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A study on Weakly Tri normal and Quasi Tri normal Rings

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Abstract A ring R is defined to be a weakly tri normal ring if for all $a \in R$ and a tripotent element t of R such that $at = 0$, implies that $Rtra$ is a nil left ideal of R. It is proved that a ring R is weakly tri normal if and only if $tR(1 + t^2) \subseteq N^*(R)$, where t is an op-tripotent element of R. Furthermore, we introduce the concept of quasi tri normal ring. A ring R is a quasi tri normal ring if $at = 0$, implies $taRt = 0$, where t is a tripotent element of R. It is observed that for a weakly tri normal ring R, (1) Every quasi tri normal ring is a weakly tri normal ring, (2) Every weakly tri normal ring is directly finite, (3) If R is a weakly tri normal π-regular ring then R is strongly π-regular.

1 Introduction

Throughout this paper, all rings are associative with identity. Let R be a ring, the op-idempotent element of R is defined as in [17]. Extending this concept op-tripotent element of R is defined, the set of tripotent element and the set of op-tripotent element of R are denoted by $T(R)$ and $T^0(R)$ respectively. Also, $N^*(R)$, $J(R)$ and $N(R)$, denote the nilradical, the Jacobson radical and the set of all nilpotent elements of R respectively. According to (Wei and Li, 2012) [17], a ring is said to be a weakly normal only when for all $a, r \in R$ and $e \in E(R)$, $ae = 0$ inferred $Rera$ is a nil left ideal of R, where $E(R)$ exemplify the set of all idempotent elements of R, they proved that R is a weakly normal ring if and only if $Rer(1 - e)$ is a nil left ideal of R for each $e \in E(R)$ and $r \in R$. According to [18], a ring R is called a quasi normal if $ae = 0$ implies that $eaRe = 0$ for $a \in N(R)$ and $e \in E(R)$.

Chen (2007) [7], reported that a ring R is said to be a semiabelian when every idempotent of R is either left semi-central or right semi-central. It was proved that in a π-regular ring R, $N(R)$ is an ideal of R under the condition that $R/J(R)$ is an abelian ring. Also, Chen(2007), studied that a semiabelian ring R is π-regular only when $N(R)$ is an ideal of R and $R/N(R)$ is regular, which extends the results of Badawi(1997). According to Badawi(1997) [2], a ring is called π-regular if for $a \in R$ there exists $n \geqslant 1$ and $b \in R$ so that $a^n = a^n b a^n$ and in case of $n = 1$ for all $a \in R$, then R is called Von Neumann regular ring and so Von Neumann regular rings are π-regular.

Cohn [8], in 1999 stated that a ring is said to be reversible if for any $a, b \in R, ab = 0 \implies ba = 0$. Baser et al. [3], extended the concept of reversible rings and they defined a ring R to be semicommutative if for any $a, b \in R, ab = 0 \implies aRb = 0$.
Zhao, et al., generalised reversible rings and introduced that a ring R is called weakly reversible [12], if $ab = 0 \implies Rbra$ is a nil left ideal of R, for all $a, b, r \in R$. It is obvious that semicommutative rings are weakly reversible. A ring R is called directly finite if for all $a, b \in R$ such that $ab = 1$ implies that $ba = 1$. Clearly reversible rings are directly finite.

Wei [16], studied the concept of left minimal element and left minimal idempotent of a ring R. Extending this concept the left minimal tripotent element of R is defined using tripotent element. A ring R is called left min-abel [16], if every left minimal tripotent element is left semicentral in R. It is obvious that, abelian rings are left min-abel.

