Palestine Journal of Mathematics

Vol. 12(2)(2023) , 141–153

Some Properties of the Fibonacci-difference Graph and Bounds for its Energy

M Deepthi Rao, Chandrashekar Adiga and Anitha N

MSC 2010 Classifications: 05C50, 05C35.

Keywords and phrases: Fibonacci-difference graph, graph properties, toeplitz matrix, energy of a graph, bounds for the energy of a graph.

AbstractThe 'Fibonacci sequence', a unique and acclaimed series of integers, albeit simple and abstract in principle, plays a significant role in modern mathematics. In this paper, we introduce the Fibonacci-difference graph G_n and investigate some of its properties. We also present various bounds for the energy of G_n. Finally we introduce a new variant of graph energy called the equi-degree energy and obtain its spectrum for G_n whenever n is a Fibonacci number.

1 Introduction

The Fibonacci sequence $\{F_n\}_{n\geq 1}$ is defined by $F_1 = 1, F_2 = 2$ and for $n \geq 3$, $F_n = F_{n-1} + F_{n-2}$. For millennia, the Fibonacci sequence has been a mainstay of mathematical theory due to its remarkable precision in modeling real-world phenomena and its ability to provide elegant solutions to technical quandaries. The simplistic sequence, whose elements are derived by adding the previous two terms together, is ostensibly unrivalled in its applicability to patterns in nature, art, and technical disciplines. It is an invaluable implement utilized in algorithm development in computer science. Unlike binary search, Fibonacci search divides large arrays into unequal intervals and uses simple addition and subtraction operations rather than the division operation used in binary search. As a result, Fibonacci search examines relatively more nearby elements in subsequent steps, narrowing down data location search. Inspired by the work of A. Arman, David S. Gunderson, Pak Ching Li's [[2]] on the Fibonacci-sum graph, in this paper we introduce the Fibonacci-difference graph and analize some of its properties.

For each positive integer $n \geq 1$, the Fibonacci-difference graph $G_n = (V, E)$ is a simple graph defined on the vertex set $V = \{v_1, v_2, ..., v_n\}$ and the edge set $E = \{v_i v_j \mid |i - j| \text{ is a Fibonacci number for } i \neq j\}$.

The graph shown in the figure below is the Fibonacci-difference graph G_8.

In this paper the results corresponding to the Fibonacci-difference graph is structured as follows: In Section 2, some properties of Fibonacci-difference graph related to the vertex degree, the degree sequence and the connectivity of the graphs are discussed. In Section 3, we derive two upper bounds and a lower bound for the energy of the Fibonacci-difference graph. In Section 4, the adjacency matrix of the Fibonacci-difference graph which is a toeplitz matrix is studied by expressing it as a sum of two known matrices and hence a bound for its energy is obtained. Section 5 investigates a new variant of graph energy called equi-degree energy for the Fibonacci-difference graph.

2 Properties of the Fibonacci-difference graph

In this section we study certain basic properties of Fibonacci-difference graph.

Proposition 2.1. *The Fibonacci-difference graph G_n has a Hamiltonian path for all $n \geq 2$. Moreover it is not unique for $n \geq 3$.*

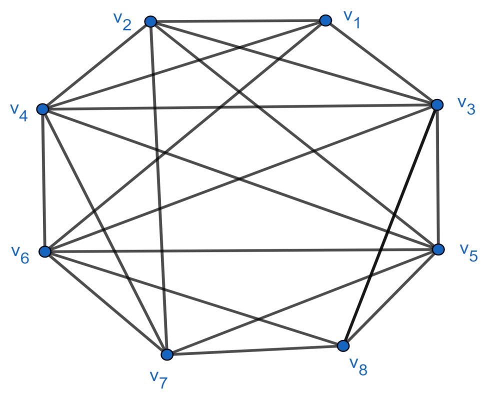

Figure 1. The Fibonacci-difference graph G_8

Proof. Since v_iv_{i+1} is an edge in G_n for $i = 1$ to $n-1$, we have $v_1, v_2, \ldots, v_n$ is a Hamiltonian path in G_n. For $n \geq 3$, $v_1, v_3, v_2, v_4, v_5, \ldots, v_n$ is also a Hamiltonian path in G_n. □

Proposition 2.2. *The Fibonacci-difference graph G_n is complete if and only if n is either 2,3 or 4.*

Proof. For $n = 2, 3$ and 4, G_n is complete. If $n \geq 5$, then there is no edge between v_1 and v_5. Hence G_n is not complete for $n \geq 5$. □

Proposition 2.3. *For $n \geq 3$, in G_n, every vertex has more than one neighbor.*

Proof. Note that v_2 and v_3 are neighbors of v_1. For, $2 \leq i \leq n-1$, v_{i-1} and v_{i+1} are neighbors of v_i. Also, v_{n-1} and v_{n-2} are adjacent to v_n. Thus each vertex has at least two neighbors. □

Proposition 2.4. *Let $n \geq 2$ and let d be such that $F_d \leq n < F_{d+1}$. Then*

$$deg(v_n) = \begin{cases} d-1 & \textit{if} \quad n = F_d, \\ d & \textit{if} \quad F_d < n < F_{d+1}. \end{cases}$$

