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Some L_P-Type Inequalities for Generalized Polar Derivative

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Abstract For the class of polynomials $\mathcal{P}$ of degree n having all their zeros in $|z| \leq k$ where $k \leq 1$, N. A. Rather *et al.* [8] prove that, for each $p > 0$ and for $\alpha \in C$ with $|\alpha| \geq k$,

$$n\big(|\alpha| - k\big)\left\{\int_0^{2\pi}\left|\frac{P\left(e^{i\theta}\right)}{D_\alpha P\left(e^{i\theta}\right)}\right|^p d\theta\right\}^{\frac{1}{p}} \leq \left\{\int_0^{2\pi}\left|1 + ke^{i\theta}\right|^p d\theta\right\}^{\frac{1}{p}}. \tag{0.1}$$

In this paper, we extend inequality (0.1) to the class of *generalized polar derivative,* defined as

$$N_\alpha^\gamma\big[P\big](z) = \sum_{k=1}^{n}\gamma_k P(z) + (\alpha - z)G_\gamma\big[P\big](z)$$

which is analogous respected to the polar derivative of a polynomial. Moreover, a numerical example is presented to show obtained results are best possible.

1 Introduction

Let $\mathcal{P}$ denote the space of all algebraic polynomials of the form $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree n and let $P'(z)$ be its derivative, $\mathcal{P}_n$ denotes the collection of all monic polynomial of degree n in $\mathcal{P}$. Let R_+^n be the set of all n tuples $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ of positive real numbers with $\sum_{k=1}^{n}\gamma_k = n$, the polynomial of degree $n-1$, $G_\gamma\big[P\big](z) = \sum_{k=1}^{n}\gamma_k\prod_{j=1,j\neq k}^{n}(z - z_j)$ is called generalized derivative [10] of $P(z) \in \mathcal{P}_n$, where $z_1, z_2, \ldots, z_n$ are the zeros of $P(z)$. For $\gamma = (1, 1, \ldots, 1)$, the generalized derivative $G_\gamma\big[P\big](z)$ reduces to ordinary derivative $P'(z)$.
The problem of the extremal properties of polynomials piqued the interest of famous chemist Mendeleev in the second half of the nineteenth century, who was looking for the bound of the derivative of a special type of polynomial. Paul Turán [11] was the first who estimated the lower bound for the maximum modulus of derived polynomial in terms of maximum modulus of polynomial. In fact he proved for polynomial $P(z)$ of degree n, if $P(z)$ has all its zeros in $|z| \leq 1$, then

$$n\max_{|z|=1}\big|P(z)\big| \leq 2\max_{|z|=1}\big|P'(z)\big|. \tag{1.1}$$

In the inequality (1.1), equality will be hold for Inequality $P(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta| \neq 0$. As an extension of (1.1), Malik [5] proved that, if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$ where $k \leq 1$, then

$$n\max_{|z|=1}\big|P(z)\big| \leq (1 + k)\max_{|z|=1}\big|P'(z)\big|. \tag{1.2}$$

In the inequality (1.2), equality will be hold for $P(z) = (z + k)^n$, where $k \leq 1$. We know that from the analysis that if $P \in \mathcal{P}_n$ then for each $p > 0$

$$\lim_{p\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}} = \max_{|z|=1}|P(z)|.$$

In literature (see [1], [3], [6]) there exists several generalizations on L_p- type inequalities. Malik [6] obtained a generalization of (1.1) in the sense that the left - hand side of (1.1) is replaced by a factor involving the integral mean of $|P(z)|$ on $|z| = 1$. In fact, he proved that, if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for each $p > 0$

$$n\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}} \leq \left\{\int_0^{2\pi}\left|1+e^{i\theta}\right|^p d\theta\right\}^{\frac{1}{p}} \max_{|z|=1}\left|P'(z)\right|. \tag{1.3}$$

The corresponding extension of (1.2), which is a generalization of (1.3), was obtained by Aziz[1] who proved that, if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$ where $k \leq 1$, then for each $p > 0$

$$n\left\{\int_0^{2\pi}\left|\frac{P\left(e^{i\theta}\right)}{P'\left(e^{i\theta}\right)}\right|^p d\theta\right\}^{\frac{1}{p}} \leq \left\{\int_0^{2\pi}\left|k+e^{i\theta}\right|^p d\theta\right\}^{\frac{1}{p}}. \tag{1.4}$$

The result is best possible and equality in above inequality holds for $P(z) = (\alpha z + \beta k)^n$ where $|\alpha| = |\beta|$.

