Palestine Journal of Mathematics

Vol. 12(2)(2023) , 169–182

Approximation properties of modified Jain-Gamma operators preserving linear function

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Communicated by Harikrishnan Panackal
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MSC 2020 Classifications: Primary 41A10; Secondary 41A25, 41A36.

Keywords and phrases: Jain-Gamma Operators, Voronovskaya type theorem, Modulus of continuity, Weighted space, Peetre K-functional, Korovkin theorem.

Abstract. In this manuscript, we study a new operator of the modified Jain-Gamma operators which preserves linear function. We investigate into one of these an operator's approximation properties and study the rate of convergence of these operator with the aid of the modulus of continuity. We also discuss approximation properties of weighted spaces. Voronovskaya type asymptotic theorem and Korovkin type theorem are discussed.

1 Introduction

Until date, different linear positive operators have been obtained by special functions of mathematics and statistical distributions. Many authors have investigated their generalizations and alterations in depth and several author's are still working on it. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 33].
First of all we define Poisson-type distribution

$$\mathcal{W}_{\alpha}(k;\beta) = \frac{\beta}{k!}(\beta + k\alpha)^{k-1}e^{-(\beta+k\alpha)}, \;\; |\alpha| < 1, \;\; 0 < \beta < \infty, \;\; k \in \mathbb{N}_0,$$

on $\mathbb{R}_+ = [0,\infty)$. This distribution was used in 1972, Jain [14] class of positive linear operators defined as

$$\mathcal{P}_n^{\alpha}(f,x) = \sum_{u=0}^{\infty} \mathcal{W}_{\alpha}(u,nx) f\left(\frac{u}{n}\right), \;\; x \in \mathbb{R}_+, \;\; n \in \mathbb{N}_0, \;\; 0 \le \alpha < 1, \tag{1.1}$$

and introduced convergence properties, where

$$\mathcal{W}_{\alpha}(u,nx) = nx(nx+u\alpha)^{u-1}\frac{e^{-(nx+u\alpha)}}{u!}, \;\; u \in \mathbb{N}_0, \tag{1.2}$$

and $\mathcal{W}_{\alpha}(u,nx) = 1$.
In 1973 Jain et al. [3] examined the roles of this operator's more general version. In 2012, Tarabie [12] using the Jain operator, defined sequences of the Jain-Beta operator and analysed a variety of statistical convergence properties. In 2013, Faracaş [18] proved a Voronovkaya type theorem for Jain operators. Again, in 2013, Mishra and et al. [22] further modified the operator and investigated the convergence properties. In 2014, Agratini [29] with the use of the continuity modulus, uniform convergence properties of linear positive operator sequences of integral type, including the Jain operator, were investigated. In 2015, Olgun et al. [9] gave a different modification of the Jain operator with the help of a ρ function, and defined the convergence properties of this operator and the Voronovkaya type theorem . In 2015, Mishra et al. [19] gave the convergence properties of a different form of Jain-Baskakov operators. In 2017, Kumar et al. [8] talk about the convergence properties of the Stancu type of Jain-Baskakov operators.

If $\alpha = 0$ in (1.1), known Szász-Mirakyan operators are obtained as follows

$$\mathcal{S}_n(f,x) = \sum_{u=0}^{\infty} e^{-nx}\frac{(nx)^u}{u!} f\left(\frac{u}{n}\right).$$

This operator with a lot of generalisations. For this, we refer a reader to [3, 20, 21, 30, 34, 35]. The generalisation of Jain operators is still a continuing process for many researchers.
The Gamma operator is among the most well-known operators. In [23], Lupaş and Muller Gamma operator is defined as

$$\mathcal{G}_n(f,x) = \int_0^{\infty} \frac{x^{n+1}}{n!} e^{-xs} s^n f\left(\frac{n}{s}\right) ds, \quad x \in (0,\infty), \quad n \in \mathbb{N}, \tag{1.3}$$

and examined into different convergence properties. Later, some investigation was performed on the Gamma operator's various convergence properties [11, 24].
In 2007, Rempulska et al. [10] extended a modified version of Gamma operator as follows

$$\mathcal{G}_{n,q}(f,x) = \int_0^{\infty} \frac{x^{n+1}}{n!} e^{-xs} s^n F_q\left(x, \frac{n}{s}\right) ds,$$

and examined at the approximation properties of differentiable functions in polynomial weighted spaces. The Gamma operator has recently been used to define new operators and the convergence properties of these operators are examined [25, 26, 6, 27, 28]. We introduced the modified Jain-Gamma [32] operator by using the expression (1.1)-(1.3).

$$A_{n,\beta}^{\alpha_n}(f;x) = \sum_{k=1}^{\infty} \frac{nx(nx+k\alpha_n)^{k-1}}{k!} e^{-(nx+k\alpha_n)} \frac{x^{n+1}}{n!} \int_0^{\infty} e^{-xs} s^n f\left(\frac{k}{xs}\right) ds + e^{-nx} f(0). \tag{1.4}$$

2 Constructions of operators

Let $\{\alpha_n\}$ be a sequence, $x \in [0,\infty)$ such that

$$\alpha_n \in [0,1)\ \forall n \in \mathbb{N}, \quad \lim_{n\to\infty} \alpha_n = 0, \tag{2.1}$$

and f be defined on the space $\mathcal{C}_B[0,\infty)$ where $\mathcal{C}_B$ is the space of all continuous and bounded function from $[0,\infty)$. We define modified Jain-Gamma operators and preserving linear function

$$\mathcal{A}_{n,\beta}^{\alpha_n}(f;x) = \sum_{k=1}^{\infty} \frac{nx(nx+k\alpha_n)^{k-1}}{k!} e^{-(nx+k\alpha_n)} \frac{x^{n+1}}{n!} \int_0^{\infty} e^{-xs} s^n f\left(\frac{k(1-\alpha_n)}{xs}\right) ds + e^{-nx} f(0). \tag{2.2}$$

