x[d(t), t]$ we arrive at

$$\overline{[x, [d(t), t]][d(t), t]} \in Z(R/P) \quad \text{for all} \ \ t, x \in I. \tag{2.26}$$

Using equation (2.25), equation (2.26) yields

$$\overline{[d(t), t]} \in Z(R/P) \quad \text{for all} \ \ t \in I. \tag{2.27}$$

By view of Lemma 2.2 we conclude that R/P is commutative or $d(R) \subseteq P$ in which case the hypothesis forces

$$\overline{d_1[x,y]} - \overline{[x,y]} \in Z(R/P) \quad \text{for all} \ \ x, y \in I. \tag{2.28}$$

Using Theorem 2.3 we obtain the required result.

Assume that

$$\overline{[d_1(x), y]} + \overline{[x, d_2(y)]} \in Z(R/P) \quad \text{for all} \ \ x, y \in I. \tag{2.29}$$

Replacing y by $[y, t]$, we obtain

$$\overline{[d_1(x), [y,t]]} + \overline{[x, [d_2(y), t]]} + \overline{[x, [y, d_2(t)]]} \in Z(R/P) \quad \text{for all} \ \ t, x, y \in I. \tag{2.30}$$

Subtracting (2.29) from (2.30), we are forced to conclude that

$$\overline{[d_1(x), [y,t]]} + \overline{[x, [d_2(y), t]]} - \overline{[x, [d_1(y), t]]} \in Z(R/P) \quad \text{for all} \ \ t, x, y \in I. \tag{2.31}$$

That is just equation (2.24), so we may argue as before that R/P is commutative or $(d_1 - d_2)(R) \subseteq P$; using equation (2.29) one can see that

$$\overline{d_1[x,y]} \in Z(R/P) \quad \text{for all} \ \ x, y \in I. \tag{2.32}$$

By view of Theorem 2.3 we conclude that R/P is commutative or $d_1(R) \subseteq P$ in which case using the hypothesis we deduce that $d_2(R) \subseteq P$. $\square$

As a consequence of Theorem 2.6, if R is a prime ring, then

$$[d_1(x), y] + [x, d_2(y)] - [x, y] \in Z(R) \ \text{ for all } \ x, y \in I$$

assures that R is a commutative integral domain. The following corollary proves that the similar conclusion remains valid on semiprime rings.

Corollary 2.7. *Let R be a 2-torsion free semiprime ring, I a nonzero ideal, d_1 and d_2 two derivations of R. Then the following assertions are equivalent:*
(1) $[d_1(x), y] + [x, d_2(y)] - [x, y] \in Z(R)$ *for all* $x, y \in I$;
(2) *R is commutative.*

Proof. Assume that $[d_1(x), y] + [x, d_2(y)] - [x, y] \in Z(R)$ for all $x, y \in I$. By view of the semiprimeness of the ring R, there exists a family Γ of prime ideals such that $\bigcap_{P \in \Gamma} P = \{0\}$, thereby obtaining $[[d_1(x), y] + [x, d_2(y)] - [x, y], r] \in P$ for all $P \in \Gamma$. Invoking Theorem 2.6, we conclude that R/P is a commutative integral domain for all $P \in \Gamma$ which, because of $\bigcap_{P \in \Gamma} P = \{0\}$, assures that R is commutative.$\square$

In [4] Ashraf and Rehman proved that, if R is a prime ring, I is a nonzero ideal of R and d is a derivation of R such that $d(x \circ y) - (x \circ y) = 0$ for all $x, y \in I$, then R is commutative.
The fundamental aim of the next theorem is to establish a generalization of the above result by investigating the behaviour of the more general expression $\overline{d_1(x) \circ y + x \circ d_2(y) - x \circ y} \in Z(R/P)$ for all $x, y \in I$. More precisely we will prove the following result.

Theorem 2.8. *Let R be a ring, I a nonzero ideal of R, P a prime ideal such that $P \subsetneq I$ and R/P is 2-torsion free, d_1 and d_2 two derivations of R. Then the following assertions are equivalent:*
(1) $\overline{d_1(x) \circ y + x \circ d_2(y) - x \circ y} \in Z(R/P)$ *for all* $x, y \in I$;
(2) R/P *is a commutative integral domain.*
Moreover, if $\overline{d_1(x) \circ y + x \circ d_2(y)} \in Z(R/P)$ *for all* $x, y \in I$, *then* R/P *is a commutative integral domain or* $\big(d_1(R) \subseteq P$ *and* $d_2(R) \subseteq P\big)$.