Proof. In G_n, the vertex v_n is adjacent to

$$v_{n-F_1}, v_{n-F_2}, \ldots, v_{n-F_{d-1}} \quad \text{if} \quad n = F_d,$$

and v_n is adjacent to

$$v_{n-F_1}, v_{n-F_2}, \ldots, v_{n-F_d} \quad \text{if} \quad F_d < n < F_{d+1}.$$

Thus

$$deg(v_n) = \begin{cases} d-1 & \text{if} \quad n = F_d, \\ d & \text{if} \quad F_d < n < F_{d+1}. \end{cases}$$

□

Proposition 2.5. *For each $n \geq 1$, G_n is connected.*

Proof. For $n = 1$, the proof is obvious. Suppose $1 \leq i < j \leq n$. Then, v_i and v_j are connected by the path $v_i, v_{i+1}, v_{i+2}, \ldots, v_j$. □

Proposition 2.6. *Let $n \geq 2$ and d be such that $F_d \leq n < F_{d+1}$. Then in G_n, $deg(v_{F_d}) = d-1+k$, where 'k' is the number of Fibonacci numbers less than or equal to $n - F_d$.*

Proof. The neighbors of v_{F_d} are

$$v_{F_d-F_1}, v_{F_d-F_2}, ..., v_{F_d-F_{d-1}},$$

and

$$v_{F_d+F_1}, v_{F_d+F_2}, ..., v_{F_d+F_k},$$

where F_k is the largest Fibonacci number such that $F_k \leq n-F_d$. Thus, $deg(v_{F_d}) = d-1+k$. □

Theorem 2.7. *Let $n \geq 2$ and let $k \in [1, n]$. Let $N(0) = 0$ and for $m \geq 1$, $N(m)$ be the number of Fibonacci numbers less than or equal to m. Then*

$$deg(v_k) = N(k-1) + N(n-k).$$

Proof. Suppose $1 \leq i < k$. Then v_k is adjacent to v_i, if $k-i$ is a Fibonacci number. Thus among $v_1, v_2, ..., v_{k-1}$ there are $N(k-1)$ vertices adjacent to v_k. Suppose $k < i \leq n$. Then v_k and v_i are adjacent if and only if $i-k$ is a Fibonacci number. Thus among $v_{k+1}, v_{k+2}, ..., v_n$ there are $N(n-k)$ vertices adjacent to v_k. Hence the result. □

Corollary 2.8. *If $F_d \leq n < F_{d+1}$, then*

$$deg(v_1) = N(0) + N(n-1) = \begin{cases} d-1 & \text{if} \quad n = F_d, \\ d & \text{if} \quad n > F_d. \end{cases}$$

Corollary 2.9.

$$deg(v_k) = deg(v_{n-k+1}).$$

Proof. We have

$$deg(v_k) = N(k-1) + N(n-k),$$

and

$$deg(v_{n-k+1}) = N(n-k) + N(k-1).$$

Hence

$$deg(v_k) = deg(v_{n-k+1}).$$

This implies

$$deg(v_1) = deg(v_n), \quad deg(v_2) = deg(v_{n-1}), \quad deg(v_3) = deg(v_{n-2}), ...$$

□

Corollary 2.10. *If n is even, there are at least two vertices having maximum degree Δ in G_n.*

Proof. Suppose n is even. Since $N(m)$ is monotonically increasing function, $v_{\frac{n}{2}}$ and $v_{\frac{n}{2}+1}$ have maximum degree in G_n. Also, we have

$$deg(v_{\frac{n}{2}}) = deg(v_{\frac{n}{2}+1})$$

and hence

$$\Delta = deg\left(v_{\frac{n}{2}}\right) = N\left(\frac{n}{2}-1\right) + N\left(\frac{n}{2}\right) = \begin{cases} 2N(\frac{n}{2}-1) & \text{if} \quad \frac{n}{2} \quad \text{is not a Fibonacci number,} \\ 2N(\frac{n}{2}-1)+1 & \text{otherwise.} \end{cases}$$

□

Corollary 2.11. *If n is odd, then $v_{\frac{n+1}{2}}$ has maximum degree in G_n. In fact*

$$\Delta = deg(v_{\frac{n+1}{2}}) = 2N\left(\frac{n-1}{2}\right).$$

Remark 2.12. 1. If n is even, then maximum degree of G_n is even if $\frac{n}{2}$ is not a Fibonacci number and maximum degree of G_n is odd if $\frac{n}{2}$ is a Fibonacci number.
2. If n is odd, then maximum degree of G_n is even.

Corollary 2.13. *Let $n \geq 2$ and $F_d \leq n-1 < F_{d+1}$. Then the number of edges in G_n is given by*

$$|E(G_n)| = nd + 2 - F_{d+2}.$$

Proof. By Theorem 2.7, we have

$$\begin{aligned}
|E(G_n)| &= \frac{1}{2}\sum_{k=1}^{n} deg(v_k) \\
&= \frac{1}{2}\sum_{k=1}^{n}\{N(k-1) + N(n-k)\} \\
&= \frac{1}{2}\{\sum_{k=1}^{n} N(k-1) + \sum_{k=1}^{n} N(n-k)\} \\
&= \sum_{k=0}^{n-1} N(k) = \sum_{k=1}^{n-1} N(k) \\
&= (F_2 - F_1) + 2(F_3 - F_2) + 3(F_4 - F_3) + \ldots + (d-1)(F_d - F_{d-1}) + d(n - F_d) \\
&= -F_1 - F_2 - F_3 - \cdots - F_d + nd \\
&= -(F_{d+2} - 2) + nd \\
&= nd + 2 - F_{d+2}.
\end{aligned}$$

□

Example 2.14. For $n = 5$, since $F_3 < 4 < F_4$, by above theorem

$$|E(G_5)| = 9.$$

Theorem 2.15. *The degree sequence of $G_{F_d}, (d \geq 6)$ is*

$$(2d-4)^{F_{d-1}-F_{d-5}}, (2d-5)^{2F_{d-5}}, \ldots, (d+2)^{2F_2}, (d+1)^{2F_1}, d^2, (d-1)^2.$$

Here, α^m means degree α appears m times.