The recurrence relations are shown in [18]. Let

$$\mathcal{S}(r,\beta,\alpha_n) = \sum_{k=0}^{\infty} \frac{(\beta+k\alpha_n)^{k+r-1}}{k!} e^{-(\beta+k\alpha_n)}, \quad r = 0,1,2..... \tag{2.3}$$

and $\beta\mathcal{S}(0,\beta,\alpha_n) = 1$ for $0 < \beta < \infty,\ |\alpha_n| < 1,\ x \in [0,\infty)$. Then, it follows

$$\mathcal{S}(r,\beta,\alpha_n) = \sum_{k=0}^{\infty} \alpha_n^k (\beta+k\alpha_n) \mathcal{S}(r-1,\beta+k\alpha_n,\alpha_n),$$

$$\mathcal{S}(r,\beta,\alpha_n) = \beta\mathcal{S}(r-1,\beta,\alpha_n) + \alpha_n \mathcal{S}(r,\beta+\alpha_n,\alpha_n),$$

$$\mathcal{S}(1,\beta,\alpha_n) = \sum_{k=0}^{\infty} \alpha_n^k = \frac{1}{1-\alpha_n}, \tag{2.4}$$

$$\mathcal{S}(2,\beta,\alpha_n)=\sum_{k=0}^{\infty}\frac{\alpha_n^k(\beta+k\alpha_n)}{1-\alpha_n}=\frac{\beta}{(1-\alpha_n)^2}+\frac{\alpha_n^2}{(1-\alpha_n)^3}, \tag{2.5}$$

$$\mathcal{S}(3,\beta,\alpha_n)=\frac{\beta^3}{(1-\alpha_n)^3}+\frac{3\beta\alpha_n^2}{(1-\alpha_n)^4}+\frac{\alpha_n^3+2\alpha_n^4}{(1-\alpha_n)^5}, \tag{2.6}$$

$$\mathcal{S}(4,\beta,\alpha_n)=\frac{\beta^3}{(1-\alpha_n)^4}+\frac{6\beta^2\alpha_n^k}{(1-\alpha_n)^5}+\frac{\beta\alpha_n^3(11\alpha_n+4)}{(1-\alpha_n)^6}+\frac{+6\alpha_n^6+8\alpha_n^5+\alpha_n^4}{(1-\alpha_n)^7}. \tag{2.7}$$

Taking $\beta=nx$.

3 Main results

Lemma 3.1. *For the operators* (2.2)*, we have:*

(i) $\mathcal{A}_{n,\beta}^{\alpha_n}(1;x)=1$;

(ii) $\mathcal{A}_{n,\beta}^{\alpha_n}(t;x)=x$;

(iii) $\mathcal{A}_{n,\beta}^{\alpha_n}(t^2;x)=\frac{nx^2}{(n-1)}+\frac{x}{(1-\alpha_n)(n-1)}$;

(iv) $\mathcal{A}_{n,\beta}^{\alpha_n}(t^3;x)=\frac{n^2x^3}{(n-1)(n-2)}+\frac{3nx^2}{(n-1)(n-2)(1-\alpha_n)}+\frac{x(2\alpha_n+1)}{(n-1)(n-2)(1-\alpha_n)^2}$;

(v) $\mathcal{A}_{n,\beta}^{\alpha_n}(t^4;x)=\frac{n^3x^4}{(n-1)(n-2)(n-3)}+\frac{6n^2x^3}{(n-1)(n-2)(n-3)(1-\alpha_n)}-\frac{nx^2(8\alpha_n+7)}{(n-1)(n-2)(n-3)(1-\alpha_n)^2}+\frac{x(6\alpha_n^2+8\alpha_n+1)}{(n-1)(n-2)(n-3)(1-\alpha_n)^3}$.

Proof. $i)$ Using the operators (2.2) , we obtain

$$\mathcal{A}_{n,\beta}^{\alpha_n}(1;x)=\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\frac{x^{n+1}}{n!}\int_0^{\infty}e^{-xs}s^n ds.$$

If $xs=t$, then it follows

$$\begin{aligned}\mathcal{A}_{n,\beta}^{\alpha_n}(1;x)&=\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\frac{x^{n+1}}{n!}\int_0^{\infty}e^{-t}\frac{t^n}{x^n}\frac{dt}{x}\\&=\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}=1,\end{aligned}$$

which proves the first result.
$ii)$ If (2.4) is used for $f(t)=t$, we have

$$\begin{aligned}\mathcal{A}_{n,\beta}^{\alpha_n}(1;x)&=\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\frac{x^{n+1}}{n!}\int_0^{\infty}e^{-xs}s^n\frac{k(1-a_n)}{xs}ds\\&=\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\frac{x^{n+1}}{n!}\int_0^{\infty}e^{-t}\frac{t^n}{x^n}\frac{k(1-a_n)}{t}\frac{dt}{x}\\&=\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\frac{k(1-\alpha_n)}{n}\\&=\sum_{k=0}^{\infty}\frac{x(1-\alpha_n)(nx+\alpha_n+k\alpha_n)^k}{k!}e^{-(nx+\alpha_n+k\alpha_n)}\\&=x(1-a_n)\mathcal{S}(1,nx+\alpha_n,\alpha_n)\\&=x(1-\alpha_n)\frac{1}{(1-\alpha_n)}=x.\end{aligned}$$