The polar derivative of a $P(z)$ of degree n with respect to a point $\alpha \in C$, is given by

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha P(z)}{\alpha} = P'(z).$$

As an extension of (1.2) to the polar derivative, Aziz and Rather [2] proved that, if all the zeros of $P(z)$ lie in $|z| \leq k$ where $k \leq 1$, then for $\alpha \in C$ with $|\alpha| \geq k$,

$$n(|\alpha| - k)\max_{|z|=1}\left|P(z)\right| \leq (1+k)\max_{|z|=1}\left|D_\alpha P(z)\right|. \tag{1.5}$$

As Aziz and Rather [2] extended inequality (1.2) to the polar derivative by similar way N.A. Rather *et al.* [8] extended inequality (1.4) to the polar derivative, in fact they proved that if all the zeros of $P(z)$ lie in $|z| \leq k$ where $k \leq 1$, then for $\alpha \in C$ with $|\alpha| \geq k$, and for each $p > 0$,

$$n\big(|\alpha| - k\big)\left\{\int_0^{2\pi}\left|\frac{P\left(e^{i\theta}\right)}{D_\alpha P\left(e^{i\theta}\right)}\right|^p d\theta\right\}^{\frac{1}{p}} \leq \left\{\int_0^{2\pi}\left|1+ke^{i\theta}\right|^p d\theta\right\}^{\frac{1}{p}}. \tag{1.6}$$

The result is best possible and equality in above inequality holds for $P(z) = (z-k)^n$.

Definition 1.1. Let R_+^n be the set of all n tuples $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ of positive real numbers with $\sum_{k=1}^n \gamma_k = n$, define

$$N_\alpha^\gamma\big[P\big](z) = \sum_{k=1}^n \gamma_k P(z) + (\alpha - z)G_\gamma\big[P\big](z)$$

with $\gamma \in R_+^n$ as the *generalized polar derivative* of $P(z) \in \mathcal{P}_n$. For $\gamma = (1, 1, \ldots, 1)$, then $N_\alpha^\gamma[P](z) = D_\alpha P(z)$.

In this direction number of papers has been recently published (see [4], [9],[10]).

In this paper, we extend the inequality (1.6) to the class of *generalized polar derivative* of a polynomial. We begin by proving the following result,

Theorem 1.2. *If $P(z) \in \mathcal{P}_n$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \leq 1$, then for $\alpha \in C$ with $|\alpha| \geq k$, and for each $p > 0$*

$$n\big(|\alpha| - k\big)\left\{ \int_0^{2\pi} \left| \frac{P\left(e^{i\theta}\right)}{N_\alpha^\gamma P\left(e^{i\theta}\right)} \right|^p d\theta \right\}^{\frac{1}{p}} \leq \left\{ \int_0^{2\pi} \left|1 + ke^{i\theta}\right|^p d\theta \right\}^{\frac{1}{p}} \tag{1.7}$$

where $\gamma \in R_+^n$.

Remark 1.3. For $\gamma = (1, 1, \ldots, 1)$ in inequality (1.7) we get inequality (1.6).

Using the fact that $\left|N_\alpha^\gamma [P] \left(e^{i\theta}\right)\right| \leq \max\limits_{|z|=1} |N_\alpha^\gamma [P] \left(e^{i\theta}\right)|, \quad 0 \leq \theta < 2\pi$, in (1.7) we obtain the following result

Corollary 1.4. *If $P(z) \in \mathcal{P}_n$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \leq 1$, then for $\alpha \in C$ with $|\alpha| \geq k$, and for each $p > 0$*

$$n\big(|\alpha| - k\big)\left\{ \int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^p d\theta \right\}^{\frac{1}{p}} \leq \left\{ \int_0^{2\pi} \left|1 + ke^{i\theta}\right|^p d\theta \right\}^{\frac{1}{p}} \max_{|z|=1} \left|N_\alpha^\gamma [P] \left(e^{i\theta}\right)\right| \tag{1.8}$$

where $\gamma \in R_+^n$.

