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On S-Nil_*-coherent rings

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Abstract Let R be a commutative ring with nonzero identity and $S \subseteq R$ be a multiplicatively closed subset of R. In this paper, we introduce and study S-Nil_*-coherent rings which are a generalization of Nil_*-coherent and S-coherent rings. An R-module M is said to be an S-Nil_*-coherent R-module if every finitely generated R-submodule of M contained in $Nil(R)M$ is S-finitely presented; and a ring R is S-Nil_*-coherent if it is S-Nil_*-coherent as R-module. Besides giving many properties of S-Nil_*-coherent rings, we generalize some results on Nil_*-coherent rings to S-Nil_*-coherent rings. Furthermore, we characterize Nil_*-coherent rings in terms of S-Nil_*-coherent rings.

1 Introduction

Throughout this paper, all rings are assumed to be commutative with nonzero identity and all modules are nonzero unital. Let R denote such a ring and S denote such a multiplicatively closed subset of R. $Reg(R)$, denotes the set of regular elements of the ring R; $Q(R) := R_{Reg(R)}$, the total quotient ring of R; $Nil(R)$ denotes the set of nilpotent elements of R (also known as the nilradical of R). For an ideal I of R and an element $a \in R$, we denote by $(I : a) = \{x \in R \mid xa \subseteq I\}$ the conductor of Ra into I. Recall that an R-module M is called a finitely presented R-module if there is an exact sequence of R-modules $F_1 \to F_0 \to M \to 0$ such that both F_0 and F_1 are finitely generated free R-modules. A finitely generated R-module M is said to be a coherent R-module if every finitely generated R-submodule of M is a finitely presented R-module; and a ring R is called a coherent ring if R is coherent as an R-module. An excellent summary of work done on coherence up to 1989 can be found in [12]. In [2], Alaoui Ismaili et al. introduced Nil_*-coherent modules over commutative rings as a new generalization of coherent modules. An R-module M is called a Nil_*-coherent R-module if every finitely generated R-submodule of $Nil(R)M$ is a finitely presented R-module; and a commutative ring R is said to be a Nil_*-coherent ring if it is Nil_*-coherent as an R-module, that is, if every finitely generated ideal of R that is contained in $Nil(R)$ is finitely presented. In [3], Anderson and Dumitrescu introduced the concept of S-finite modules as follows: an R-module M is called an S-finite module if there exist a finitely generated R-submodule N of M and $s \in S$ such that $sM \subseteq N$. Recently, in [7], Bennis and El Hajoui investigated the S-versions of finitely presented modules and coherent modules which are called, respectively, S-finitely presented modules and S-coherent modules. An R-module M is called an S-finitely presented module for some multiplicatively closed subset S of R if there exists an exact sequence of R-modules $0 \to K \to F \to M \to 0$, where F is a finitely generated free R-module and K is an S-finite R-module. Moreover, an R-module M is said to be S-coherent, if it is finitely generated and every finitely generated submodule of M is S-finitely presented. They showed that the S-coherent rings have a characterization similar to the classical one given by Chase for coherent rings (see [7, Theorem 3.8]).

Some of our results use the $R \propto M$ construction. Let R be a ring and M be an R-module. Then $R \propto M$, the *trivial* (*ring*) *extension of* R *by* M, is the ring whose additive structure is that of the external direct sum $R \oplus M$ and whose multiplication is defined by $(r_1, m_1)(r_2, m_2) := (r_1r_2, r_1m_2 + r_2m_1)$ for all $r_1, r_2 \in R$ and all $m_1, m_2 \in M$. The basic properties of trivial ring extensions are summarized in the books [12, 13]. Mainly, trivial ring extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. See for instance [4, 5, 6, 14, 15].