$iii)$ If (2.5) is used for $f(t)=t^2$, we have

$$\mathcal{A}_{n,\beta}^{\alpha_n}(t^2;x)=\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\frac{x^{n+1}}{n!}\int_0^{\infty}e^{-xs}s^n\frac{k^2(1-\alpha_n)^2}{x^2s^2}ds.$$
$$=\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\frac{k^2(1-\alpha_n)^2}{n(n-1)},$$

taking $k^2=k(k-1)+k$, write last equation in the following form

$$\mathcal{A}_{n,\beta}^{\alpha_n}(t^2;x)=\frac{x(1-\alpha_n)^2}{n-1}\sum_{k=2}^{\infty}\frac{(nx+k\alpha_n)^{k-1}}{(k-2)!}e^{-(nx+k\alpha_n)}+\frac{x(1-\alpha_n)^2}{n-1}\sum_{k=1}^{\infty}\frac{(nx+k\alpha_n)^{k-1}}{(k-1)!}e^{-(nx+k\alpha_n)}$$
$$=\frac{x(1-\alpha_n)^2}{n-1}\mathcal{S}(2,nx+2\alpha_n,\alpha_n)+\frac{x(1-\alpha_n)^2}{(n-1)}\mathcal{S}(1,nx+\alpha_n,\alpha_n)$$
$$=\frac{nx^2(1-\alpha_n)^2}{(n-1)(1-\alpha_n)^2}+\frac{x(1-\alpha_n)^2}{(n-1)(1-\alpha_n)^3}$$
$$=\frac{nx^2}{(n-1)}+\frac{x}{(n-1)(1-\alpha_n)}.$$

$iv)$ If (2.6) is applied for $f(t)=t^3$, we have

$$\mathcal{A}_{n,\beta}^{\alpha_n}(t^3;x)=\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\frac{x^{n+1}}{n!}\int_0^{\infty}e^{-xs}s^n\frac{k^3(1-\alpha_n)^3}{x^3s^3}ds.$$
$$=\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\frac{k^3(1-\alpha_n)^3}{n(n-1)(n-2)},$$

taking $k^3=k(k-1)(k-2)+3k^2-2k$, the last equation is written as follows

$$\mathcal{A}_{n,\beta}^{\alpha_n}(t^3;x)=\frac{x(1-\alpha_n)^3}{(n-1)(n-2)}\left[\sum_{k=3}^{\infty}\frac{(nx+k\alpha_n)^{k-1}}{(k-3)!}e^{-(nx+k\alpha_n)}+3\sum_{k=2}^{\infty}\frac{(nx+k\alpha_n)^{k-1}}{(k-2)!}e^{-(nx+k\alpha_n)}\right.$$
$$\left.+\sum_{k=1}^{\infty}\frac{(nx+k\alpha_n)^{k-1}}{(k-1)!}e^{-(nx+k\alpha_n)}\right]$$
$$=\frac{x(1-\alpha_n)^3}{(n-1)(n-2)}\left[\mathcal{S}(3,nx+3\alpha_n,\alpha_n)+3\mathcal{S}(2,nx+2\alpha_n,\alpha_n)+\mathcal{S}(1,nx+\alpha_n,\alpha_n)\right]$$
$$=\frac{n^2x^3(1-\alpha_n)^3}{(n-1)(n-2)(1-\alpha_n)^3}+\frac{3nx^2(1-\alpha_n)^3}{(n-1)(n-2)(1-\alpha_n)^4}+\frac{x(1-\alpha_n)^3(2\alpha_n+1)}{(n-1)(n-2)(1-\alpha_n)^5}$$
$$=\frac{n^2x^3}{(n-1)(n-2)}+\frac{3nx^2}{(n-1)(n-2)(1-\alpha_n)}+\frac{x(1-\alpha_n)^3(2\alpha_n+1)}{(n-1)(n-2)(1-\alpha_n)^2}.$$

$v)$ If (2.7) is applied for $f(t)=t^4$, we have

$$\mathcal{A}_{n,\beta}^{\alpha_n}(t^4;x)=\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\frac{x^{n+1}}{n!}\int_0^{\infty}e^{-xs}s^n\frac{k^4(1-\alpha_n)^4}{x^4s^4}ds$$
$$=\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\frac{k^4(1-\alpha_n)^4}{n(n-1)(n-2)(n-3)}.$$

Since $k^4 = k(k-1)(k-2)(k-3) + 6k^3 - 11k^2 + 6k$, we can write last equation as

$$\mathcal{A}_{n,\beta}^{\alpha_n}(t^4;x) = \frac{x(1-\alpha_n)^4}{(n-1)(n-2)(n-3)}\Bigg[\sum_{k=4}^{\infty}\frac{(nx+k\alpha_n)^{k-1}}{(k-4)!}e^{-(nx+k\alpha_n)} + 6\sum_{k=3}^{\infty}\frac{(nx+k\alpha_n)^{k-1}}{(k-3)!}e^{-(nx+k\alpha_n)}$$
$$+ 7\sum_{k=2}^{\infty}\frac{(nx+k\alpha_n)^{k-1}}{(k-2)!}e^{-(nx+k\alpha_n)} + \sum_{k=1}^{\infty}\frac{(nx+k\alpha_n)^{k-1}}{(k-1)!}e^{-(nx+k\alpha_n)}\Bigg]$$
$$= \frac{x(1-\alpha_n)^4}{(n-1)(n-2)(n-3)}\big[\mathcal{S}(4, nx+4\alpha_n, \alpha_n) + 6\mathcal{S}(3, nx+3\alpha_n, \alpha_n)$$
$$+ 7\mathcal{S}(2, nx+2\alpha_n, \alpha_n) + \mathcal{S}(1, nx+\alpha_n, \alpha_n)\big]$$
$$= \frac{n^3x^4(1-\alpha_n)^4}{(n-1)(n-2)(n-3)(1-\alpha_n)^4} + \frac{6n^2x^3(1-\alpha_n)^4}{(n-1)(n-2)(n-3)(1-\alpha_n)^5}$$
$$- \frac{nx^2(1-\alpha_n)^4(8\alpha_n+7)}{(n-1)(n-2)(n-3)(1-\alpha_n)^6} + \frac{x(1-\alpha_n)^4(6\alpha_n^2+8\alpha_n+1)}{(n-1)(n-2)(n-3)(1-\alpha_n)^7}$$