Next, we obtain a generalization of above Corollary 1 in the sense that maximum of $\left|N_\alpha^\gamma [P] (z)\right|$ on the boundary of $|z| \leq 1$ on the right side of (1.8) is replaced by a factor involving integral mean of $\left|N_\alpha^\gamma [P] (z)\right|$ on the boundary of $|z| \leq 1$. In fact, we prove

Theorem 1.5. *If $P(z) \in \mathcal{P}_n$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \leq 1$, then for $\alpha \in C$ with $|\alpha| \geq k$, and for each $p > 0, \quad s > 1, \quad t > 1$, with $s^{-1} + t^{-1} = 1$,*

$$n\big(|\alpha| - k\big)\left\{ \int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^p d\theta \right\}^{\frac{1}{p}} \leq \left\{ \int_0^{2\pi} \left|1 + ke^{i\theta}\right|^{ps} d\theta \right\}^{\frac{1}{ps}} \left\{ \int_0^{2\pi} \left|N_\alpha^\gamma P\left(e^{i\theta}\right)\right|^{pt} d\theta \right\}^{\frac{1}{pt}} \tag{1.9}$$

where $\gamma \in R_+^n$.

2 Computations and analysis

In this section, we present some examples which not only show validity of our results but also show that obtained results are best possible.

Example 2.1. Let $P(z) = z(z^2 - 1)$, clearly all the zeros of $P(z)$ in $|z| \leq 1$. Now

$$G_\gamma [P] (z) = \gamma_1(z^2 - 1) + \gamma_2 z(z + 1) + \gamma_3 z(z - 1).$$

Taking different values of $(\gamma_1, \gamma_2, \gamma_3)$ we get different polynomials, in particular if we let $(\gamma_1, \gamma_2, \gamma_3) = (2, 1/2, 1/2)$ such that $\sum_{j=1}^{3} \gamma_j = 3$, we have

$$G_\gamma [P] (z) = 3z^2 - 2.$$

Also by the definition of generalized polar derivative, we have

$$N_\alpha^\gamma\left[P\right](z) = \sum_{j=1}^{3}\gamma_j P(z) + (\alpha - z)G_\gamma\left[P\right](z).$$

For $\alpha = 2$, we have

$$N_\alpha^\gamma\left[P\right](z) = 6z^2 - z + 4.$$

Letting $p \to \infty$ in Theorem 1.2, for $k = 1$ the L.H.S and R.H.S of inequality (1.7) take the values 12/11 and 2 respectively in the similar manner the L.H.S and R.H.S take the value 12 and 22. In both cases the above example shows that inequality (1.7) and inequality (1.9) are preserved.

Next, by the help of an example we show that inequality (1.7) and inequality (1.9) are best possible.

Example 2.2. Let $P(z) = z^3 - z^2 + z - 1$, clearly all the zeros of $P(z)$ lie in closed unit disk $|z| \le 1$. Now

$$G_\gamma\left[P\right](z) = \gamma_1(z^2 + 1) + \gamma_2(z - 1)(z + i) + \gamma_3(z - 1)(z - i).$$

Taking different values of $(\gamma_1, \gamma_2, \gamma_3)$ we get different polynomials, in particular if we let $(\gamma_1, \gamma_2, \gamma_3) = (2, 1/2, 1/2)$ such that $\sum_{j=1}^{3}\gamma_j = 3$, we have

$$G_\gamma\left[P\right](z) = 3z^2 - z + 2.$$

Also by the definition of generalized polar derivative, we have

$$N_\alpha^\gamma\left[P\right](z) = \sum_{j=1}^{3}\gamma_j P(z) + (\alpha - z)G_\gamma\left[P\right](z).$$

For $\alpha = 2$, we have

$$N_\alpha^\gamma\left[P\right](z) = 4z^2 - z + 1.$$

Letting $p \to \infty$ in Theorem 1.2, for $k = 1$ both L.H.S and R.H.S in inequality (1.7) take the same value 2, in the similar manner both L.H.S and R.H.S in inequality (1.9) take the value 12. Which shows that both inequality (1.7) and inequality (1.9) are sharp.

3 Lemmas

For the proof of main results, we need the following Lemmas.

Lemma 3.1. *Every convex set containing all the zeros of P(z) also contains the zeros $G_\gamma[P](z)$, for all $\gamma \in R_+^n$.*

The Lemma 3.1 is due to N. A. Rather *et al.* [10].