In present article, we define S-Nil_*-coherent rings as a new generalization of S-coherent and Nil_*-coherent rings. If R is a ring and S is a multiplicatively closed subset of R, then we say that an R-module M is S-Nil_*-coherent if every finitely generated R-submodule of $Nil(R)M$ is S-presented; and a ring R is S-Nil_*-coherent if it is S-Nil_*-coherent as R-module, that is, every finitely generated ideal of R which is contained in $Nil(R)$ is S-finitely presented. Among many results of this paper, we give a necessary and sufficient condition for a ring to be S-Nil_*-coherent (see Theorem 2.6). Also, we characterize Nil_*-coherent rings in terms of S-Nil_*-coherent rings. Moreover, we study the S-Nil_*-coherence property under homomorphism, direct products and localization (see Theorem 2.8, Proposition 2.13, Theorem 2.14 and Proposition 2.15). Also, we study some particular cases of the trivial ring extension and examine conditions under which $R \propto M$ is an $(S \propto M)$-Nil_*-coherent ring (see Theorem 2.16). Finally, we investigate the S-Nil_*-coherence property on amalgamated algebras.

2 Main results

We shall begin with the following definition:

Definition 2.1. Let R be a ring and S be a multiplicatively closed subset of R.

(1) An R-module M is said to be an S-Nil_*-coherent if every finitely generated R-submodule of $Nil(R)M$ is S-presented.

(2) R is called an S-Nil_*-coherent ring if it is S-Nil_*-coherent as R-module.

Remark 2.2. Let R be a ring and S be a multiplicatively closed subset of R. Then the followings hold:

(1) Every Nil_*-coherent R-module is an S-Nil_*-coherent module. The converse is also true when $S \subseteq U(R)$.

(2) Every S-coherent R-module is an S-Nil_*-coherent module.

(3) If R is an S-Noetherian ring, then every module over R is an S-Nil_*-coherent module.

The following example shows that the converse of Remark 2.2(1) is not true, in general.

Example 2.3. Let M be a countable direct sum of copies of $\mathbb{Z}/2\mathbb{Z}$, $R = \mathbb{Z} \propto M$ and $S = \{(2,0)^n \mid n \in \mathbb{N}\}$ is a multiplicatively closed subset of R. Then, R is an S-coherent ring and hence R is an S-Nil_*-coherent ring. However, R is not a Nil_*-coherent. Indeed, let $m = (\bar{1}, \bar{0}, \ldots) \in M$. So, $(0 : (0, m)) = 2\mathbb{Z} \propto M$ is not finitely generated.

For each multiplicatively closed subset $S \subseteq R$, $S^* := \{a \in R \mid \frac{a}{1} \text{ is a unit of } R_S\}$ denotes the saturation of S. Note that S^* is a multiplicatively closed subset containing S.

Proposition 2.4. *Let R be a ring and M be an R-module. Then the following statements are satisfied:*

(1) *If $S_1 \subseteq S_2$ are multiplicatively closed subsets of R and M is an S_1-Nil_*-coherent module, then M is an S_2-Nil_*-coherent module.*

(2) *M is an S-Nil_*-coherent module if and only if M is an S^*-Nil_*-coherent module, with S^* is the saturation of S.*

Proof. (1) It is explicit.
(2) If M is an S-Nil_*-coherent module, then M is an S^*-Nil_*-coherent module since $S \subseteq S^*$. For the converse, it suffices to prove that every submodule S^*-presented is S-presented. This, in turn, follows from the fact that any S^*-finite module is an S-finite. □

Proposition 2.5. *Let R be a ring and S be a multiplicatively closed subset of R such that $S \subseteq Reg(R)$. If M is an S-Nil_*-coherent R-module, then M_S is a Nil_*-coherent R_S-module.*

Proof. Let L be a finitely generated R_S-submodule of M_S such that $L \subseteq Nil(R_S)M_S$. So, L is of the form N_S, where N is a finitely generated R-submodule of M. Moreover, by hypothesis, $Nil(R_S)M_S \cong Nil(R)_S M_S \cong (Nil(R)M)_S$, and hence $N \subseteq Nil(R)M$. Since M is an S-Nil_*-coherent R-module, we then have N is an S-finitely presented module. Which proves that L is a finitely presented R_S-module and thus M_S is a Nil_*-coherent R_S-module, as required. □

The following theorem provides a necessary and sufficient condition for a ring to be an S-Nil_*-coherent ring.