In this study, the results appeared in Wei and Li [17, 18], Chen [7] are extended and generalized using the concept of tripotent element. The objective is to study and to define a new type

of ring called a weakly tri normal ring using tripotent element. It is interesting to note that using tripotent element which is not an idempotent in a weakly normal ring [17], it is seen that every condition of weakly normal rings is also satisfied and hence it is named as weakly tri normal ring.
Thus, a ring R is called weakly tri normal if for all $a, r \in R$ and $t \in T(R)$ with $at = 0$, implies $Rtra$ is a nil left ideal of R.
For example let, $R = M_{2\times2}(\mathbf{R})$ be an upper triangular matrix ring over a real number field $\mathbf{R}$. Then $\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ is tripotent but not idempotent, such that $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$,
implies $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$.
Thus, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \in N(M_{2\times2}(\mathbf{R}))$.
So, $R = M_{2\times2}(\mathbf{R})$ is a weakly tri normal ring.
Clearly, idempotents are also tripotents and so, every weakly normal ring is weakly tri normal ring. And hence, weakly reversible ring, abelian ring are weakly tri normal ring.

In a similar way like weakly tri normal ring, another new type of ring termed as Quasi tri normal ring is defined.
Thus, a ring R is called Quasi tri normal if $at = 0$, implies $taRt = 0$ for $a \in N(R)$ and $t \in T(R)$.
For example Let, $R = M_{2\times2}(\mathbf{R})$ be a full matrix ring over a real number field $\mathbf{R}$. Then $\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ is tripotent but not idempotent, such that $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$,
implies $\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$.
So, $R = M_{2\times2}(\mathbf{R})$ is a quasi tri normal ring. From the above it is proved that Quasi tri normal rings are weakly tri normal but the converse does not hold.

Throughout the study it is found that R is a weakly tri normal ring if and only if $tR(1+t^2) \subseteq N^*(R)$, where $t \in T^0(R)$ and moreover it is proved that a ring R is quasi tri normal if and only if $t^2R(1-t^2)Rt^2 = 0$, for all $t \in T(R)$. A ring R is quasi tri normal if and only if $t^2R(1+t^2)Rt^2 = 0$, for all $t \in T^0(R)$. And various properties of weakly tri normal rings as well as quasi tri normal rings are established.

2 Weakly Tri normal rings

In this section some characterizations of weakly tri normal rings are discussed.

Theorem 2.1. *The following conditions are equivalent for a ring R.*

(1) R is a weakly tri normal ring.

(2) $Rtr(1-t^2)$ is a nil left ideal of R for all $t \in T(R)$, $r \in R$.

(3) $tR(1-t^2) \subseteq N^*(R)$ for any $t \in T(R)$.

Proof. (1) $\implies$ (2) since, $t^3 = t \implies t - t^3 = 0 \implies (1-t^2)t = 0$. Therefore, $Rtr(1-t^2)$ is a nil left ideal of R.
(2) $\implies$ (3). Let, $t \in T(R)$ then $Rtr(1-t^2) \subseteq N^*(R)$ for all $r \in R$, by(2). Thus, $tR(1-t^2) \subseteq N^*(R)$.
(3) $\implies$ (1). Let us assume that, $at^2 = 0$, $a \in R$ and $t \in T(R)$. Then $t = t^3 \implies (1-t^2)t = 0$ and $a = a - at^2 = a(1-t^2)$. Now, $tRa = tRa(1-t^2) \subseteq tR(1-t^2)$, since $tR(1-t^2) \subseteq N^*(R)$. So, $tRa = tRa(1-t^2) \subseteq tR(1-t^2) \subseteq N^*(R) \implies tRa(1-t^2) \subseteq N^*(R) \implies tRra(1-t^2) \subseteq N^*(R)$, $r \in R \implies Rtra(1-t^2) \subseteq N^*(R) \implies Rtra \subseteq N^*(R)$. So, for any $r \in R$ and $t \in T(R)$, we get $Rtra$ is a nil left ideal of R. Hence, R is a weakly tri normal ring. □

Theorem 2.2. *A ring R is a weakly tri normal ring if and only if $tN(R)(1-t^2) \subseteq N^*(R)$ for any $t \in T(R)$.*

Proof. Let R be a weakly tri normal ring. Since $N(R)$ is the set of all nilpotent elements of R. So, $tN(R)(1-t^2)$ is a nil right ideal of R for any $t \in T(R)$, which implies that $tN(R)(1-t^2) \subseteq N^*(R)$.