Next, we need the following Lemma (See [7, page 36]).

Lemma 3.2. *Let $S(z)$ and $P(z)$ be two analytic functions in $|z| \le 1$ such that $S(z)$ is subordination to the function $P(z)$ and let $P(z)$ be univalent in the same disk. then, for all $p > 0$*

$$\left\{\int_0^{2\pi}\left|S\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}} \le \left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}}.$$

Lemma 3.3. *If $P(z) \in \mathcal{P}_n$, has all its zeros in $|z| \leq k$, $k \leq 1$, and $\gamma \in R_+^n$ with $\sum_{j=1}^{n} \gamma_j = n$, then for $|z| = 1$*

$$\Big|G_\gamma\big[Q\big](z)\Big| \leq k\Big|nQ(z) - zG_\gamma\big[Q\big](z)\Big| \tag{3.1}$$

where $Q(z) = z^n\overline{P\big(1/\overline{z}\big)}$.

Proof of lemma 3.3. Since all the zeros of a polynomial P(z) lie in $|z| \leq k$, where $k \leq 1$, we can write

$$P(z) = \prod_{j=1}^{n}(z - z_j), \qquad where \quad |z_j| \leq k, \;\; j = 1, 2, \ldots, n.$$

Then, the polynomial $F(z) = P(kz)$ has all its zeros in $|z| \leq 1$, for $\gamma \in R_+^n$. We have,

$$\frac{zG_\gamma\big[F\big](z)}{F(z)} = \sum_{j=1}^{n} \frac{\gamma_j z}{z - \zeta_j} \qquad where \qquad \zeta_j = \frac{z_j}{k} \quad and \quad |\zeta_j| \leq 1, \quad 1 \leq k \leq n. \tag{3.2}$$

So that for the points $e^{i\theta}$, $\quad 0 \leq \theta < 2\pi$, other than the zeros of $F(z)$, we have

$$Re\Bigg\{\frac{e^{i\theta}G_\gamma\big[F\big]\left(e^{i\theta}\right)}{F\left(e^{i\theta}\right)}\Bigg\} = Re\Bigg\{\sum_{j=1}^{n}\frac{\gamma_j e^{i\theta}}{e^{i\theta} - \zeta_j}\Bigg\}$$

$$= \sum_{j=1}^{n}\gamma_j Re\Bigg\{\frac{e^{i\theta}}{e^{i\theta} - \zeta_j}\Bigg\}$$

$$\geq \frac{n}{2}.$$

Which implies

$$Re\Bigg\{\frac{e^{i\theta}G_\gamma\big[F\big]\left(e^{i\theta}\right)}{nF\left(e^{i\theta}\right)}\Bigg\} \geq \frac{1}{2}.$$

For the points $e^{i\theta}$, $\quad 0 \leq \theta < 2\pi$, which are not the zeros of $F(z)$, we have

$$\left|1 - \frac{e^{i\theta}G_\gamma\big[F\big]\left(e^{i\theta}\right)}{nF\left(e^{i\theta}\right)}\right| \leq \left|\frac{e^{i\theta}G_\gamma\big[F\big]\left(e^{i\theta}\right)}{nF\left(e^{i\theta}\right)}\right|$$

equivalently

$$\Big|nF\left(e^{i\theta}\right) - e^{i\theta}G_\gamma\big[F\big]\left(e^{i\theta}\right)\Big| \leq \Big|G_\gamma\big[F\big]\left(e^{i\theta}\right)\Big|. \tag{3.3}$$

For the $e^{i\theta}$, $\quad 0 \leq \theta < 2\pi$, which are not the zeros of $F(z)$. Since the inequality (3.3) is trivially true for points $e^{i\theta}$, $\quad 0 \leq \theta < 2\pi$, which are zeros of $F(z)$, therefore, it follows that

$$\Big|nF(z) - zG_\gamma\big[F\big](z)\Big| \leq \Big|G_\gamma\big[F\big](z)\Big| \quad for \quad |z| = 1. \tag{3.4}$$

Since $F(z) = P(kz)$, it follows from (3.2)

$$G_\gamma\big[F\big](z) = F(z)\sum_{j=1}^{n}\frac{\gamma_j}{z - \frac{z_j}{k}} = kP(kz)\sum_{j=1}^{n}\frac{\gamma_j}{kz - z_j} = kG_\gamma\big[P\big](kz).$$