Theorem 2.6. *Let R be a ring and S be a multiplicatively closed subset of R. Then the followings are equivalent:*

(1) *R is an S-Nil_*-coherent ring.*

(2) *$(0 : a)$ is an S-finite ideal of R for each $a \in Nil(R)$ and the intersection of any two finitely generated ideals of R which are contained in $Nil(R)$ is an S-finite ideal.*

Proof. Suppose that R is an S-Nil_*-coherent ring. Take an element $a \in Nil(R)$. Via the exact sequence $0 \to (0 : a) \to R \to Ra \to 0$, we can see that $(0 : a)$ is an S-finite ideal. Now, let I and J be two finitely generated ideals of R that are contained in $Nil(R)$. Consider the exact sequence $0 \to I \cap J \to I \bigoplus J \to I + J \to 0$. Since $I + J$ is an S-finitely presented ideal of R, we conclude that $I \cap J$ is an S-finite ideal. Conversely, let I be a finitely generated ideal of R such that $I \subseteq Nil(R)$. So, $I = \sum_{i=1}^{n} Ra_i$ for some $a_1, \ldots, a_n \in R$. By induction on n, we will prove that I is an S-finitely presented ideal. For $n = 1$, we are done. For the induction step (with $n > 1$), consider the exact sequence $0 \to (\sum_{i=1}^{n-1} Ra_i) \cap Ra_n \to (\sum_{i=1}^{n-1} Ra_i) \bigoplus Ra_n \to I \to 0$. By hypothesis, we have $(\sum_{i=1}^{n-1} Ra_i) \cap Ra_n$ is an S-finite ideal. Also, $(\sum_{i=1}^{n-1} Ra_i) \bigoplus Ra_n$ is an S-finitely presented ideal of R since $(\sum_{i=1}^{n-1} Ra_i)$ and Ra_n are S-finitely presented ideals. It follows that I is an S-finitely presented ideal of R by [7, Theorem 2.4]. This completes the proof. □

Let P be a prime ideal of a ring R. We say that an R-module M is a P-finitely presented if it is $(R \setminus P)$-finitely presented. Also, M is called a P-Nil_*-coherent module if M is an $(R \setminus P)$-Nil_*-coherent module.

Theorem 2.7. *Let R be a ring and M be an R-module. Then the followings are equivalent:*

(1) *M is a Nil_*-coherent module.*

(2) *M is a P-Nil_*-coherent module for each $P \in Spec(R)$.*

(3) *M is an $\mathfrak{m}$-Nil_*-coherent module for each $\mathfrak{m} \in Max(R)$.*

Proof. (1) ⇒ (2) By Remark 2.2.
(2) ⇒ (3) It is obvious.
To prove (3) ⇒ (1), we only need show that an R-submodule N of M is finitely presented if and only if N is an $\mathfrak{m}$-finitely presented module for each $\mathfrak{m} \in Max(R)$. Suppose that N is an $\mathfrak{m}$-finitely presented module for each $\mathfrak{m} \in Max(R)$. So, N is a finitely generated module. Consider the following exact sequence of R-modules $0 \to K \to R^n \to N \to 0$ for some positive integer n. By hypothesis, K is an $\mathfrak{m}$-finite module for each $\mathfrak{m} \in Max(R)$. This yields that K is a finitely generated module by a similar arguments of [3, Proposition 12]. Hence N is finitely presented. This completes the proof. □

Theorem 2.8. *Let R be a ring, S be a multiplicatively closed subset of R and $0 \to M_1 \xrightarrow{v} M_2 \xrightarrow{u} M_3 \to 0$ be an exact sequence of R-modules. Then the following statements are satisfied:*

(1) *Assume that M_1 is a finitely generated module such that $v(M_1) \subseteq Nil(R)M_2$ and M_2 is an S-Nil_*-coherent module, then M_3 is an S-Nil_*-coherent module.*