$$= \frac{n^3x^4}{(n-1)(n-2)(n-3)} + \frac{6n^2x^3}{(n-1)(n-2)(n-3)(1-\alpha_n)}$$
$$- \frac{nx^2(8\alpha_n+7)}{(n-1)(n-2)(n-3)(1-\alpha_n)^2} + \frac{x(6\alpha_n^2+8\alpha_n+1)}{(n-1)(n-2)(n-3)(1-\alpha_n)^3}.$$

□

which completes the proof.

Theorem 3.2. *Let* $f \in \mathcal{C}_E[0,\infty),\ n \in \mathbb{N}$ *and* $x \in [0,\infty)$. *Then we have*

$$\lim_{n\to\infty}\left(\mathcal{A}_{n,\beta}^{\alpha_n}(f;x) - f(x)\right) = 0.$$

Proof. It is obvious from the Lemma (3.1). □

Lemma 3.3. *For the operators* (2.2)*, the inequality*

$$\mathcal{A}_{n,\beta}^{\alpha_n}((t-x)^2;x) \le N^*(n,\alpha_n)\frac{x^2+x}{n-1}$$

holds, where $N_i = (n,\alpha_n),\ i = 1\ N^* = max(N_i)$.

Proof. Note $x^s/(1+x)^l \le x^s\ \forall\ x \ge 0,\ l \le s, l, s = 1, 2$. From Lemma (3.1) and from linearity of the operators (2.2) it follows

$$\mathcal{A}_{n,\beta}^{\alpha_n}((t-x)^2;x) = \left(\frac{n}{(n-1)} - 1\right)x^2 + \frac{1}{(n-1)(1-\alpha_n)}x$$
$$= \frac{1}{(n-1)}x^2 + \frac{1}{n-1}\frac{1}{(1-\alpha_n)}x$$

Let $N^* = max\{1, N_1\}$

$$= \frac{x^2}{(n-1)}.1 + \frac{x}{n-1}N_1(n,\alpha_n)$$
$$= \frac{x^2}{(n-1)}N^* + \frac{x}{n-1}N_1(n,\alpha_n)$$
$$= \frac{x^2}{(n-1)}N^* + \frac{x}{n-1}N^* \le N^*(n,\alpha_n).\frac{x^2+x}{n-1}.$$

□

4 Rate of Convergence

Let us now discuss the operators convergence properties and defined by (2.2) using Peetre's K-functional and the modulus of continuity. We also present the convergence of the operator for $f \in Lip_N(\zeta)$.

Theorem 4.1. *Let* $f \in \mathcal{C}_E[0,\infty)$ *and* $x \in [0,\infty),\ n \in \mathbb{N}$. *Then we have*

$$\left|\mathcal{A}_{n,\beta}^{\alpha_n}(f;x) - f(x)\right| \leq N^{**}\omega\left(f; \sqrt{\frac{x^2+x}{(n-1)}}\right),$$

where $N^{**} = 1 + \sqrt{N^*}$.

Proof. Use the definition modulus of continuity and their well-known properties and the linearity of the operators (2.2), we can write

$$\begin{aligned}
\left|\mathcal{A}_{n,\beta}^{\alpha_n}(f;x) - f(x)\right| &= \left|\sum_{k=1}^{\infty} \frac{nx(nx+k\alpha_n)^{k-1}}{k!} e^{-(nx+k\alpha_n)} \frac{x^{n+1}}{n!} \int_0^\infty e^{-xs} s^n f\left(\frac{k(1-\alpha_n)}{xs}\right) - f(x) ds\right| \\
&\leq \sum_{k=1}^{\infty} \frac{nx(nx+k\alpha_n)^{k-1}}{k!} e^{-(nx+k\alpha_n)} \frac{x^{n+1}}{n!} \int_0^\infty e^{-xs} s^n \left| f\left(\frac{k(1-\alpha_n)}{xs}\right) - f(x)\right| ds \\
&\leq \sum_{k=1}^{\infty} \frac{nx(nx+k\alpha_n)^{k-1}}{k!} e^{-(nx+k\alpha_n)} \frac{x^{n+1}}{n!}
\end{aligned}$$

$$\begin{aligned}
&\times \int_0^\infty e^{-xs} s^n \left(1 + \frac{|k(1-\alpha_n)/(xs) - x|}{\delta}\right) \omega(f;\delta) ds \\
&= \omega(f;\delta) + \sum_{k=1}^{\infty} \frac{nx(nx+k\alpha_n)^{k-1}}{k!} e^{-(nx+k\alpha_n)} \frac{x^{n+1}}{n!} \frac{1}{\delta} \omega(f;\delta) \\
&\times \int_0^\infty e^{-xs} s^n \left|\frac{k(1-\alpha_n)}{xs} - x\right| ds.
\end{aligned}$$