Proof. We are given that

$$\overline{d_1(x) \circ y + x \circ d_2(y) - x \circ y} \in Z(R/P) \quad \text{for all} \ \ x, y \in I. \tag{2.33}$$

If $Z(R/P) = \{\overline{0}\}$, then the hypothesis reduces to

$$\overline{d_1(x) \circ y + x \circ d_2(y) - x \circ y} = \overline{0} \quad \text{for all} \ \ x, y \in I. \tag{2.34}$$

Replacing y by yx, we obtain

$$\overline{d_1(x) \circ (yx) + x \circ (d_2(y)x) + x \circ (yd_2(x)) - x \circ (yx)} = \overline{0} \quad \text{for all} \ \ x, y \in I. \tag{2.35}$$

Therefore

$$\overline{(d_1(x) \circ y)x - y[d_1(x), x] + (x \circ d_2(y))x + x \circ (yd_2(x)) - (x \circ y)x} = \overline{0} \tag{2.36}$$

for all $x, y \in I$. Combining (2.34) together with (2.36) one can see that

$$\overline{-y[d_1(x), x] + (x \circ y)d_2(x) + y[d_2(x), x]} = \overline{0} \quad \text{for all} \ \ x, y \in I. \tag{2.37}$$

Writing ty instead of y, we obviously get

$$\overline{[x, t]yd_2(x)} = \overline{0} \quad \text{for all} \ \ t, x, y \in I \tag{2.38}$$

and thus

$$\overline{[x, t]}(I/P)\overline{d_2(x)} = \overline{0} \quad \text{for all} \ \ t, x \in I. \tag{2.39}$$

Using Brauer's trick we are forced to conclude that R/P is commutative which is a contradiction or $d_2(R) \subseteq P$ in which case the hypothesis becomes

$$d_1(x) \circ y - x \circ y \in P \quad \text{for all} \ \ x, y \in I. \tag{2.40}$$

Replacing y by yx, we obtain

$$(d_1(x) \circ y)x - y[d_1(x), x] - (x \circ y)x \in P \quad \text{for all} \ \ x, y \in I. \tag{2.41}$$

By view of equation (2.40) we find that

$$y[d_1(x), x] \in P \quad \text{for all} \ \ x, y \in I \tag{2.42}$$

which leads to

$$yR[d_1(x), x] \subseteq P \quad \text{for all} \ \ x, y \in I. \tag{2.43}$$

Since $P \subsetneq I$ it follows that $[d_1(x), x] \in P$ for all $x \in I$. Consequently $d_1(R) \subseteq P$ or R/P is commutative which is a contradiction. Hence the hypothesis reduces to $x \circ y \in P$ and thus for $y = yr$ we obtain $[x, y]r \in P$, so that $[x, y]R[x, y] \subseteq P$ for all $x, y \in I$ which leads to conclude that I/P is commutative. Therefore R/P is commutative, a contradiction.
Hence $Z(R/P) \neq \{\overline{0}\}$; substituting yz for y in (2.33), where $\overline{z} \in Z(R/P)$, one can verify that

$$\overline{\big(d_1(x) \circ y\big)z + \big(x \circ d_2(y)\big)z + (x \circ y)d_2(z) - y[x, d_2(z)] - (x \circ y)z} \in Z(R/P) \tag{2.44}$$

for all $x, y \in I$. In light of (2.33), one can see that

$$\overline{(x \circ y)d_2(z) - y[x, d_2(z)]} \in Z(R/P) \quad \text{for all} \ \ x, y \in I. \tag{2.45}$$

Writing z instead of x in the last equation, we arrive at

$$\overline{yd_2(z)z} \in Z(R/P) \quad \text{for all} \;\; y \in I. \tag{2.46}$$

Since $Z(R/P) \neq \{\overline{0}\}$, it follows that $\overline{yd_2(z)} \in Z(R/P)$; substituting ty instead of y one can conclude that $\overline{t} \in Z(R/P)$ and thus R/P is commutative or $\overline{yd_2(z)} = \overline{0}$ for all $y \in I$ in which case we obtain $\overline{d_2(z)} = \overline{0}$ because of $P \subsetneq I$. Taking $\overline{y} \in Z(R/P)$ in equation (2.33) we find that

$$2\overline{(d_1(x) - x)y} \in Z(R/P) \quad \text{for all} \;\; x, y \in I \tag{2.47}$$

thereby obtaining

$$\overline{d_1(x) - x} \in Z(R/P) \quad \text{for all} \;\; x \in I \tag{2.48}$$

and thus

$$\overline{[d_1(x), x]} = \overline{0} \quad \text{for all} \;\; x \in I. \tag{2.49}$$

Hence R/P is commutative or $d_1(R) \subseteq P$.
If $d_1(R) \subseteq P$, then equation (2.48) yields $\overline{x} \in Z(R/P)$ for all $x \in I$. Hence R/P is commutative.