(2) *If M_1 is an S-coherent module and M_3 is a Nil_*-coherent module, then M_2 is an S-Nil_*-coherent module.*

(3) *If M_2 is an S-Nil_*-coherent module, then so is M_1.*

Proof. (1) Let N_3 be a finitely generated submodule of M such that $N_3 \subseteq Nil(R)M_3$. So, there is an exact sequence of R-modules $0 \to T_3 \to R^m \to N_3 \to 0$, where m is a positive integer and T_3 is an R-module. On the other hand, fix an exact sequence $0 \to T_1 \to R^n \to M_1 \to 0$, for some positive integer n and R-module T_1. Then, by the Horseshoe Lemma, we obtain the following exact commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_1 & \longrightarrow & T_2 & \longrightarrow & T_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R^n & \longrightarrow & R^{n+m} & \longrightarrow & R^m & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & u^{-1}(N_3) & \longrightarrow & N_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Since $u^{-1}(N_3)$ is finitely generated of the S-Nil_*-coherent module M_2 and $u^{-1}(N_3) \subseteq u^{-1}(Nil(R)M_3) \subseteq Nil(R)u^{-1}(M_3) = Nil(R)M_2 + \ker(u) \subseteq Nil(R)M_2$, we have that $u^{-1}(N_3)$ is S-finitely presented. So, by [7, Proposition 2.3], T_2 is S-finite, and hence T_3 is S-finite. Thus N_3 is S-finitely presented.
(2) Take a finitely generated N_2 of M_2 such that $N_2 \subseteq Nil(R)M_2$ and consider the exact sequence of R-modules $0 \to \ker(u|_{N_2}) \to N_2 \to u(N_2) \to 0$. Then, $u(N_2)$ is finitely generated R-submodule of $Nil(R)M_3$. As M_3 is an Nil_*-coherent module, we get that $u(N_2)$ is finitely presented. Which implies that $\ker(u|_{N_2})$ is finitely generated and so $\ker(u|_{N_2})$ is S-finitely presented since M is S-coherent. Therefore, by [7, Theorem 2.4], N_2 is S-finitely presented.
(3) It is clear that every submodule of an S-Nil_*-coherent module is also an S-Nil_*-coherent module. This completes the proof. □

Corollary 2.9. *Let R be a ring, S be a multiplicatively closed subset of R and I be an ideal of R such that $I \cap S = \emptyset$. Then the following statements hold:*

(1) *If R an S-Nil_*-coherent ring and I is a finitely generated ideal such that $I \subseteq Nil(R)$, then R/I is a $\pi(S)$-Nil_*-coherent ring, with $\pi : R \to R/I$ is the canonical epimorphism.*

(2) *If I is an S-coherent R-module and R/I is a Nil_*-coherent ring, then R is an S-Nil_*-coherent ring.*

Proof. (1) By applying Theorem 2.8 to the exact sequence $0 \to I \to R \to R/I \to 0$, we obtain that R/I is an S-Nil_*-coherent R-module. Now, let J/I be a finitely generated ideal of R/I such that $J/I \subseteq Nil(R/I)$. We must prove that J/I is a $\pi(S)$-finitely presented ideal of R/I. By [7, Proposition 2.6], it is enough to show that J/I is an S-finitely presented R-module. This, in turn, follows from the fact that J/I is a finitely generated R-submodule of R/I and $J/I \subseteq Nil(R)(R/I)$.
(2) Since R/I is a Nil_*-coherent ring, then R/I is a Nil_*-coherent R-module by the proof of [2, Corollary 2.3]. The result follows from Theorem 2.8 (2). □

Let R be a ring and S be a multiplicatively closed subset of R. We say that an R-module M is an S-pseudo coherent module if every finitely generated submodule of M is S-finitely presented.