The Cauchy-Schwarz inequality is applied to the last expression on the right side of this inequality, first for the integral, then for the sum, to obtain

$$\begin{aligned}
\left|\mathcal{A}_{n,\beta}^{\alpha_n}(f;x) - f(x)\right| &\leq \omega(f;\delta) + \frac{1}{\delta}\omega(f;\delta) \left\{ \sum_{k=1}^{\infty} \frac{nx(nx+k\alpha_n)^{k-1}}{k!} e^{-(nx+k\alpha_n)} \right. \\
&\left. \times \left(\int_0^\infty \frac{x^{n+1}}{n!} e^{-xs} s^n \left(\frac{k(1-\alpha_n)}{xs} - x\right)^2 ds \right)^{\frac{1}{2}} \left(\int_0^\infty \frac{x^{n+1}}{n!} e^{-xs} s^n 1^2 ds \right)^{\frac{1}{2}} \right\} \\
&= \omega(f;\delta) + \frac{1}{\delta}\omega(f;\delta) \sqrt{\mathcal{A}_{n,\beta}^{\alpha_n}((t-x)^2;x)} \\
&\leq \omega(f;\delta) + \frac{1}{\delta}\omega(f;\delta) \sqrt{N^*} \sqrt{\frac{x^2+x}{(n-1)}}.
\end{aligned}$$

We take $\delta = \sqrt{(x^2+x)/(n-1)}$, then it follows

$$\left|\mathcal{A}_{n,\beta}^{\alpha_n}(f;x) - f(x)\right| \leq \omega\left(f; \sqrt{\frac{x^2+x}{(n-1)}}\right)(1 + \sqrt{N^*}) \leq N^{**}\omega\left(f; \sqrt{\frac{x^2+x}{(n-1)}}\right),$$

finally complete the proof. □

Let $\mathcal{C}_E[0,\infty)$ denote the space of real valued continuous and bounded functions on the interval $[0,\infty)$, with the norm $||f|| = \sup_{0<x\leq\infty} |f(x)|$. For every $\delta > 0$, Peetre's K-functional is defined by

$$K_2(f;\delta) = \inf_{g \in \mathcal{C}_E^2(0,\infty)} \{||f-h|| + \delta||h''||\},$$

where $\mathcal{C}_E^2(0,\infty) = \{h \in \mathcal{C}_E(0,\infty) : h', h'' \in \mathcal{C}_E(0,\infty)\}$. There exist a real positive number $D > 0$ such that

$$K_2(f;\delta) \leq D\omega_2(f;\sqrt{\delta}) \tag{4.1}$$

where ω_2 is the second order modulus of smoothness of f, defined by

$$\omega(f;\delta) = \sup_{0<h\leq\delta} \sup_{0<x\leq\infty} |f(x+2a) - 2f(x+a) + f(x)|.$$

Lemma 4.2. *Let $h \in \mathcal{C}_E^2(0,\infty)$. Then we have*

$$|\mathcal{A}_{n,\beta}^{\alpha_n}(h;x) - h(x)| \leq \delta_n(x)||h''||,$$

where

$$\delta_n(x) = \mathcal{A}_{n,\beta}^{\alpha_n}((t-x)^2;x)$$

Proof. For the operators $\mathcal{A}_{n,\beta}^{\alpha_n}(f;x)$, we get

$$\mathcal{A}_{n,\beta}^{\alpha_n}((t-x);x) = \mathcal{A}_{n,\beta}^{\alpha_n}(t;x) - x\mathcal{A}_{n,\beta}^{\alpha_n}(1;x) = 0$$

Let $h \in \mathcal{C}_E^2(0,\infty)$ and $x \in (0,\infty)$. By Taylor's formula of h, we may write

$$h(t) - h(x) = (t-x)h'(x) + \int_x^t (t-u)h''(u)du, \;\; t \in [0,\infty).$$

Apply the operator $\mathcal{A}_{n,\beta}^{\alpha_n}$ to this equality, we obtain

$$\begin{aligned}\mathcal{A}_{n,\beta}^{\alpha_n}(h(t)-h(x);x) &= \mathcal{A}_{n,\beta}^{\alpha_n}((t-x)h'(x);x) + \mathcal{A}_{n,\beta}^{\alpha_n}\left(\int_x^t (t-u)h''(u)du;x\right)\\ &= \mathcal{A}_{n,\beta}^{\alpha_n}\left(\int_x^t (t-u)h''(u)du;x\right).\end{aligned}$$

By using the inequality

$$\left|\int_x^t (t-u)h''(u)du\right| \leq \int_x^t |(t-u)|||h''(u)||du \leq \frac{(t-x)^2}{2}||h''(u)|| \leq (t-x)^2||h''(u)||,$$

in account of this inequality, we can determine that

$$|\mathcal{A}_{n,\beta}^{\alpha_n}((h;x)-h(x))| \leq \left\{\mathcal{A}_{n,\beta}^{\alpha_n}((t-x)^2;x)\right\}||h''|| = \delta_n(x)||h''||.$$

□

Theorem 4.3. *Let $f \in \mathcal{C}_E[0,\infty)$. $\forall\, x \in [0,\infty)$, $\exists$ a constant $E > 0$ such that*

$$|\mathcal{A}_{n,\beta}^{\alpha_n}(f;x) - f(x)| \leq E\omega_2(f;\sqrt{\delta_n(x)}),$$

where

$$\delta_n(x) = \mathcal{A}_{n,\beta}^{\alpha_n}((t-x)^2;x)$$

Proof. For the operators $\mathcal{A}_{n,\beta}^{\alpha_n}$, we write

$$\mathcal{A}_{n,\beta}^{\alpha_n}(f;x) - f(x) = \mathcal{A}_{n,\beta}^{\alpha_n}((f-h;x) + (f-h)(x)\mathcal{A}_{n,\beta}^{\alpha_n}(h-h(x);x), \tag{4.2}$$

and

$$|\mathcal{A}_{n,\beta}^{\alpha_n}(f;x) - f(x)| \leq |\mathcal{A}_{n,\beta}^{\alpha_n}(f-h;x)| + |(f-h)(x)| + |\mathcal{A}_{n,\beta}^{\alpha_n}(h;x) - h(x)|$$