Proof. For $m = 0, 1, 2, \ldots, (d-4)$, define

$$A_m = B_m \cup C_m$$

where

$$B_0 = \{v_1\}, C_0 = \{v_{F_d}\},$$

and

$$B_m = \{v_k \mid F_m < k \leq F_{m+1}\},$$

$$C_m = \{v_k | F_d - F_{m+1} + 1 \leq k < F_d - F_m + 1\}, (1 \leq m \leq d-4).$$

Also, define $A_{d-3} = \{v_k | F_{d-3} + 1 \leq k \leq F_d - F_{d-3}\}$.
It is easy to check that $B_m \cap C_m = \phi$ for $m = 0, 1, 2, ..., (d-4)$, $A_i \cap A_j = \phi$ for $i \neq j$ and $V(G_{F_d}) = \bigcup_{m=0}^{d-3} A_m$.
Note that, $|B_0| = |C_0| = 1$, $|B_1| = |C_1| = 1$, $|B_m| = |C_m| = F_{m-1}$, $(2 \leq m \leq d-4)$.
Hence, $|A_0| = 2$, $|A_1| = 2$, $|A_m| = |B_m| + |C_m| = 2F_{m-1}$, $(2 \leq m \leq d-4)$.
Moreover, $|A_{d-3}| = F_d - 2F_{d-3} = F_{d-1} - F_{d-5}$.

Now, we shall show that degree of each vertex in A_m is same.

Case (i): Degree of each vertex in A_0 is $d-1$.
In fact neighbors of v_1 are $v_{F_1+1}, v_{F_2+1}, \ldots, v_{F_{d-1}+1}$ and hence $d(v_1) = d-1$. Also $d(v_{F_d}) = d(v_1) = d-1 \quad (\because d(v_k) = d(v_{F_d-k+1}))$.
Thus degree $(d-1)$ repeat 2 times.

Case (ii): Degree of each vertex in A_1 is d.
The neighbors of v_2 are $v_{F_1+2}, v_{F_2+2}, \ldots, v_{F_{d-1}+2}, v_{2-F_1}$.
So, $d(v_2) = d$. Also, $d(v_{F_d-1}) = d(v_2) = d$.
Hence two vertices have degree d.

Case (iii): Degree of each vertex in A_m, $(2 \le m \le d-4)$ is $(d-1+m)$.
Suppose $v_k \in B_m$, i.e., $F_m < k \le F_{m+1} \le F_{d-3}$.
The neighbors of v_k are $v_{F_1+k}, v_{F_2+k}, \ldots, v_{F_{d-1}+k}, v_{k-F_1}, v_{k-F_2}, \ldots, v_{k-F_m}$.
Thus, $d(v_k) = d-1+m$. Since $d(v_k) = d(v_{F_d-k+1})$, it follows that degree of each vertex in C_m $(2 \le m \le d-4)$ is also $(d-1+m)$. Hence $(d-1+m)$ repeat $|A_m| = 2F_{m-1}$ times $(2 \le m \le d-4)$.

Case (iv): Degree of each vertex in A_{d-3} is $(2d-4)$.
Suppose $v_k \in A_{d-3}$. Then, $k \in [F_{d-3}+1, F_d - F_{d-3}]$.
Since $[F_{d-3}+1, F_d - F_{d-3}] = [F_{d-3}+1, F_{d-2}) \cup [F_{d-2}, F_{d-1}) \cup [F_{d-1}, F_d - F_{d-3}]$,
$k \in [F_{d-3}+1, F_{d-2})$ or $k \in [F_{d-2}, F_{d-1})$ or $k \in [F_{d-1}, F_d - F_{d-3}]$.
Suppose $k \in [F_{d-3}+1, F_{d-2})$. We shall show that $d(v_k) = 2d-4$.
In fact neighbors of v_k are $v_{k+F_1}, v_{k+F_2}, \ldots, v_{k+F_{d-1}}, v_{k-F_1}, v_{k-F_2}, v_{k-F_{d-3}}$.
Thus $d(v_k) = 2d-4$.
Similarly in other two cases we can show that $d(v_k) = 2d-4$. Hence, $(2d-4)$ repeat $|A_{d-3}| = F_d - 2F_{d-3} = F_{d-1} - F_{d-5}$ times.
So, the degree sequence of $G_{F_d} (d \ge 6)$ is

$$(2d-4)^{F_{d-1}-F_{d-5}}, (2d-5)^{2F_{d-5}}, \ldots, (d+2)^{2F_2}, (d+1)^{2F_1}, d^2, (d-1)^2.$$

□

Example 2.16. The degree sequence of G_{F_6} is $\{8^7, 7^2, 6^2, 5^2\}$.
The degree sequence of G_{F_7} is $\{10^{11}, 9^4, 8^2, 7^2, 6^2\}$.