Replacing $F(z)$ by $P(kz)$ and $G_\gamma[F](z)$ by $kG_\gamma[P](kz)$ in(3.4), we obtain

$$\Big|nP(kz) - zkG_\gamma[P](kz)\Big| \leq k\Big|G_\gamma[P](kz)\Big| \quad for \quad |z| = 1. \tag{3.5}$$

Also $k \leq 1$, we take in particular $z = \frac{e^{i\theta}}{k}$, $\quad 0 \leq \theta < 2\pi$ in (3.5) to get

$$\Big|nP\left(e^{i\theta}\right) - e^{i\theta}G_\gamma[P]\left(e^{i\theta}\right)\Big| \leq k\Big|G_\gamma[P]\left(e^{i\theta}\right)\Big|.$$

This shows that,

$$\Big|nP(z) - zG_\gamma[P](z)\Big| \leq k\Big|G_\gamma[P](z)\Big| \quad for \quad |z| = 1. \tag{3.6}$$

Now,

$$\begin{aligned} nP(z) - zG_\gamma[P](z) &= \sum_{j=1}^{n} \gamma_j P(z) - zG_\gamma[P](z) \\ &= P(z)\sum_{j=1}^{n}\left(\gamma_j - \frac{z\gamma_j}{z - z_j}\right) \\ &= -P(z)\sum_{j=1}^{n}\left(\frac{\gamma_j z_j}{z - z_j}\right). \end{aligned} \tag{3.7}$$

Also,

$$\begin{aligned} z^{n-1}\overline{G_\gamma[Q](1/\overline{z})} &= -z^n\overline{Q(1/\overline{z})}\sum_{j=1}^{n}\left(\frac{\gamma_j z_j}{z - z_j}\right). \\ &= -P(z)\sum_{j=1}^{n}\left(\frac{\gamma_j z_j}{z - z_j}\right). \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), for $|z| = 1$ we have

$$\Big|G_\gamma[Q](z)\Big| = \Big|nP(z) - zG_\gamma[P](z)\Big|. \tag{3.9}$$

Similarly, we get

$$\Big|G_\gamma[P](z)\Big| = \Big|nQ(z) - zG_\gamma[Q](z)\Big|. \tag{3.10}$$

From (3.6), (3.9) and (3.10) we get for $|z| = 1$

$$\Big|G_\gamma[Q](z)\Big| = k\Big|nQ(z) - zG_\gamma[Q](z)\Big|.$$

This proves Lemma 3.3. □

4 Proofs

Proof of Theorem 1.2. By the definition of *generalized polar derivative* and for every $\alpha \in C$, with $|\alpha| \geq k$, and $\sum_{k=1}^{n} \gamma_k = n$, we have for $|z| = 1$

$$\Big|N_\alpha^\gamma[P](z)\Big| = \Big|\sum_{k=1}^{n} \gamma_k P(z) + (\alpha - z)G_\gamma[P](z)\Big|$$

$$= \Big|nP(z) + (\alpha - z)G_\gamma\big[P\big](z)\Big|$$

$$\geq |\alpha|\Big|G_\gamma\big[P\big](z)\Big| - \Big|nP(z) - zG_\gamma\big[P\big](z)\Big|.$$

Using inequality (3.6) we obtain,

$$\Big|N_\alpha^\gamma\big[P\big](z)\Big| \geq |\alpha|\Big|G_\gamma\big[P\big](z)\Big| - k\Big|G_\gamma\big[P\big](z)\Big|$$

$$\Big|N_\alpha^\gamma\big[P\big](z)\Big| \geq \big(|\alpha| - k\big)\Big|G_\gamma\big[P\big](z)\Big|. \tag{4.1}$$