Suppose that

$$\overline{d_1(x) \circ y + x \circ d_2(y)} \in Z(R/P) \quad \text{for all} \;\; x, y \in I. \tag{2.50}$$

If $Z(R/P) = \{\overline{0}\}$, then the hypothesis becomes

$$d_1(x) \circ y + x \circ d_2(y) \in P \quad \text{for all} \;\; x, y \in I. \tag{2.51}$$

Substituting yx for y, we arrive at

$$(d_1(x) \circ y)x - y[d_1(x), x] + (x \circ d_2(y))x + x \circ (yd_2(x)) \in P \quad \text{for all} \;\; x, y \in I. \tag{2.52}$$

Combining (2.52) together with (2.51) one can see that

$$-y[d_1(x), x] + (x \circ y)d_2(x) + y[d_2(x), x] \in P \quad \text{for all} \;\; x, y \in I. \tag{2.53}$$

That is just equation (2.37), so we may argue as before that $d_2(R) \subseteq P$ and the hypothesis forces

$$d_1(x) \circ y \in P \quad \text{for all} \;\; x, y \in I. \tag{2.54}$$

Replacing y by yx we find that

$$y[d_1(x), x] \in P \quad \text{for all} \;\; x, y \in I \tag{2.55}$$

which leads to

$$yR[d_1(x), x] \subseteq P \quad \text{for all} \;\; x, y \in I. \tag{2.56}$$

Hence $[d_1(x), x] \in P$ for all $x \in I$ and thus $d_1(R) \subseteq P$.
Now if $Z(R/P) \neq \{\overline{0}\}$; substituting yz for y in (2.50), where $\overline{z} \in Z(R/P) \setminus \{\overline{0}\}$, we arrive at

$$\overline{\big(d_1(x) \circ y\big)z + \big(x \circ d_2(y)\big)z + (x \circ y)d_2(z) - y[x, d_2(z)]} \in Z(R/P) \tag{2.57}$$

for all $x, y \in I$. By view of equation (2.50), it follows that

$$\overline{(x \circ y)d_2(z) - y[x, d_2(z)]} \in Z(R/P) \quad \text{for all} \;\; x, y \in I. \tag{2.58}$$

So we may argue as in equation (2.45) to get $\overline{d_2(z)} = \overline{0}$ in which case the hypothesis assures that for $\overline{y} \in Z(R/P) \setminus \{\overline{0}\}$ we have

$$2\overline{d_1(x)y} \in Z(R/P) \quad \text{for all} \;\; x, y \in I \tag{2.59}$$

in such a way that

$$\overline{d_1(x)} \in Z(R/P) \quad \text{for all} \;\; x \in I. \tag{2.60}$$

Commuting the last equation with $\overline{x}$ we obtain

$$\overline{[d_1(x), x]} \in Z(R/P) \quad \text{for all} \quad x \in I \tag{2.61}$$

hence R/P is commutative or $d_1(R) \subseteq P$. On the other hand, taking $\overline{x} \in Z(R/P)$ in the hypothesis we are forced to conclude that $2\overline{xd_2(y)} \in Z(R/P)$, so that by the same technics used in equation (2.59) one can prove that R/P is commutative or $d_2(R) \subseteq P$.□
As a consequence of Theorem 2.8, if R is a prime ring, then $d_1(x) \circ y + x \circ d_2(y) - x \circ y \in Z(R)$ for all $x, y \in I$, assures that R is commutative. The following corollary extends this result on semiprime rings.

Corollary 2.9. *Let R be a 2-torsion free semiprime ring, I a nonzero ideal, d_1 and d_2 two derivations of R. The following assertions are equivalent:*
(1) *$d_1(x) \circ y + x \circ d_2(y) - x \circ y \in Z(R)$ for all $x, y \in I$;*
(2) *R is commutative.*

Proof. Assume that R is a 2-torsion free semiprime ring. Then there exists a family Γ of prime ideals such that $\bigcap_{P\in\Gamma} P = \{0\}$. Suppose there exists two derivations d_1 and d_2 such that $[d_1(x)\circ y+x\circ d_2(y)-x\circ y, r] = 0$ for all $x, y \in I, r \in R$. Accordingly, $[d_1(x)\circ y+x\circ d_2(y)-x\circ y, r] \in P$ for all $P \in \Gamma$ and thus Theorem 2.6 yields R/P is commutative for all $P \in \Gamma$. Therefore, for all $x, y \in R$ we have $[x, y] \in P$ for all $P \in \Gamma$ so that $[x, y] = 0$ for all $x, y \in R$.□

The following example proves that the condition `R is 2-torsion free` is necessary in Theorems 2.3, 2.6 and 2.8.