Theorem 2.17. *If $n \ge 3$, then G_n has no pendent vertices.*

Proof. G_3 has no pendent vertices. For $n \ge 4$ and $F_d \le n < F_{d+1}$, the minimum degree δ of G_n is given by

$$\delta = deg(v_1) = deg(v_n) = \begin{cases} d-1 & \text{if} \quad n = F_d, \\ d & \text{if} \quad n > F_d. \end{cases}$$

Thus

$$deg(v_k) \in \{d-1, d, d+1, \ldots, \Delta\},$$

and

$$2 \le d-1 \le \delta.$$

Thus

$$deg(v_k) \ne 1 \quad \text{for} \quad 1 \le k \le n.$$

□

Theorem 2.18. *For $n \ge 3$, G_n is not bipartite.*

Proof. G_2 is bipartite. For $n \ge 3$, v_1v_2, v_1v_3, v_2v_3 are adjacent in G_n which implies G_n contains a triangle. Hence G_n is not bipartite. □

Proposition 2.19. *For $n \ge 3$, girth of G_n is 3.*

Proof. In $G_n (n \ge 3)$, (v_1, v_2, v_3) is a cycle. More generally, (v_i, v_{i+1}, v_{i+2}) for $1 \le i \le (n-2)$, is a 3-cycle. Hence, the length of the shortest cycle in G_n is 3. □

Theorem 2.20. *Let n be a Fibonacci number. Then G_{n+1} contains a cycle of length $n+1$.*

Proof. Observe that $(v_1, v_2, v_3, \ldots, v_{n+1}, v_1)$ is a cycle in G_{n+1} of length $n+1$. □

Theorem 2.21. *Let $n \geq 5$. If $C = (v_1, v_2, \ldots, v_m)$ is a cycle in Fibonacci-difference graph G_n, then there do exist edges $(v_i v_k)$ and $(v_j v_l)$ in C with $i < j < k < l$. That is, there exists crossing chords inside C.*

Proof. If $C = (v_1, v_2, \ldots, v_m)$ is a cycle in G_n, $(n \geq 5)$. Then $\{v_2 v_4\}$ and $\{v_3 v_5\}$ are the crossing chords inside C. □

Theorem 2.22. *G_n is non outer planar.*

Proof. For $n \geq 4$, G_n contains a complete graph K_4 and also number of edges in G_n is greater than $2n - 3$ for $n > 1$. Therefore G_n is non outer planar. □

3 Bounds for energy of G_n

The eigen structure of the toeplitz matrices is a task and often required in variety of problems, including trigonometric moment problems, optimum filtering, stochastic processes and signal processing. The adjacency matrix of the graph G_n denoted by $A(G_n)$ is a symmetric toeplitz matrix of order n with the elements of the first row as $a_1, a_2, \ldots a_n$, where

$$a_i = \begin{cases} 1 & \text{if} \quad |i-1| \quad \text{is a Fibonacci number,} \\ 0 & \text{otherwise.} \end{cases}$$

The purpose of this section is to obtain bounds for the energy of G_n.
Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots \geq \lambda_n$ be the eigenvalues of $A(G_n)$ and m be the number of edges in G_n. As well known,

$$\sum_{i=1}^{n} \lambda_i = 0,$$

$$\sum_{i=1}^{n} \lambda_i^2 = 2m,$$

and

$$detA = \prod_{i=1}^{n} \lambda_i.$$

Also, The energy of G_n, denoted by $\varepsilon(G_n)$ is defined as

$$\varepsilon(G_n) = \sum_{i=1}^{n} |\lambda_i|.$$

This concept was introduced by I. Gutman and extensive research has been done on energy of graphs. We make use of the following well known lemmas to prove our results.

Lemma 3.1. *[5] Let G be a graph with $n \geq 2$ vertices and m-edges. Then for $1 \leq r \leq n$, we have*

$$\sqrt{\frac{2m(n-1)}{nr}} \geq \lambda_r \geq \sqrt{\frac{2m(r-1)}{n(n-r+1)}}.$$

Lemma 3.2 ([5]). *Let G be a simple graph with n vertices and having degree sequence $d_1 \geq d_2 \geq d_3 \geq \ldots \geq d_n$. Then, $\lambda_1 \geq \frac{d_1 + d_2}{\sqrt{2n}}$.*

Lemma 3.3. *[5] We have $\lambda_1(G) \leq \sqrt{2m - (n-1)d_n + (d_n - 1)d_1}$, where d_1 and d_n are the maximum and the minimum degrees of the vertices of the graph G respectively.*

Theorem 3.4. *For the graph G_n, we have*

$$\varepsilon(G_n) \leq \sqrt{\frac{(n-1)(4mn-(d_1+d_2)^2)}{2n}} + \sqrt{\frac{2m(n-1)}{n}},$$

where m is the number of edges and d_1 and d_2 are the first and the second largest degrees respectively.