Theorem 2.10. *Let R be a ring, S be a multiplicatively closed subset of R and $\{M_i \mid 1 \leq i \leq n\}$ be a family of R-modules. Then $M := \bigoplus_{i=1}^{n} M_i$ is an S-Nil_*-coherent R-module if and only if M_i is an S-Nil_*-coherent R-module, for each $i = 1, \ldots, n$.*

Proof. The necessity follows from Theorem 2.8(3). For the converse, we will prove the assertion for $n = 2$. Take a finitely generated submodule N of M such that $N \subseteq Nil(R)M$. For $i = 1, 2$, note $\pi_i : M \to M_i$ the canonical projection and set $N_i := \pi_i(N)$. One can see that N_1 and N_2 are finitely generated R-submodules of $Nil(R)M_1$ and $Nil(R)M_2$, respectively. Hence, N_1 and N_2 are S-pseudo coherent R-modules. By a similar argument of [7, Proposition 3.2], we have $N_1 \bigoplus N_2$ is an S-pseudo coherent R-module. Thus, N is an S-finitely presented R-module since N can be seen as an R-submodule of $N_1 \bigoplus N_2$, as required. □

Corollary 2.11. *Let R be a ring and S be a multiplicatively closed subset of R. If R admits a finitely generated faithful S-Nil_*-coherent R-module M, then R is an S-Nil_*-coherent ring.*

Proof. Let $M := \sum_{i=1}^{n} Rm_i$ and let $\phi : R \to \bigoplus_{i=1}^{n} Rm_i$ be the homomorphism of R-modules, given by $\phi(r) = \sum_{i=1}^{n} rm_i$. Since M is faithful, we conclude that ϕ is injective. Moreover, the fact that M is an S-Nil_*-coherent R-module gives that Rm_i is an S-Nil_*-coherent R-module for each $1 \leq i \leq n$. Hence $\bigoplus_{i=1}^{n} Rm_i$ is an S-Nil_*-coherent R-module and so is R, as needed. □

Corollary 2.12. *Let R be a ring, S be a multiplicatively closed subset of R and M be a finitely generated R-module. If N is an S-Nil_*-coherent R-module, then so is $Hom_R(M, N)$.*

Proof. Since M is a finitely generated module, then there exists an exact sequence of R-modules $0 \to K \to R^n \to M \to 0$. Hence $Hom_R(M, N)$ can be seen as an R-submodule of $Hom_R(R^n, N) \cong N^n$ because $Hom(., N)$ is left-exact. By Theorem 2.10, we have N^n is an S-Nil_*-coherent R-module and thus $Hom_R(M, N)$ is also an S-Nil_*-coherent R-module. □

Let $\{R_i \mid 1 \leq i \leq n\}$ be a family of rings and $R := R_1 \times \cdots \times R_n$ be the direct product of those rings. If S_i is a multiplicatively closed subset of R_i for each $i = 1, \ldots, n$, then $S := S_1 \times \cdots \times S_n$ is a multiplicatively closed subset of R.

Proposition 2.13. *Suppose that S_i is a multiplicatively closed subset of a ring R_i for each $i = 1, \ldots, n$. Let $R := R_1 \times \cdots \times R_n$ and $S := S_1 \times \cdots \times S_n$. Then R is an S-Nil_*-coherent ring if and only if R_i is an S_i-Nil_*-coherent ring for each $i = 1, \ldots, n$.*

Proof. Suppose that R is an S-Nil_*-coherent ring. Let I_1 be a finitely generated ideal of R_1 such that $I_1 \subseteq Nil(R_1)$. Then $I := I_1 \times 0 \times \cdots \times 0$ is finitely generated ideal of R such that $I \subseteq Nil(R)$. By assumption, we get that I is an S-finitely presented ideal of R, which implies that I_1 is an S_1-finitely presented ideal of R_1. Thus R_1 is an S-Nil_*-coherent ring. Similarly, we prove that R_i is an S_i-Nil_*-coherent ring, for $i = 2, \ldots, n$. For the converse, let $I := I_1 \times \cdots \times I_n$ be a finitely generated ideal of R such that $I \subseteq Nil(R)(= Nil(R_1) \times \cdots \times Nil(R_n))$. Hence, by hypothesis, I_i is an S_i-presented ideal of R_i, for each $i = 1, \ldots, n$. This yields that I is an S-finitely presented ideal of R. Hence R is an S-Nil_*-coherent ring, as desired. □