For the converse part, for any $t \in T(R)$, $(tR(1-t^2))(tR(1-t^2)) = (tR-tRt^2)(tR-tRt^2) = tRtR - tRtRt^2 - tRt^2tR + tRt^2tRt^2 = tRtR - tRtRt^2 - tRtR + tRtRt^2 = 0$, as $t^3 = t$. So, $tR(1-t^2) \subseteq N(R)$. Hence, $tR(1-t^2) = t(tR(1-t^2))(1-t^2) \subseteq tN(R)(1-t^2) \subseteq N^*(R) \implies tR(1-t^2) \subseteq N^*(R)$. Thus, R is a weakly tri normal ring. □

An element t of a ring R is called an op-tripotent element if $t^3 = -t$, the set of all op-tripotent elements is denoted by $T^0(R)$. In general op-tripotent is not tripotent. For example let, $R = M_{2\times 2}(\mathbf{C})$ then $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in R$ is op-tripotent but not tripotent.

An element $t \in R$ is called potent if there exists some positive integer $n \geqslant 2$ such that $t^n = t$, clearly tripotent is potent but every potent element is not tripotent. For example, $R = M_{2\times 2}(\mathbf{C})$ then $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in R$ is potent but not tripotent, as $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}^5 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ but $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}^3 \neq \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, the set of all potent elements of R is denoted by $PT(R)$.

Theorem 2.3. *A ring R is a weakly tri normal if and only if $tR(1+t^2) \subseteq N^*(R)$, where $t \in T^0(R)$.*

Proof. Let R be a weakly tri normal ring, since t is op-tripotent, $t^3 = -t \implies t + t^3 = 0 \implies (1+t^2)t = 0$. Therefore, $Rtr(1+t^2)$ is a nil left ideal for any $r \in R$. Since, $N^*(R)$ is the sum of nil ideals so, $Rtr(1+t^2) \subseteq N^*(R) \implies tRr(1+t^2) \subseteq N^*(R)$, as tripotents are central. Thus, $tR(1+t^2) \subseteq N^*(R)$, $r \in R$.

Conversely let, $at^2 = 0$, $a \in R$ and $t \in T^0(R)$. Then $(1+t^2)t = 0$ and $a = a + at^2 = a(1+t^2)$. Now, $tRa = tRa(1+t^2) \subseteq tR(1+t^2)$, since $tR(1+t^2) \subseteq N^*(R)$. So, $tRa = tRa(1+t^2) \subseteq tR(1+t^2) \subseteq N^*(R) \implies tRa(1+t^2) \subseteq N^*(R) \implies tRra(1+t^2) \subseteq N^*(R)$, $r \in R \implies Rtra(1+t^2) \subseteq N^*(R) \implies Rtra \subseteq N^*(R)$. So, for any $r \in R$ and $t \in T^0(R)$, $Rtra$ is a nil left ideal of R. Hence, R is a weakly tri normal ring. □

Theorem 2.4. *R is a weakly tri normal ring if and only if $tR\big(1 - t^{p(t)-1}\big) \subseteq N^*(R)$ for any $t \in PT(R)$ and $p(t)$ is the minimal positive integer.*