Proof. Let

$$f(x) = x^2 - kx + 1.$$

The function $f(x)$ is increasing for $x \geq \frac{k}{2}$ and decreasing for $x < \frac{k}{2}$. So, $f(x) \geq f(\frac{k}{2})$.

$$x^2 + \frac{k^2}{4} \geq kx.$$

Or equivalently,

$$x \leq \frac{x^2}{k} + \frac{k}{4}, \quad k > 0. \tag{3.1}$$

We have,

$$\varepsilon(G_n) = \lambda_1 + \sum_{i=2}^{n} |\lambda_i|.$$

Using (3.1), we have

$$\varepsilon(G_n) \leq \lambda_1 + \sum_{i=2}^{n} \frac{|\lambda_i|^2}{k} + \sum_{i=2}^{n} \frac{k}{4}.$$

Employing Lemma 3.1 and Lemma 3.2 in the above inequality we deduce

$$\varepsilon(G_n) \leq \sqrt{\frac{2m(n-1)}{n}} + \frac{2m}{k} - \frac{(d_1+d_2)^2}{2nk} + \frac{k(n-1)}{4}.$$

Let

$$g(x) = \sqrt{\frac{2m(n-1)}{n}} + \frac{2m}{x} - \frac{(d_1+d_2)^2}{2nx} + \frac{x(n-1)}{4}. \tag{3.2}$$

Then we can verify that the point at which the function $g(x)$ attains its minimum value is

$$x = \sqrt{\frac{2(4mn-(d_1+d_2)^2)}{n(n-1)}}. \tag{3.3}$$

On substituting (3.3) in (3.2) we get the required result. □

Lemma 3.5. *[4] If G is a graph with n vertices with λ_1 being the largest eigenvalue, then*

$$\lambda_1(G) \geq \sqrt{d_1}$$

where d_1 is the highest degree.

Lemma 3.6. *[3] If G is a graph with n vertices and the clique number ω, then,*

$$\lambda_1 \leq \sqrt{\frac{2m(\omega-1)}{\omega}}.$$

Theorem 3.7. *If G_n is non-singular, then we have*

$$\varepsilon(G_n) \leq \sqrt{\frac{3m}{2}} + \frac{1}{4}ln\left(\frac{\sqrt{\frac{3m}{2}}}{|detA|}\right) + \frac{m}{2} - \frac{d_1}{4} + 2(n-1),$$

where m is the number of edges and d_1 is the highest degree in G_n.

Proof. If G_n is non-singular, we have $|\lambda_i| > 0$ for $i = 1, 2, \ldots, n$. Thus,

$$|detA| = \prod_{i=1}^{n} |\lambda_i| > 0.$$

Also we have

$$\sum_{i=1}^{n} {\lambda_i}^2 = 2m.$$

Now consider the function $f(x) = x^2 - 4x - lnx, \quad x > 0$.
Since $f'(x) = 2x - 4 - \frac{1}{x}$, $f(x)$ increases for $x \geq 1 + \frac{\sqrt{6}}{2}$ and decreases for $x < 1 + \frac{\sqrt{6}}{2}$. Thus,

$$f(x) \geq f\left(1 + \frac{\sqrt{6}}{2}\right).$$

This implies,

$$x^2 - 4x - lnx \geq \left(1 + \frac{\sqrt{6}}{2}\right)^2 - 4\left(1 + \frac{\sqrt{6}}{2}\right) - ln\left(1 + \frac{\sqrt{6}}{2}\right),$$

which implies,

$$x \leq \frac{x^2}{4} - \frac{lnx}{4} + \frac{ln\left(1 + \frac{\sqrt{6}}{2}\right)}{4} + \frac{2\sqrt{6} + 3}{8},$$

or

$$x \leq \frac{x^2}{4} - \frac{lnx}{4} + 2. \tag{3.4}$$

Using (3.4), Lemma 3.5 and Lemma 3.6 we have,

$$\begin{aligned}
\varepsilon(G_n) &= \lambda_1 + \sum_{i=2}^{n} |\lambda_i| \\
&\leq \lambda_1 + \sum_{i=2}^{n} \frac{|\lambda_i|^2}{4} - \sum_{i=2}^{n} \frac{ln|\lambda_i|}{4} + \sum_{i=2}^{n} 2 \\
&\leq \sqrt{\frac{3m}{2}} + \frac{1}{4}(2m - {\lambda_1}^2) - \sum_{i=1}^{n} \frac{ln|\lambda_i|}{4} + \frac{ln(\lambda_1)}{4} + 2(n-1) \\
&\leq \sqrt{\frac{3m}{2}} + \left(\frac{m}{2} - \frac{d_1}{4}\right) - \frac{ln|detA|}{4} + \frac{1}{4} ln\left(\sqrt{\frac{3m}{2}}\right) + 2(n-1).
\end{aligned}$$

Hence the result. □

Theorem 3.8. *We have*

$$\varepsilon(G_n) \geq \frac{2md_1 - n\sqrt{2m - (n-1)d_n + (d_n - 1)d_1}}{d_1\sqrt{2m - (n-1)d_n + (d_n - 1)d_1} - 1}.$$

Proof. Let $p_1, p_2, p_3, \ldots, p_n$ and $q_1, q_2, q_3 \ldots q_n$ be the real numbers for which there exists real constants r and R such that for each i, $i = 1, 2, 3 \ldots n$ and
$rp_i \leq q_i \leq Rp_i$, the following inequality is valid [7]:

$$\sum_{i=1}^{n} q_i^2 + rR \sum_{i=1}^{n} p_i^2 \leq (r + R) \sum_{i=1}^{n} p_i q_i. \tag{3.5}$$