Example 1.

Let us consider $R = M_2(\mathbb{Z}/2\mathbb{Z})$, it is straightforward to check that R is a noncommutative prime ring with $char R = 2$. Moreover, if we take $d : R \longrightarrow R$ the inner derivation defined on R by $d(X) = [A, X]$ where $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then for all $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $Y = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in R$, we find that

$$d[X, Y] - [X, Y] = \begin{pmatrix} b\gamma - \beta c & 0 \\ 0 & b\gamma - \beta c \end{pmatrix} \in Z(R).$$

Hence d satisfies the conditions of Theorems 2.3, 2.6 and 2.8 but R is not commutative.

The following example proves that the condition of the `primeness` imposed on the ideal is necessary in Theorems 2.3, 2.6 and 2.8.

Example 2.

Let us consider $\mathcal{R} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in R \right\}$; where R is a noncommutative ring, $I = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ an ideal of $\mathcal{R}$. It is clear that $\mathcal{R}$ is a 2-torsion free non-prime ring. Moreover, if we take the same derivation defined in Example 1 then for all $X, Y \in R$ one can check that $d(X \circ Y) - X \circ Y = d[X, Y] - [X, Y] = 0$ for all $x, y \in I$. Hence d satisfies the conditions of Theorems 2.3, 2.6 and 2.8 but R is not commutative.

References

[1] S. Ali, N. A. Dar, *On ∗-centralizing mappings in rings with involution*, Georgian J. Maths. 21 (2014), no. 1, 25-28.

[2] S. Ali and S. Huang, On derivation in semiprime rings, Algebr. Represent. Theor. 15 (2012), no. 6, 1023-1033.

[3] F. A. A. Almahdi, A. Mamouni and M. Tamekkante. *A Generalization of Posner's Theorem on Derivations in Rings*. Indian J. Pure Appl. Math. 51 (2020), no. 1, 187-194.

[4] M. Ashraf and N. Rehman, On commutativity of rings with derivation, Results Math. 42 (2002), no. 1-2, 3-8.

[5] H. E. Bell and M. N. Daif, On commutativity and strong commutativity preserving maps, Canad. Math. Bull. 37 (1994), 443-447.

[6] H. E. Bell, M. N. Daif, On derivations and commutativity in prime rings, Acta Math. Hungar. 66 (1995), 337-343.

[7] M. Bresar and C. R. Miers, Strong commutativity preserving mappings of semiprime rings, Canad. Math. Bull. 37 (1994), 457-460.

[8] M. N. Daif, H. E. Bell, Remarks on derivations on semiprime rings, Internat. J. Math. Math. Sci. 15 (1992), 205-206.

[9] I. N. Herstein , A note on derivations, Canad. Math. Bull. 21 (1978), 369–370.

[10] M. Hongan, A note on semiprime rings with derivation, Internat. J. Math. Math. Sci. 20 (1997), no. 2, 413-415.

[11] A. Mamouni, L. Oukhtite and H. Elmir, New classes of endomorphisms and some classification theorems. Comm. Algebra 48 (2020), no. 1, 71-82.

[12] A. Mamouni, L. Oukhtite and B. Nejjar, Differential identities on prime rings with involution, J. Algebra Appl. 17 (2018), no. 9, 1850163, 11pp.

[13] A. Mamouni, L. Oukhtite and B. Nejjar, On $*$-semiderivations and $*$-generalized semiderivations, J. Algebra Appl. 16 (2017), no. 4, 1750075, 8 pp.

[14] B. Nejjar, A. Kacha, A. Mamouni and L. Oukhtite, Commutativity theorems in rings with involution, Comm. Algebra 45 (2017), no. 2, 698-702.

[15] L. Oukhtite, A. Mamouni and M. Ashraf, Commutativity theorems for rings with differential identities on Jordan ideals, Comment. Math. Univ. Carolin. 54 (2013), no. 4, 447-457

[16] E. C. Posner, Derivations in prime rings, Proc. Amer. Math. Soc. 8 (1957), 1093- 1100.

Author information

L. Oukhtite, A. Mamouni, and H. El Mir, Department of Mathematics, Faculty of Science and Technology of Fez, Box 2202, University S. M. Ben Abdellah Fez
Department of Mathematics, Faculty of Sciences, University Moulay Ismaïl, Meknes, Morocco.
E-mail: a.mamouni.fste@gmail.com

Received: 2022-01-22
Accepted: 2022-04-03