Proof. Suppose R be a weakly tri normal ring. So for all $a, r \in R$ and $t \in PT(R)$, $at = 0$, implies $Rtra$ is a nil left ideal of R. Since every tripotent is potent, so $t \in PT(R)$, $t^{p(t)} = t \implies t - t^{p(t)} = 0 \implies \big(1 - t^{p(t)-1}\big)t = 0$. Since R is a weakly tri normal so, $Rtr\big(1 - t^{p(t)-1}\big)$ is a nil left ideal of R. Therefore, $Rtr\big(1 - t^{p(t)-1}\big) \subseteq N^*(R) \implies tRr\big(1 - t^{p(t)-1}\big) \subseteq N^*(R) \implies tR\big(1 - t^{p(t)-1}\big) \subseteq N^*(R)$, as $Rr \subseteq R$. Clearly converse part also holds. □

Theorem 2.5. *Weakly tri normal rings are directly finite.*

Proof. Let, R be a weakly tri normal ring such that $xy = 1$, for any $x, y \in R$. Also let, $t \in T(R)$ and $t = yx$. Then $xt = x$. Using Theorem 2.1, $tR(1-t^2) \subseteq N^*(R)$. Now, $1 - t^2 = xy(1-t^2) = xty(1-t^2) \in N^*(R)$. This implies that $1 - t^2 = 0 \implies 1 = t^2 = (yx)(yx) = y(xy)x = yx$. Therefore, $yx = 1$. Hence, R is directly finite. □

Corollary 2.6. *It is observed that, the converse part of the Theorem 2.5, does not hold in general.*

Proof. For $n \geqslant 2$ let the full matrix ring $R = M_{n\times n}(\mathbf{R})$ over the field of real number $\mathbf{R}$ is directly finite. But, $R = M_{n\times n}(\mathbf{R})$ is not a weakly tri normal ring for $n \geqslant 2$.
For $n = 2$ it can be easily shown. For $n = 3$,
$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ be a tripotent element and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ be any element in $M_{3\times 3}(\mathbf{R})$.

Then, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$. Now for $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ in $M_{3\times 3}(\mathbf{R})$.

We get, $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

Therefore, $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} \notin N(M_{3\times 3}(\mathbf{R}))$.

This shows that, $R = M_{3\times 3}(\mathbf{R})$ is not a weakly tri normal ring. Similarly the result can be proved for $n \geqslant 4$. □

Corollary 2.7. *As a consequence of Corollary 2.6, we get the Corollary 2.7, if R is a weakly tri normal ring, then the full matrix ring $M_{n\times n}(R)$ is not weakly tri normal for $n \geqslant 2$.*

Theorem 2.8. *Weakly tri normal rings are left min-abel.*

Proof. Let t^2 be the left minimal tripotent element of R and $a \in R$. Let, $h = at^2 - t^2at^2$. If $h \neq 0$, then $th = 0$ and $ht^2 = h$ and $Rh = Rt$. Since R is a weakly tri normal ring, by Theorem 2.1, $R(1-t^2)rt \subseteq N^*(R)$ for any $r \in R$. So, $Rt = Rh = R(1-t^2)h = R(1-t^2)ht \subseteq N^*(R)$, which is a contradiction. Thus, $h = 0$, this implies t^2 is left semicentral in R, hence R is a left min-abel. □

By [[17], Corollary 2.6(1), (3)], the following corollary follows from Theorem 2.1.

Corollary 2.9. *(1) The subrings and finite direct products of weakly tri normal rings are weakly tri normal.*

(2) Let R be a weakly tri normal ring. If $t \in T(R)$ such that $Rt^2R = R$ then $t^2 = 1$.

Lemma 2.10. *Let R be a ring and I be an ideal of R such that R/I is a weakly tri normal ring. If $I \subseteq N(R)$, then R is a weakly tri normal ring.*