Now,

$$|\mathcal{A}_{n,\beta}^{\alpha_n}(f;x) - f(x)| \leq ||f-h|| + ||f-h|| + |\mathcal{A}_{n,\beta}^{\alpha_n}(h;x) - h(x)|,$$

From Lemma (4.2), we obtain

$$|\mathcal{A}^{\alpha_n}_{n,\beta}(f;x) - f(x)| \leq 2||f-h|| + \delta_n(x)||h''|| \\ \leq 2\{||f-h|| + \delta_n(x)||h''||\}$$

Taking the infimum $\forall\ h \in \mathcal{C}^2_B(0,\infty)$ on the right-hand side of the last inequality and considering (4.1), we get

$$|\mathcal{A}^{\alpha_n}_{n,\beta}(f;x) - f(x)| \leq 2L_2(f;\delta_n(x)) \\ \leq 2C\omega_2(f;\sqrt{\delta_n(x)}) \\ = B\omega_2(f;\sqrt{\delta_n(x)}),$$

which completes the proof. □

Theorem 4.4. *Let $f \in \mathcal{C}_E[0,\infty)$. $\forall\ x \in [0,\infty)$, there exist a real positive real number $L > 0$ such that*

$$|\mathcal{A}^{\alpha_n}_{n,\beta}(f;x) - f(x)| \leq L\omega_2(f;\sqrt{\delta_n(x)}).$$

Proof. For $x \in [0,\infty)$, By Taylor's expansion,
$h(t) = h(x) + (t-x)h'(x) + \int_x^t (t-u)h''(u)du.$
Applying $\mathcal{A}^{\alpha_n}_{n,\beta}$, on both sides of the above equation, we have

$$\mathcal{A}^{\alpha_n}_{n,\beta}(h;x) - h(x) = h'(x)\mathcal{A}^{\alpha_n}_{n,\beta}((t-x);x) + \mathcal{A}^{\alpha_n}_{n,\beta}\left(\int_x^t (t-u)h''(u)du;x\right).$$

Thus, we get

$$|\mathcal{A}^{\alpha_n}_{n,\beta}(h;x) - h(x)| \leq \mathcal{A}^{\alpha_n}_{n,\beta}\left(\left|\int_x^t (t-u)h''(u)du\right|;x\right) \leq \mathcal{A}^{\alpha_n}_{n,\beta}((t-x)^2;x)||h''|| \\ = \delta(x)||h''||. \tag{4.3}$$

Since

$$|\mathcal{A}^{\alpha_n}_{n,\beta}(f;x)| \leq \sum_{k=1}^{\infty} \frac{nx(nx+k\alpha_n)^{k-1}}{k!} e^{-(nx+k\alpha_n)} \frac{x^{n+1}}{n!} \\ \times \int_0^\infty e^{-xs} s^n \left| f\left(\frac{k(1-\alpha_n)}{xs}\right)\right| ds + e^{-nx}|f(0)| \leq ||f||, \tag{4.4}$$

using (4.3) and (4.4), we obtain

$$|\mathcal{A}^{\alpha_n}_{n,\beta}(f;x) - f(x)| \leq |\mathcal{A}^{\alpha_n}_{n,\beta}(f-h;x) - (f-h)(x)| + |\mathcal{A}^{\alpha_n}_{n,\beta}(h;x) - h(x)| \\ \leq 2||f-h|| + \delta_n(x)||h''||,$$

taking infimum over all $h \in \mathcal{C}^2_E(0,\infty)$, we get

$$|\mathcal{A}^{\alpha_n}_{n,\beta}(f;x) - f(x)| \leq L\omega_2(f;\delta_n(x)),$$

taking inequality (4.1), we get

$$|\mathcal{A}^{\alpha_n}_{n,\beta}(f;x) - f(x)| \leq L\omega_2(f;\sqrt{\delta_n(x)}).$$

which proves the theorem. □

Theorem 4.5. *Let $f \in \mathcal{C}_E[0,\infty)$ and $0 < \zeta \leq 1$. Then if $f \in Lip_N(\zeta)$, that is, the inequality*

$$|f(t) - f(x)| \leq N|t-x|^\zeta, \quad \forall\ x,t \in (0,\infty),$$

then $\forall\ x \in [0,\infty)$ and N>0 is a constant, we have

$$|\mathcal{A}^{\alpha_n}_{n,\beta}(f;x) - f(x)| \leq N\delta_n^{\frac{\zeta}{2}}(x),$$

where $\delta_n = \mathcal{A}^{\alpha_n}_{n,\beta}((t-x)^2;x)$.