Let R and T be two rings and $\phi : R \to T$ be a ring homomorphism making T an R-module. Recall from [12] that R is called a retract of T (via ϕ) if there is a ring homomorphism $\psi : T \to R$ satisfying $\psi \circ \phi = id_R$; ψ is called a retraction of ϕ. It will be convenient to view ϕ as an inclusion map.

Theorem 2.14. *Let R and T be two rings, $\phi : R \to T$ be a ring homomorphism making R a retract of T and let ψ be a retraction of ϕ. If T is an S-Nil_*-coherent ring for some multiplicatively closed subset S of T such that* $\ker(\psi) \cap S = \emptyset$*, then R is a $\psi(S)$-Nil_*-coherent ring.*

Proof. Let $I := \sum_{i=1}^{n} Ra_i$ be a finitely generated ideal of R such that $I \subseteq Nil(R)$. Since T is an S-Nil_*-coherent ring, we then have IT is an S-finitely presented ideal of T. Take the exact sequence of T-modules $0 \to \ker(u) \to T^n \xrightarrow{u} IT \to 0$, where $u((t_i)_i) = \sum_{i=1}^{n} t_i a_i$. It follows that $\ker(u)$ is an S-finite submodule of T^n. Now, we consider the R-epimorphism $\psi^n : T^n \to R^n$ given by $\psi^n((t_i)_i) = (\psi(t_i))_i$. We will prove that $\psi^n(\ker(u))$ is a $\psi(S)$-finite R-module. Indeed, since $\ker(u)$ is an S-finite T-module, then $s\ker(u) \subseteq J \subseteq \ker(u)$ for some finitely generated T-submodule J of $\ker(u)$ and $s \in S$. Hence $\psi(s)\psi^n(\ker(u)) \subseteq \psi^n(J) \subseteq \psi^n(\ker(u))$ and $\psi^n(J)$ is a finitely generated R-module, we are done. On the other hand, let $v : R^n \to I$ be the canonical epimorphism of R-modules. One can prove that $\ker(v) = \psi^n(\ker(u))$ and thus I is a $\psi(S)$-finitely presented ideal of R, as desired. □

Proposition 2.15. *Let R be a ring and V be a multiplicatively closed subset of R such that $V \subseteq Reg(R)$. If R is an S-Nil_*-coherent ring for some multiplicatively closed subset S of R, then R_V is an S_V-coherent ring.*

Proof. Let J be a finitely generated ideal of R_V such that $J \subseteq Nil(R_V)$. Then there is a finitely generated R-submodule I of $Nil(R)$ such that $J = I_V$. Since R is an S-Nil_*-coherent ring, then I is an S-finitely presented ideal. Using [7, Lemma 3.10], we conclude that $J = I \otimes_R R_V$ is an S_V-finitely presented ideal of R_V, as needed. □

Theorem 2.16. *Let $(R, \mathfrak{m})$ be a local ring, S be a multiplicatively closed subset of R and M be an R-module such that $\mathfrak{m}M = 0$. Then $R \propto M$ is an $(S \propto M)$-Nil_*-coherent ring if and only if R is an S-Nil_*-coherent ring and M is an S-finite R-module.*