Proof. Let, $a, t \in R$ and $t \in T(R)$ with $at = 0$. So in $\bar{R} = R/I$, $\bar{a}\bar{t} = \bar{0}$. Since $\bar{R} = R/I$ is weakly tri normal so, $\bar{R}\bar{t}\bar{r}\bar{a}$ is a nil left ideal of $\bar{R}$. So, there exists $n \geqslant 1$ for any $\bar{x} \in \bar{R}$ such that $(\bar{x}\bar{t}\bar{r}\bar{a})^n = \bar{0}$. Thus, $(xtra)^n \in I$, since $I \subseteq N(R)$, there exists $m \geqslant 1$ such that for $x \in R$ we get $(xtra)^{nm} = 0$. This implies that $xtra \in N(R)$ for all $x \in R$. Hence $Rtra$ is a nil left ideal of R for all $r \in R$. So R is a weakly tri normal ring. □

Theorem 2.11. *Let S and T be two rings, and M be a (S, T)-bimodule. Let, $R = \begin{pmatrix} S & M \\ 0 & T \end{pmatrix}$. Then R is a weakly tri normal ring if and only if S and T are weakly tri normal.*

Proof. If R is a weakly tri normal ring then S and T are also weakly tri normal rings, by Corollary 2.9.(1).

Conversely, let S and T be weakly tri normal rings. Let $I = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$. Then I is an ideal of R and $R/I \cong S \times T$ is a weakly normal ring by Corollary 2.9.(1) and the hypothesis. Since $I \subseteq N(R)$, by Lemma 2.10, R is a weakly tri normal ring. □

The following corollary directly follows from the above Theorem 2.11 and induction on n.

Corollary 2.12. *A ring R is weakly tri normal if and only if for any $n \geqslant 1$, the $n \times n$ upper triangular matrix ring $M_{n\times n}(R)$ is a weakly tri normal ring.*

3 Quasi Tri normal rings

In this section some properties of quasi tri normal rings are discussed and a relation between quasi tri normal ring and weakly tri normal ring is derived.

Theorem 3.1. *A ring R is a quasi tri normal if and only if $t^2R(1-t^2)Rt^2 = 0$, for all $t \in T(R)$.*

Proof. For any $a \in R$, let $h = t^2a - t^2at^2$. Then $ht^2 = (t^2a - t^2at^2)t^2 = t^2at^2 - t^2at^4 = t^2at^2 - t^2at^2 = 0$, $t^2h = t^2(t^2a - t^2at^2) = t^4a - t^4at^2 = t^2a - t^2at^2 = h$ and $h^2 = (t^2a - t^2at^2)(t^2a - t^2at^2) = 0$, as $t^3 = t$. Let R be a quasi tri normal ring, then $t^2hRt^2 = 0 \implies hRt^2 = 0$. Since $h = t^2a(1-t^2)$, so $t^2a(1-t^2)Rt = hRt^2 = 0$, which shows that $t^2a(1-t^2)Rt^2 = 0$ for all $a \in R$.

Conversely let, $t \in T(R)$ such that $at^2 = 0$, where $a \in N(R)$. Then by the hypothesis $t^2aRt^2 = t^2a(1-t^2)Rt^2 \in t^2R(1-t^2)Rt^2$. Thus $t^2aRt^2 = 0$, as $t^2R(1-t^2)Rt^2 = 0$. So R is a quasi tri normal ring. □

Theorem 3.2. *A ring R is a quasi tri normal if and only if $t^2R(1+t^2)Rt^2 = 0$, for all $t \in T^0(R)$.*

Proof. Let R be a quasi tri normal ring. Since $t \in T^0(R)$, then $t^3 = -t$. Let $h = t^2a + t^2at^2$, where $a \in R$. Then $t^2h = -h$, $ht^2 = 0$ and $h^2 = 0$, as $t^3 = -t$. Since R is a quasi tri normal ring. So, $t^2hRt^2 = 0 \implies -hRt^2 = 0 \implies hRt^2 = 0$. Also, $h = t^2a(1+t^2)$ implies that $hRt^2 = t^2a(1+e^2)Rt^2 = 0 \implies t^2R(1+t^2)Rt^2 = 0$, for any $a \in R$. Clearly converse also holds. □

Corollary 3.3. *(1) A ring R is a quasi tri normal if and only if for any $t \in T(R)$, $t^2yxt^2 = t^2yt^2xt^2$, $x, y \in R$.*

(2) Let R be a quasi tri normal ring. If $t \in T(R)$ is such that $RtR = R$ then $t^2 = 1$.