Equality in equation (3.5) holds if and only if for at least one i, $1 \leq i \leq n$, $rp_i = q_i = Rp_i$.
For $p_i = \frac{1}{d_1}$, $r = -1$, $R = d_1\sqrt{2m - (n-1)d_n + (d_n - 1)d_1}$ and $q_i = |\lambda_i|$ in equation (3.5) we

get,

$$\sum_{i=1}^{n}|\lambda_i|^2-\frac{n}{d_1}\sqrt{2m-(n-1)d_n+(d_n-1)d_1}$$
$$\leq\left(\sqrt{2m-(n-1)d_n+(d_n-1)d_1}-\frac{1}{d_1}\right)\sum_{i=1}^{n}|\lambda_i|.$$

Using the fact that $\sum_{i=1}^{n}\lambda_i^2=2m$, in the above equation, we get

$$\frac{2md_1-n\sqrt{2m-(n-1)d_n+(d_n-1)d_1}}{d_1}$$
$$\leq\left(\sqrt{2m-(n-1)d_n+(d_n-1)d_1}-\frac{1}{d_1}\right)\sum_{i=1}^{n}|\lambda_i|.$$

Hence the result. □

4 Bounds for the energy of the graph G_{F_d}

Graphs with Fibonacci number of vertices are mathematically similar to hypercube graphs. These graphs support efficient protocol for routing and broadcasting in distributed computations. They serve as a network topological descriptor in parallel computing. In this section, we derive a new bound for the energy of the graph G_{F_d} .

4.1 Splitting of a toeplitz matrix as the sum of Circulant and Skew Circulant matrices

Using the concept explained in the papers [8] and [6], we split the adjacency matrix $A(G_{F_d})$ as the sum of a circulant matrix C and a skew-circulant matrix S whose first rows $(c_1,c_2,c_3,\ldots,c_n)$ and $(c'_1,c'_2,c'_3,\ldots,c'_n)$ are given by

$$c_1=a_1=0, c_2=\frac{a_2+a_n}{2}, c_3=\frac{a_3+a_{n-1}}{2},\ldots,c_{n-1}=\frac{a_{n-1}+a_3}{2}, c_n=\frac{a_n+a_2}{2},$$

and

$$c'_1=a_1=0, c'_2=\frac{a_2-a_n}{2}, c'_3=\frac{a_3-a_{n-1}}{2},\ldots,c'_{n-1}=\frac{a_{n-1}-a_3}{2}, c'_n=\frac{a_n-a_2}{2}.$$

Here $a_1,a_2,a_3,\ldots,a_n$ are the first row elements of $A(G_{F_d})$. If $\nu_1,\nu_2,\nu_3,\ldots,\nu_n$ are non-increasing eigenvalues of C and $\tau_1,\tau_2,\tau_3,\ldots,\tau_n$ are the non-increasing eigenvalue of S, then

$$\nu_j=\sum_{k=1}^{n}c_k e^{\frac{2\pi i(j-1)(k-1)}{n}} \quad \text{for} \quad j=1,2,\ldots,n$$

and

$$\tau_j=\sum_{k=1}^{n}c'_k e^{\frac{\pi i(2j-1)(k-1)}{n}} \quad \text{for} \quad j=1,2,\ldots,n.$$

Nikiforov [9] recognized that the energy of the graph is equal to the sum of the singular values of its adjacency matrix and hence

$$\sum_{j=1}^{n}s_j(C)=\sum_{j=1}^{n}|\nu_j|$$

and
$$\sum_{j=1}^{n} s_j(S) = \sum_{j=1}^{n} |\tau_j|$$
where $s_j(C), j = 1, 2, 3, \ldots, n$ are the singular values of C and $s_j(S), j = 1, 2, 3, \ldots, n$ are the singular values of S respectively. Now we state the Fan's lemma [10] which we will be using to prove our main result.

Lemma 4.1. *Let X , Y and Z be the square matrices of order n such that $Z = X + Y$. Then*
$$\sum_{i=1}^{n} s_i(Z) \leq \sum_{i=1}^{n} s_i(X) + \sum_{i=1}^{n} s_i(Y).$$
Equality holds if and only if there exists an orthogonal matrix P, such that PX and PY are both positive semi-definite.

Theorem 4.2. *If λ_j for $j = 1, 2, 3, \ldots, n$ are the eigenvalues of the toeplitz adjacency matrix $A(G_{F_d})$ and ν_j and τ_j are the eigenvalues of circulant and skew-circulant matrices respectively, then*
$$\varepsilon(G_{F_d}) \leq 2n(d-1).$$

Proof. Using Fan's Lemma 4.1, we have,
$$\sum_{j=1}^{n} s_j(C+S) \leq \sum_{j=1}^{n} s_j(C) + \sum_{j=1}^{n} s_j(S)$$
$$\begin{aligned}
\sum_{j=1}^{n} |\lambda_j| &\leq \sum_{j=1}^{n} |\nu_j| + \sum_{j=1}^{n} |\tau_j| \\
&\leq \sum_{j=1}^{n} \left(\sum_{k=1}^{n} |c_k e^{\frac{2\pi i(j-1)(k-1)}{n}}| + \sum_{k=1}^{n} |c'_k e^{\frac{\pi i(k-1)(2j-1)}{n}}| \right) \\
&\leq n \left(\sum_{k=1}^{n} |c_k| + \sum_{k=1}^{n} |c'_k| \right) \\
&= n \left(\sum_{k=2}^{n} |\frac{a_k + a_{n-k+2}}{2}| + \sum_{k=2}^{n} |\frac{a_k - a_{n-k+2}}{2}| \right) \\
&\leq n \left(\sum_{k=2}^{n} |a_k| + \sum_{k=2}^{n} |a_{n-k+2}| \right) \\
&= 2n \sum_{k=2}^{n} |a_k| \\
&= 2n(d-1).
\end{aligned}$$