Proof. Let $f \in \mathcal{C}_E[0,\infty) \cap Lip_N(\zeta)$. By using linearity and monotonicity of the operators $\mathcal{A}_{n,\beta}^{\alpha_n}$, we get

$$\begin{aligned}|\mathcal{A}_{n,\beta}^{\alpha_n}(f;x) - f(x)| &\leq \mathcal{A}_{n,\beta}^{\alpha_n}\big(|f(t)-f(x)|;x) \leq N\mathcal{A}_{n,\beta}^{\alpha_n}(|t-x|^{\zeta};x)\\ &= N\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\frac{x^{n+1}}{n!}\int_0^{\infty}e^{-xs}s^n\left|\frac{k(1-\alpha_n)}{xs}-x\right|^{\zeta}ds.\end{aligned}$$

Applying the Holder inequality to the right side twice in succession with $p = 2/\zeta$, $q = 2/(2-\zeta)$, we obtain

$$\begin{aligned}\mathcal{A}_{n,\beta}^{\alpha_n}(f;x) - f(x) &\leq N\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\frac{x^{n+1}}{n!}\int_0^{\infty}e^{-xs}s^n\left|\frac{k(1-\alpha_n)}{xs}-x\right|^{\zeta}1ds\\ &\leq N\left(\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\frac{x^{n+1}}{n!}\int_0^{\infty}e^{-xs}s^n\left|\frac{k(1-\alpha_n)}{xs}-x\right|^{2}ds\right)^{\frac{\zeta}{2}}\\ &\times\left(\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\frac{x^{n+1}}{n!}\int_0^{\infty}e^{-xs}s^n\right)^{\frac{(2-\zeta)}{2}}\\ &\leq N\left(\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\frac{x^{n+1}}{n!}\int_0^{\infty}e^{-xs}s^n\left|\frac{k(1-\alpha_n)}{xs}-x\right|^{2}ds\right)^{\frac{\zeta}{2}},\end{aligned}$$

and

$$\begin{aligned}|\mathcal{A}_{n,\beta}^{\alpha_n}(f;x) - f(x)| &\leq N\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\left(\int_0^{\infty}\frac{x^{n+1}}{n!}e^{-xs}s^n\left(\frac{k(1-\alpha_n)}{xs}-x\right)^{\frac{4}{\zeta}}ds\right)^{\frac{\zeta}{2}}\\ &\times\left(\int_0^{\infty}\frac{x^{n+1}}{n!}e^{-xs}s^n ds\right)^{\frac{(2-\zeta)}{2}}\\ &\leq N\left(\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\frac{x^{n+1}}{n!}\int_0^{\infty}e^{-xs}s^n\left(\frac{k(1-\alpha_n)}{xs}-x\right)^{2}ds\right)^{\frac{\zeta}{2}}\\ &= N\mathcal{A}_{n,\beta}^{\alpha_n}(t-x)^2;x)^{\frac{\zeta}{2}} = N\delta_n^{\frac{\zeta}{2}}(x).\end{aligned}$$

which is the desired result. □

5 Weighted approximation properties

Now, we'll discuss various definitions and theorems in this section.
Let $H_\gamma[0,\infty)$ denote the space of all functions having the property and $\gamma(x) = 1 + x^2$.

$$|f(x)| \leq N_f\gamma(x),$$

where N_f is a positive constant on f functions and $x \in [0,\infty)$. The norm on $H_\gamma[0,\infty)$ is defined as

$$||f||_\gamma = \sup_{0\leq x\leq\infty}\frac{|f(x)|}{1+x^2}.$$

$\mathcal{C}_\gamma[0,\infty)$ denotes the space of all continuous functions belonging to $H_\gamma[0,\infty)$ and $\mathcal{C}_\gamma^0[0,\infty)$ denotes the subspace of all functions $f \in \mathcal{C}_\gamma[0,\infty)$ for which

$$\lim_{x\to\infty}\frac{|f(x)|}{\gamma(x)} = 0.$$

The basic theorem for approximation of weighted spaces is given by Gadjiev [31].

Theorem 5.1. *Let $\{\mathcal{A}_n\}$ be a sequence of positive linear operators defined from $\mathcal{C}_\gamma^0[0,\infty)$ to $H_\gamma[0,\infty)$, and satisfying the conditions $\lim_{n\to\infty}||\mathcal{A}_n(t^u;x)-x^u||_\gamma=0, u=0,1,2$. Then for any $f\in\mathcal{C}_\gamma^0[0,\infty)$,*

$$\lim_{n\to\infty}||\mathcal{A}_n(f;x)-f(x)||_\gamma=0.$$

It is shown in [31] that a sequence of linear positive operators $\mathcal{A}_n$ is defined from $\mathcal{C}_\gamma^0[0,\infty)$ to $H_\gamma[0,\infty)$ if and only if $||\mathcal{A}_n(\gamma;x)||_\gamma\le N_\gamma$ where N_γ is a positive constant.

Theorem 5.2. *Let $\{A_{n,\beta}^{\alpha_n}\}$ be a sequence of positive linear operators. For each $f\in\mathcal{C}_\gamma^0[0,\infty)$, we have*

$$\lim_{n\to\infty}||A_{n,\beta}^{\alpha_n}(f;x)-f(x)||_\gamma=0.$$

Proof. Using Lemma (3.1), we get

$$\begin{aligned}\sup_{0\le x\infty}\frac{|A_{n,\beta}^{\alpha_n}(\gamma;x)|}{1+x^2}&=\sup_{0\le x\infty}\frac{|A_{n,\beta}^{\alpha_n}(1+t^2;x)|}{1+x^2}=\sup_{0\le x\infty}\frac{|A_{n,\beta}^{\alpha_n}(1;x)+A_{n,\beta}^{\alpha_n}(t^2;x)|}{1+x^2}\\&=\sup_{0\le x<\infty}\frac{\left|1+\frac{nx^2}{(n-1)}+\frac{x}{(n-1)(1-\alpha_n)}\right|}{1+x^2}\\&\le 1+\frac{n}{(n-1)}+\frac{1}{(n-1)(1-\alpha_n)}.\end{aligned}$$