(3) Semi-abelian rings are quasi tri normal. But the converse does not hold in general.

Proof. (1) Let $x, y \in R$, $t \in T(R)$ and $t = yx$. Let R be a quasi tri normal then by Theorem 3.1, $t^2R(1-t^2)Rt^2 = 0$. Thus, $t^2y(1-t^2)xt^2 = 0 \implies t^2yxt^2 = t^2yt^2xt^2$. The converse part is clear.
(2) Since R is a quasi tri normal, by Theorem 3.1, $t^2R(1-t^2)Rt^2 = 0$. Since $Rt^2R = R$ so, $R(1-t^2)R = Rt^2R(1-t^2)Rt^2R = 0$. Thus, $t^2 = 1$.
(3) Let, $t \in T(R)$ be right semicentral, then $t^2R(1-t^2)Rt^2 = t^2Rt^2(1-t^2)Rt^2 = 0$, and if t is left semicentral then $t^2R(1-t^2)Rt^2 = t^2R(1-t^2)t^2Rt^2 = 0$, by Theorem 3.1. So, R is a quasi tri normal ring. For the converse part, let R_1 and R_2 be semiabelian rings which are not abelian. If $t_1 \in T(R_1)$ is right semicentral which is not central and $t_2 \in T(R_2)$ be left semicentral which is not central, then the tripotent (t_1, t_2) is neither right nor left semicentral in $R_1 \oplus R_2$. Hence $R_1 \oplus R_2$ is not semiabelian but, $R_1 \oplus R_2$ is a quasi tri normal ring. □

Theorem 3.4. *Quasi tri normal rings are directly finite.*

Proof. Let $x, y \in R$ be such that, $xy = 1$. Also let $t \in T(R)$ and $t^2 = yx$. Then $xt^2 = xyx = x$ and $t^2y = yxy = y$. Let R be a quasi tri normal. Then by Theorem 3.1, $t^2R(1-t^2)Rt^2 = 0$. Thus, $t^2y(1-t^2)xt^2 = 0$. Therefore, $y(1-t^2)x = 0 \implies yx = yt^2x$. Now, $1 = xyxy = x(yx)y = x(yt^2x)y = xyt^2xy = t^2 = yx$. Thus, $xy = 1 \implies yx = 1$. Hence R is directly finite. □

Theorem 3.5. *Quasi tri normal rings are left min-abel.*

Proof. Let t^2 be the left minimal tripotent element of R and $a \in R$. Let $h = at^2 - t^2at^2$. If $h \neq 0$, then $t^2h = 0$ and $ht^2 = h$ and $h \in N(R)$. Since, R is a quasi tri normal, this implies $ht^2Rt^2 = 0 \implies hRt^2 = 0$, a contradiction. Hence $h = 0$, which implies t^2 is left semicentral in R, hence R is left min-abel. □

Theorem 3.6. *Quasi tri normal rings are weakly tri normal.*

Proof. Let R be a quasi tri normal ring and $t \in T^0(R)$ then $t^3 = -t$. Let $h = t^2a + t^2at^2$, for any $a \in R$. Then $t^2h = -h$, $ht^2 = 0$ and $h^2 = 0$, as $t^3 = -t$. Since R is a quasi tri normal ring, $t^2hRt^2 = 0 \implies -hRt^2 = 0 \implies hRt^2 = 0$. Also, $h = t^2a(1+t^2)$ implies that $hRt^2 = t^2a(1+t^2)Rt^2 = 0 \implies t^2R(1+t^2)Rt^2 = 0$, for any $a \in R$ ...(1). Now, $Rt^2R(1+t^2)Rt^2R(1+t^2) = R(t^2R(1+t^2)Rt^2)R(1+t^2) = 0$, by (1). So, $Rt^2R(1+t^2)$ is a nilpotent left ideal of R. Hence R is a weakly tri normal ring. □

The following example shows that the converse of Theorem 3.6 does not hold in general.

Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a division ring.

Considering the tripotent $t = t_{11} + t_{22}$, i.e, $t = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix}$, then by calculating it is observed that $t^2R(1-t^2)Rt \neq 0$, thus R is not a quasi tri normal by Theorem 3.1. But by Corollary 2.12, R is weakly tri normal ring. Again, by YU[[19], Pro-2.1], R is left quasi-duo ring, so R is left min-abel. Thus the above example also shows that the converse of the Theorem 3.5 does not hold.

4 Some applications

Generalizing the notion defined by [17], the following concepts are defined using tripotent element.

A ring R is called left tripotent reflexive if $aRt = 0 \implies tRa = 0$, for all $a \in R$ and $t \in T(R)$; A ring R is called strongly left tripotent reflexive if $at = 0 \implies ta = 0$, for all $a \in J(R)$, $t \in T(R)$; and weakly left tripotent reflexive if $at = 0 \implies ta = 0$, for all $a \in R$ and left semicentral tripotent t of R.

By using the concept of tripotent in the results appeared in [17], the following results are obtained.

Lemma 4.1. *Strongly left tripotent reflexive rings are left tripotent reflexive and left tripotent reflexive rings are weakly left tripotent reflexive.*

Proof. Firstly, R is a strongly left tripotent reflexive ring, then for all $a \in J(R)$ and $t \in T(R)$, $at = 0 \implies ta = 0$. Let, $aRt = 0$ where $a \in R$ and $t \in T(R)$. If possible let, $tRa \neq 0$ then there exists $b \in R$ such that $tba \neq 0$. Now, $(tbaR)^2 = tbaRtbaR = tb(aRt)baR = 0$. Therefore, $tba \in J(R)$. Since, R is a strongly left tripotent reflexive ring, So, $tba \in J(R)$, $(tba)t = 0 \implies t(tba) = 0$. So, $tba = 0$. Hence, $tRa = 0$ and so R is a left tripotent reflexive ring. For the second part let R be a left tripotent reflexive ring, so $aRt = 0 \implies tRa = 0$ for all $a \in R$ and $t \in T(R)$. Let, $at = 0$, where t is a left semicentral tripotent element of R. Since, $aRt = atRt = 0$ and R is a left tripotent reflexive ring. So, $tRa = 0$. Thus $ta = 0$. Hence R is a weakly left tripotent reflexive ring.

□

Lemma 4.2. *Let R be a weakly tri normal ring and $x \in R$. If x is Von Neumann regular then $x \in Rx^2 \cap x^2R$.*

Proof. Let R be a weakly tri normal ring and $x \in R$ is Von Neumann regular then there exists $y \in R$ such that $x = xyx$. Let, $t = yx$. Then $t^3 = (yx)^3 = yxyxyx = yxyx = yx = t$. So, $t^3 = t \in R$ is a tripotent and $x = xyx = xt$, since R is a weakly tri normal ring, so $R(1-t^2)x$ is a nil left ideal of R. Thus, there exists $n \geqslant 1$ such that $(y(1-t^2)x)^n = 0$, $y \in R$. Since, $y(1-t^2)x = yx - yt^2x = t - yt^2x$. So, there exists $a \in R$ such that,$(y(1-t^2)x)^n = (t-yt^2x)^n = t - at^2x = 0 \implies t = at^2x$. Hence, $x = xt = xat^2x = xa(yx)^2x = xayxyxx = xayxx = xayx^2 \in Rx^2$, as $x = xyx$. Therefore, $x \in Rx^2$. Again let, $t = xy$ then $x = tx$ and $t^3 = (xy)^3 = xyxyxy = xyxy = xy = t$, so $t^3 = t \in R$ is tripotent. So by Theorem 2.1, $tR(1-t^2) \subseteq N^*(R)$, implies that $x(1-t^2) = tx(1-t^2) \in tR(1-t^2) \subseteq N^*(R)$. Thus, $x(1-t^2)y \in N^*(R)$; $y \in R$. So, there exists $m \geqslant 1$ such that, $(x(1-t^2)y)^m = 0$. Since, $x(1-t^2)y = xy - xt^2y = t - xt^2y$. So, there exists $b \in R$ such that, $(x(1-t^2)y)^m = (t - xt^2y)^m = t - xt^2b = 0 \implies t = xt^2b$. Therefore, $x = tx = xt^2bx = x(xy)^2bx = xxyxybx = xxybx = x^2ybx \in x^2R \implies x \in x^2R$. Thus, $x \in Rx^2 \cap x^2R$. □