Proof. Suppose that $R \propto M$ is an $(S \propto M)$-Nil_*-coherent ring. Let $\phi : R \to R \propto M$ be the usual embedding (given by $\phi(r) = (r, 0)$) and let $\psi : R \propto M \to R$ be the R-module homomorphism, given by $\psi(r, m) = r$. Note that $\psi \circ \phi = id_R$, so R is a module retract of $R \propto M$ (via ϕ). Hence, by Theorem 2.14, R is an S-Nil_*-coherent ring since $S \propto M$ is disjoint with $\ker(\psi) = 0 \propto M$. Now, we will prove that M is an S-finite module over R. We may assume, without loss of generality, that $M \neq 0$. Pick a nonzero element $m \in M$. Since $R \propto M$ is an $(S \propto M)$-Nil_*-coherent ring, we then have $(0 : (0, m)) = \mathfrak{m} \propto M$ is an $(S \propto M)$-finite ideal of $R \propto M$. So, there exist an element $(s, m_1) \in S \propto M$ and a finite subset $\{(r_1, e_1), \ldots, (r_n, e_n)\}$ of $R \propto M$ such that $(s, m_1)(\mathfrak{m} \propto M) \subseteq ((r_1, e_1), \ldots, (r_n, e_n)) \subseteq \mathfrak{m} \propto M$. It follows that $sM \subseteq (e_1, \ldots, e_n)$, and thus M is an S-finite module.
Conversely, let J be a finitely generated ideal of $R \propto M$ such that $J \subseteq Nil(R \propto M)$ and let $X := \{(r_i, e_i) \mid 1 \leq i \leq n\}$ be a minimal generating set of J, where $r_i \in Nil(R)$ and $e_i \in M$. Consider the exact sequence of $(R \propto M)$-modules:

$$0 \to \ker(u) \to (R \propto M)^n \xrightarrow{u} J \to 0$$

where $u((a_i, f_i)_i) = \sum_{i=1}^{n}(a_i, f_i)(r_i, e_i) = \sum_{i=1}^{n}(a_i r_i, a_i e_i)$. Moreover, the minimality of X yields $\ker(u) = \{(a_i, f_i)_{1 \leq i \leq n} \in (R \propto M)^n \mid \sum_{i=1}^{n} a_i r_i = 0\}$. Now, let $I := \sum_{i=1}^{n} Rr_i$ and consider the R-module epimorphism $v : R^n \to I$, given by $v((b_i)_i) = \sum_{i=1}^{n} b_i r_i$. Then $\ker(v)$ is an S-finite R-module; that is, $s\ker(v) \subseteq (x_1, \ldots, x_l) \subseteq \ker(v)$ for some finite subset $\{x_1, \ldots, x_l\} \subseteq R$ and $s \in S$. Also, we have $tM \subseteq (e_1, \ldots, e_p)$ for some finite subset $\{e_1, \ldots, e_p\} \subseteq M$ and $t \in S$. Notice that $\ker(u) = \ker(v) \propto M$. It follows that

$$(st, 0)\ker(u) \subseteq ((x_1, 0), \ldots, (x_l, 0), (0, e_1), \ldots, (0, e_p)) \subseteq \ker(u).$$

Thus $\ker(u)$ is an $(S \propto M)$-finite ideal of $R \propto M$. This completes the proof. □

Corollary 2.17. *Let $(R, \mathfrak{m})$ be a local ring and S be a multiplicatively closed subset of R. Then R is an S-Nil_*-coherent ring if and only if $R \propto R/\mathfrak{m}$ is an $(S \propto R/\mathfrak{m})$-$Nil_*$-coherent ring.*

Next, we explore a different context, namely, the trivial ring extension of a domain by its quotient field.

Proposition 2.18. *Let R be a domain which is not a field, and K its quotient field. Then $R \propto K$ is not an S-Nil_*-coherent ring for every multiplicatively closed subset S of R.*

Proof. The result follows since $(0 : (0, 1)) = 0 \propto K$ is not an $(S \propto K)$-finite ideal. □

Let A and B be two rings, J be an ideal of B and $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f B = \{(a, f(a) + j) \mid a \in A \text{ and } j \in J\}$$

called the amalgamation of A with B along J with respect to f. This construction has been first indroduced and studied D'Anna, Finocchiaro, and Fontana in [8, 9]. In particular, if I is an ideal of A and $id_A : A \to A$ is the identity homomorphism on A, then $A \bowtie I = A \bowtie^{id_A} I = \{(a, a + i) \mid a \in R \text{ and } i \in I\}$ is the amalgamated duplication of A along J (introduced and studied by D'Anna and Fontana in [10]). The basic properties of the amalgamation extension are summarized in [11].