Hence the result. □

5 Equi-degree energy

Many authors have defined various types of graph energy. Inspired by the work of Color energy by C. Adiga et al. [1], in this paper we introduce a new graph energy called equi-degree energy of a graph G. We define the equi-degree adjacency matrix $[e_{ij}]$, $i, j = 1, 2, 3, \ldots, n$. Where,
$$e_{ij} = \begin{cases} 1 & \text{if} \quad d(v_i) = d(v_j) \quad \text{for} \quad i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

The graph G_{F_d} has many vertices with equal degree. We can observe that the equi-degree matrix is a real symmetric matrix. In the next theorem we shall show that equi-degree eigenvalues of G_{F_d} are all integers.

Theorem 5.1. *The equi-degree eigenvalues of G_{F_d} graph are $F_{d-1} - F_{d-5} - 1$, $2F_{d-5} - 1, \ldots,$ $2F_2 - 1$, $1(3 \quad times)$, $-1(F_d + 2 - d) \quad times)$, and hence the equi-degree energy is given by*

$$\varepsilon(EDE(G_{F_d})) = 2(F_d + 2 - d).$$

Proof. Let $E = (e_{ij})_{F_d \times F_d}$ where,

$$e_{ij} = \begin{cases} 1 & \text{if} \quad d(v_i) = d(v_j) \quad \text{for} \quad i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

The rows of $E - \lambda I$ are as follows:
$R_1 = (e_{11}, e_{12}, \ldots, e_{1F_d})$, where

$$e_{1j} = \begin{cases} -\lambda & \text{if} \quad j = 1, \\ 1 & \text{if} \quad j = F_d, \\ 0 & \text{otherwise.} \end{cases}$$

$R_2 = (e_{21}, e_{22}, \ldots, e_{2F_d})$, where

$$e_{2j} = \begin{cases} -\lambda & \text{if} \quad j = 2, \\ 1 & \text{if} \quad j = F_d - 1, \\ 0 & \text{otherwise.} \end{cases}$$

$R_3 = (e_{31}, e_{32}, \ldots, e_{3F_d})$, where

$$e_{3j} = \begin{cases} -\lambda & \text{if} \quad j = 3, \\ 1 & \text{if} \quad j = F_d - 2, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$E_s = \{F_s{+}1, F_s{+}2, \ldots, F_{s+1}, F_d{-}F_{s+1}{+}1, F_d{-}F_{s+1}{+}2, \ldots, F_d{-}F_s\}, \quad \text{for} \quad (3 \leq s \leq d{-}4)$$

and

$$E_{d-3} = \{F_{d-3} + 1, F_{d-3} + 2, \ldots, F_d - F_{d-3}\}.$$

Observe that $|E_s| = 2F_{s-1}$ for $(3 \leq s \leq d - 4)$ and $|E_{d-3}| = F_{d-1} - F_{d-5}$.
If $m \in E_s, (3 \leq s \leq d - 4)$, then
$R_m = (e_{m1}, e_{m2}, \ldots, e_{mF_d})$ where

$$e_{mj} = \begin{cases} -\lambda & \text{if} \quad j = m, \\ 1 & \text{if} \quad j \in E_s, \quad j \neq m, \\ 0 & \text{otherwise.} \end{cases}$$

If $m \in E_{d-3}$ then $R_m = (e_{m1}, e_{m2}, \ldots, e_{mF_d})$ where

$$e_{mj} = \begin{cases} -\lambda & \text{if} \quad j = m, \\ 1 & \text{if} \quad j \in E_{d-3}, \quad j \neq m, \\ 0 & \text{otherwise.} \end{cases}$$

$$R_{F_d-2} = (0, 0, 1, 0, 0, \ldots, 0, -\lambda, 0, 0),$$

$$R_{F_d-1} = (0,1,0,0,\ldots,0,-\lambda,0),$$

and

$$R_{F_d} = (1,0,0,\ldots,0,0,-\lambda).$$

Replacing

$$\begin{gathered}R_1 \text{ by } R_1 + R_{F_d},\ R_2 \text{ by } R_2 + R_{F_d-1},\ R_3 \text{ by } R_3 + R_{F_d-2},\\ R_{F_2+1} \text{ by } R_{F_3+1} + R_{F_3+2} + \ldots + R_{F_4} + R_{F_d-F_4+1} + R_{F_d-F_4+2} + \ldots + R_{F_d-F_3},\\ \vdots \\ R_{F_{d-4}+1} \text{ by } R_{F_{d-4}+1} + R_{F_{d-4}+2} + \ldots + R_{F_{d-3}} + R_{F_d-F_{d-3}+1} + \ldots + R_{F_d-F_{d-4}},\\ R_{F_{d-3}+1} \text{ by } R_{F_{d-3}+1} + R_{F_{d-3}+2} + \ldots + R_{F_d-F_{d-3}},\end{gathered}$$

and after some simplifications we get

$$|E-\lambda I| = (1-\lambda)^3(2F_2-1-\lambda)(2F_3-1-\lambda)\ldots(2F_{d-5}-1-\lambda)(F_{d-1}-F_{d-5}-1-\lambda)|F|$$