Let R be a ring and an element $a \in R$ is called π-regular [11], if there exists $b \in R$ such that $a^n = a^n b a^n$; $n \geqslant 1$. For $n = 1$, a is called Von Neumann regular. Also, a is called strongly π-regular, if $a^n = ba^{n+1}$. In case of $n = 1$, a is called strongly regular. A ring R is said to be Von Neumann regular, strongly regular, π-regular and strongly π-regular, if every elements of R is Von Neumann regular, strongly regular, π-regular and strongly π-regular respectively. A ring R is said to be unit-regular, if for any $a \in R$ such that $a = aua$, where $u \in U(R)$, group of units of R. Clearly unit regular implies Von Neumann regular.

Lemma 4.3. *If R is a weakly tri normal π-regular ring then R is strongly π-regular.*

Proof. Let R be a π-regular weakly tri normal ring. So, for any $x \in R$, there exists $n \geqslant 1$ and for $y \in R$ such that $x^n = x^n y x^n$. This implies that x^n is Von Neumann regular. Since, R is a weakly tri normal ring, so by Lemma 4.2, we have $x^n \in R(x^n)^2 \subseteq Rx^{(n+1)}$. This shows that R is a strongly π-regular ring. □

Lemma 4.4. *Let R be a quasi tri normal ring and $x \in R$. If x is Von Neumann regular then x is strongly regular.*

Proof. If x is Von Neumann regular, then for some $y \in R$, we get, $x = xyx$. Let, $t \in T(R)$ then, $t = yx$, then $t^3 = yxyxyx = yxyx = yx = t$, $t^2 = yxyx = yx = t$ and $x = xt$. Since, $t = t^3 = t^2tt^2 = t^2yxt^2 = t^2yt^2xt^2$, by Corollary 3.3. Thus, $t = t^2yt^2xt^2 = tytxt = tyyxxt = tyyxx = ty^2x^2$. So, we have $x = xt = xty^2x^2 = xy^2x^2$. Similarly, we can show $x = x^2y^2x$. Hence x is strongly regular. □

Corollary 4.5. *If a is π-regular, then there exists a tripotent, $t \in T(R)$ such that ta is Von Neumann regular.*

Proof. If a is π-regular then there exists $n \geqslant 1$, such that $a^n = a^n u a^n$, where $u \in U(R)$, this implies that a^n is Von Neumann regular. So by Lemma 4.4, a^n is strongly regular. Let $t = a^n u$, then $t^3 = a^n u a^n u a^n u = (a^n u a^n) u a^n u = a^n u a^n u = a^n u = t$. Thus, $t^3 = t$, t is a tripotent. Also, $a^n = ta^n$ and $t = a^n u \implies a^n = tu^{-1} = tv$, as $v = u^{-1}$. Since, $(ta)(a^{n-1}u)(ta) = t(a^n)uta = ttvuta = tt1ta = t^3a = ta$. This shows that ta is Von Neumann regular.

□

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