Since $P(z)$ has all its zeros in $|z| \leq k, \quad k \leq 1$, it follows by Lemma 3.1 that all the zeros of $G_\gamma\big[P\big](z)$ also lie in $|z| \leq k, \quad k \leq 1$. This shows that the polynomial

$$z^{n-1}\overline{G_\gamma\big[P\big]\Big(\frac{1}{\overline{z}}\Big)} = nQ(z) - zG_\gamma\big[Q\big](z)$$

has all its zeros in $|z| \leq \frac{1}{k}$. Therefore, it follows from Lemma 3.3 that the function

$$A(z) = \frac{zG_\gamma\big[Q\big](z)}{k\Big(nQ(z) - zG_\gamma\big[Q\big](z)\Big)}$$

is analytic in $|z| \leq 1$ and $|A(z)| \leq 1$ for $|z| = 1$. Furthermore, $A(0) = 0$. Thus the function $1 + kA(z)$ is subordinate to the function $1 + kz$ in $|z| \leq 1$. Hence by the application of Lemma 3.2, it follows for each $p > 0$

$$\left\{\int_0^{2\pi} \Big|1 + kA\left(e^{i\theta}\right)\Big|^p d\theta\right\}^{\frac{1}{p}} \leq \left\{\int_0^{2\pi} \Big|1 + ke^{i\theta}\Big|^p d\theta\right\}^{\frac{1}{p}}. \tag{4.2}$$

Now

$$1 + kA(z) = \frac{nQ(z)}{\Big(nQ(z) - zG_\gamma\big[Q\big](z)\Big)}.$$

Which gives with the help of inequality (3.10)

$$n\big|Q(z)\big| = \big|1 + kA(z)\big|\big|nQ(z) - zG_\gamma\big[Q\big](z)\big|$$

$$= \big|1 + kA(z)\big|\big|G_\gamma\big[P\big](z)\big|. \tag{4.3}$$

Since $|P(z)|$ = $|Q(z)|$ for $|z| = 1$, therefore from (4.3), we get for $|z| = 1$

$$n\big|P(z)\big| = \big|1 + kA(z)\big|\big|G_\gamma\big[P\big](z)\big|. \tag{4.4}$$

Combining (4.1) and (4.4), we get for $\alpha > k$ and $|z| = 1$

$$n\big(|\alpha| - k\big)\big|P(z)\big| \leq \big|1 + kA(z)\big|\Big|N_\alpha^\gamma\big[P\big](z)\Big|. \tag{4.5}$$

From (4.2) and (4.5), we deduce for each $p > 0$,

$$n^p\big(|\alpha| - k\big)^p \int_0^{2\pi} \left|\frac{P\left(e^{i\theta}\right)}{N_\alpha^\gamma\big[P\big]\left(e^{i\theta}\right)}\right|^p d\theta \leq \int_0^{2\pi} \Big|1 + ke^{i\theta}\Big|^p d\theta.$$

Which gives,

$$n\big(|\alpha| - k\big)\left\{\int_0^{2\pi} \left|\frac{P\left(e^{i\theta}\right)}{N_\alpha^\gamma\big[P\big]\left(e^{i\theta}\right)}\right|^p d\theta\right\}^{\frac{1}{p}} \leq \left\{\int_0^{2\pi} \Big|1 + ke^{i\theta}\Big|^p d\theta\right\}^{\frac{1}{p}}.$$

This completes the proof of Theorem 1.2. □

Proof of Theorem 1.5. Proceeding as in Theorem A we obtain from (4.5) for each $p > 0$,

$$n(|\alpha|-k)\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}} \leq \left\{\int_0^{2\pi}\left|\left(1+kA(e^{i\theta})\right)N_\alpha^\gamma[P]\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}}.$$

Using Holder's inequality for $s > 1, \quad t > 1$ with $s^{-1} + t^{-1} = 1$

$$n(|\alpha|-k)\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}} \leq \left\{\int_0^{2\pi}\left|1+kA(e^{i\theta})\right|^{ps} d\theta\right\}^{\frac{1}{ps}}\left\{\int_0^{2\pi}\left|N_\alpha^\gamma P\left(e^{i\theta}\right)\right|^{pt} d\theta\right\}^{\frac{1}{pt}}.$$

Using inequality (4.2) with replacing p by ps, we get

$$n(|\alpha|-k)\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}} \leq \left\{\int_0^{2\pi}\left|1+ke^{i\theta}\right|^{ps} d\theta\right\}^{\frac{1}{ps}}\left\{\int_0^{2\pi}\left|N_\alpha^\gamma[P]\left(e^{i\theta}\right)\right|^{pt} d\theta\right\}^{\frac{1}{pt}}.$$

This completes the proof of Theorem 1.5. □

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