where F is $F_d \times F_d$ matrix whose rows are

$$\begin{aligned}R'_1 &= R_1 \text{ with } -\lambda \text{ replaced by } 1\\ R'_2 &= R_2 \text{ with } -\lambda \text{ replaced by } 1\\ R'_3 &= R_3 \text{ with } -\lambda \text{ replaced by } 1\\ R'_{F_3+1} &= R_{F_3+1} \text{ with } -\lambda \text{ replaced by } 1\\ R'_{F_4+1} &= R_{F_4+1} \text{ with } -\lambda \text{ replaced by } 1\\ &\ \vdots\\ R'_{F_{d-4}+1} &= R_{F_{d-4}+1} \text{ with } -\lambda \text{ replaced by } 1\\ R'_{F_{d-3}+1} &= R_{F_{d-3}+1} \text{ with } -\lambda \text{ replaced by } 1\end{aligned}$$

and

$$R'_m = R_m \text{ if } m \neq 1,2,3,F_3+1,F_4+1,\ldots,F_{d-3}+1.$$

Perform the following row operations on F:

$$\begin{gathered}R'_{F_d} \to (-R'_1) + R'_{F_d}\\ R'_{F_d-1} \to (-R'_2) + R'_{F_d-1}\\ R'_{F_d-2} \to (-R'_3) + R'_{F_d-2}\\ R'_k \to (-R'_{F_s+1}) + R'_k, \quad k \in E_s, \quad k \neq F_s+1, (3 \leq s \leq d-4)\\ R'_{F_{d-3}+2} \to (-R'_{F_{d-3}+1}) + R'_{F_{d-3}+2}\\ \vdots\\ R'_{F_d-F_{d-3}} \to (-R'_{F_{d-3}+1}) + R'_{F_d-F_{d-3}}.\end{gathered}$$

After some simplifications we obtain

$$\begin{aligned}|F| &= (-1-\lambda)^{3+(2F_2-1)+(2F_3-1)+\ldots+(2F_{d-5}-1)+(F_{d-1}-F_{d-5}-1)}|C|\\ &= (-1-\lambda)^{F_d+2-d}|C|,\end{aligned}$$

where C is an upper triangle matrix whose diagonal entries are 1. Hence $|C| = 1$. Thus equidegree specturm of G_{F_d} is

$$\begin{pmatrix}F_{d-1}-F_{d-5}-1, & 2F_{d-5}-1, & 2F_{d-4}-1, & \ldots, & 2F_2-1, & 1, & -1\\ 1, & 1, & 1, & \ldots, & 1, & 3, & F_d+2-d\end{pmatrix}.$$

□

This completes the proof.

References

[1] C. Adiga, E. Sampathkumar, M.A. Sriraj, A.S. Shrikanth, *Color Energy of a Graph*, Proc. Jangjeon Math. Soc., **16(3)**, (2013), 335-351.

[2] A. Arman, David S. Gundersona, Pak Ching Li, *Properties of the Fibonacci-sum graph*, **arXiv:1710.10303**, math.CO, (2017).

[3] Pintu Bhunia, Santanu Bag , Kallol Paul , *Bounds for eigenvalues of the adjacency matrix of a graph*, J. Interdiscip. Math., **22(4)**, (2019), 415-431.

[4] O. Favaron, M. Maheo and J. F. Sacle, *Some eigenvalue properties in graphs (Conjecture of Graffitic-11)*, Discrete. Math., **111**, (1993), 197-220.

[5] Akbar Jahanabani, Honnathan Rodríguez Zambrano, *Koolen-Moulton-type upper bounds on the energy of a graph*, MATCH Commun. Math. Comput. Chem., **83**, (2020), 497-518.

[6] Zhongyun Liu, Siheng Chen, Weijin Xu, Yulin Zhang, *The eigen-structures of real(skew) circulant matrices with some applications*, Comput. Appl. Math., **38:178**, (2019).

[7] Ž. Milovanović, E.I. Milovanović, A. Zakić, *A Short Note on Graph Energy*, MATCH Commun. Math. Comput. Chem., **72**, (2014), 179-182.

[8] Michael K. Ng, *Circulant and skew-circulant splitting methods for Toeplitz systems*, J. Comput. Appl. Math., **159(1)**, (2003), 101- 108.

[9] V. Nikiforov, *The energy of graphs and matrices*, J. Math. Anal. Appl., **326**, (2007), 1472-1475.

[10] Wasin So, María Robbianob, Nair Maria Maia de Abreuc, Ivan Gutman, *Applications of a theorem by Ky Fan in the theory of graph energy*, Linear Algebra Appl., **432**, (2010), 2163-2169.

Author information

M Deepthi Rao, Department of Science and Humanities, PES University,100ft. Ring Road, BSK 3rd Stage, Bengaluru - 560085.,.
E-mail: `deepthirao@pes.edu`

Chandrashekar Adiga, Department of Studies in Mathematics, University of Mysore, Manasagangothri, Mysuru - 570 006., INDIA.
E-mail: `c_adiga@hotmail.com`

Anitha N, Department of Science and Humanities, PES University, 100ft. Ring Road, BSK 3rd Stage, Bengaluru - 560085, INDIA.
E-mail: `nanitha@pes.edu`

Received: 2021-08-10.
Accepted: 2022-01-29.