Theorem 2.19. *Let $f : A \to B$ be a ring homomorphism, J be an ideal of B and S be a multiplicatively closed subset of A such that* $\ker(f) \cap S = \emptyset$*. Let $R := A \bowtie^f J$ and $S' := \{(s, f(s)) \mid s \in S\}$. Then the following statements hold:*

(1) *If R is an S'-Nil_*-coherent ring, then A is an S-Nil_*-coherent ring.*

(2) *If $f^{-1}(J)$ is a finitely generated ideal of A such that $f^{-1}(J) \subseteq Nil(A)$ and R is an S'-Nil_*-coherent ring, then $f(A) + J$ is an $f(S)$-Nil_*-coherent ring.*

(3) *Assume that $f^{-1}(J)$ and J are finitely generated ideals of A and $f(A) + J$, respectively, and $f^{-1}(J) \subseteq Nil(A)$. If A is an S-Nil-coherent ring and $f(A) + J$ is a Nil_*-coherent ring then R is an S'-Nil_*-coherent ring.*

Proof. (1) Since A is a module retract of R via ϕ, where $\phi : R \to A \bowtie^f J$, $a \mapsto (a, f(a))$. So, by Theorem 2.14, A is an S-Nil_*-coherent ring.
(2) By assumption, we have $f^{-1}(J) \times \{0\}$ is a finitely generated ideal of R and $f^{-1}(J) \times \{0\} \subseteq Nil(R)$. Therefore, $f(A) + J \cong R/(f^{-1}(J) \times \{0\})$ is an $f(S)$-Nil_*-coherent ring by Corollary 2.9.
(3) Our task is to show that R is an S'-Nil_*-coherent ring. By Corollary 2.9(2), it remains to prove that $f^{-1}(J) \times \{0\}$ is an S'-coherent R-module. It is clear that $f^{-1}(J) \times \{0\}$ is a finitely generated R-module. On the other hand, let K be a finitely generated R-submodule of $f^{-1}(J) \times \{0\}$. So, $K := I \times \{0\}$ for some finitely generated ideal I of A. Write $I := \sum_{i=1}^n Aa_i$, where $a_i \in I$ for each i. Consider the natural exact sequence of A-modules $0 \to \ker(u) \to A^n \xrightarrow{u} I \to 0$, where $u((\alpha_i)_i) = \sum_{i=1}^n \alpha_i a_i$. Since $I \subseteq Nil(A)$ and A is an S-Nil_*-coherent ring, we have that $\ker(u)$ is an S-finite A-module. As $K = \sum_{i=1}^n R(a_i, 0)$, we get the following short exact sequence of R-modules $0 \to \ker(v) \to R^n \xrightarrow{v} K \to 0$, where $v : R^n \to K$ is given by

$$v((\alpha_i, f(\alpha_i) + j_i)_i) = \sum_{i=1}^n (\alpha_i, f(\alpha_i) + j_i)(a_i, 0) = \left(\sum_{i=1}^n \alpha_i a_i, 0 \right).$$

Note that $R^n \cong A^n \bowtie^{f^n} J^n$, where $f^n : A^n \to B^n$ is defined by $f^n((\alpha_i)_i) = (f(\alpha_i))_i$. Therefore, since $\ker(u)$ is an S-finite, then there are $s \in S$ and a finitely generated A-module F such that $s \ker(u) \subseteq F \subseteq \ker(u)$. Hence $(s, f(s))(\ker(u) \bowtie^{f^n} J^n) \subseteq F \bowtie^{f^n} J^n \subseteq \ker(u) \bowtie^{f^n} J^n$. By [1, Lemma 2.4], we have $\ker(u) \bowtie^{f^n} J^n$ is an S'-finite module and so is $\ker(v)$ because $\ker(v) \cong \ker(u) \bowtie^{f^n} J^n$ as R-modules. This completes the proof. □

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