There exists a positive constant $\mathcal{D}$ such that for each n and $0\le\alpha_n<1$

$$\frac{n}{(n-1)}+\frac{1}{(n-1)(1-\alpha_n)}<\mathcal{D}.$$

Hence we may write $\sup_{0\le x<\infty}\frac{|A_{n,\beta}^{\alpha_n}(\gamma;x)|}{1+x^2}=||A_{n,\beta}^{\alpha_n}(\gamma;x)||_\gamma\le 1+\mathcal{D}$, which shows that $\{A_{n,\beta}^{\alpha_n}\}$ is a sequence of positive linear operators defined from $D_\gamma^0[0,\infty)$ to $H_\gamma[0,\infty)$.
Considering the results (2.4) and (2.5) obtained above, for $u=0,1,2$
It is clear that $\lim_{n\to\infty}||A_{n,\beta}^{\alpha_n}(t^v;x)-x^v||_\gamma=0,\ u=0,1,2$
Thus, the proof is completed. □

Theorem 5.3. *Let $f\in\mathcal{C}_H[0,\infty)$ and $x\in[0,\infty),\ n\in\mathbb{N}$. For the operators*

$$\mathcal{A}_{n,\beta}^{\alpha_n}(f;x)=\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\frac{x^{n+1}}{n!}\int_0^\infty e^{-xs}s^nf\left(\frac{k(1-\alpha_n)}{xs}\right)ds+e^{-nx}f(0).$$

and

$$\mathcal{B}_{n,\beta}^{\alpha_n}(f;x)=\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}f\left(\frac{k(1-\alpha_n)}{n}\right),$$

the inequality

$$|\mathcal{A}_{n,\beta}^{\alpha_n}(f;x)-\mathcal{B}_{n,\beta}^{\alpha_n}(f;x)|\le\omega(f;\delta)\theta(x),$$

holds true, where

$$\theta(x)=1+\frac{1}{\delta}\sqrt{\frac{x^2}{(n-1)}+\frac{x}{n(n-1)(1-\alpha_n)}}$$

$$\delta=\sqrt{\frac{x^2}{(n-1)}+\frac{x}{n(n-1)(1-\alpha_n)}}.$$

Proof. We determine the definition and properties of modulus of continuity, we have

$$\begin{aligned}\mathcal{A}_{n,\beta}^{\alpha_n}(f;x)-\mathcal{B}_{n,\beta}^{\alpha_n}(f;x)&=\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\frac{x^{n+1}}{n!}\\&\quad\times\int_0^\infty e^{-xs}s^n\left(f\left(\frac{k(1-\alpha_n)}{xs}\right)-f\left(\frac{k(1-\alpha_n)}{n}\right)\right)ds,\end{aligned}$$

from which it follows

$$
\begin{aligned}
|\mathcal{A}_{n,\beta}^{\alpha_n}(f;x)-\mathcal{B}_{n,\beta}^{\alpha_n}(f;x)| &\leq \sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\frac{x^{n+1}}{n!}\\
&\times\int_0^{\infty}e^{-xs}s^n\left|f\left(\frac{k(1-\alpha_n)}{xs}\right)-f\left(\frac{k(1-\alpha_n)}{n}\right)\right|ds\\
&\leq \sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\frac{x^{n+1}}{n!}\int_0^{\infty}e^{-xs}s^n\omega\left(f;\frac{\frac{k(1-\alpha_n)}{xs}-\frac{k(1-\alpha_n)}{n}}{\delta}\delta\right)d\\
&\leq \omega(f;\delta)+\frac{1}{\delta}\omega(f;\delta)\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\frac{x^{n+1}}{n!}\\
&\times\int_0^{\infty}e^{-xs}s^n\left|\frac{k(1-\alpha_n)}{xs}-\frac{k(1-\alpha_n)}{n}\right|ds.
\end{aligned}
$$

The Cauchy-Schwarz inequality is applied to the second expression on the right side of this inequality, first for the integral, then for the sum, to obtain

$$
|\mathcal{A}_{n,\beta}^{\alpha_n}(f;x)-\mathcal{B}_{n,\beta}^{\alpha_n}(f;x)| \leq \omega(f;\delta)+\frac{1}{\delta}\omega(f;\delta)\sqrt{\mathcal{A}_{n,\beta}^{\alpha_n}\left(\left(\left(\frac{k(1-\alpha_n)}{xs}\right)-\left(\frac{k(1-\alpha_n)}{n}\right)^2;x\right)}.
$$

If we calculate $\mathcal{A}_{n,\beta}^{\alpha_n}\big((k(1-\alpha_n)/xs)-k(1-\alpha_n)/n\big)^2;x\big)$, we show that

$$
\begin{aligned}
\mathcal{A}_{n,\beta}^{\alpha_n}(f;x)\left(\left(\frac{k(1-\alpha_n)}{xs}\right)-\left(\frac{k(1-\alpha_n)}{n}\right)^2;x\right) &= k^2(1-\alpha_n)^2\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\frac{x^{(n+1)}}{n!}\\
&\times\int_0^{\infty}e^{-xs}s^n\left(\frac{1}{x^2s^2}-\frac{2}{xs^2}+\frac{1}{n^2}\right)ds.
\end{aligned}
$$

If we say $xs=t$, then it follows

$$
\begin{aligned}
\mathcal{A}_{n,\beta}^{\alpha_n}(f;x)\left(\left(\frac{k(1-\alpha_n)}{xs}\right)-\left(\frac{k(1-\alpha_n)}{n}\right)^2;x\right) &= \sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\frac{k^2(1-\alpha_n)^2}{n(n-1)}\\
&-\frac{2}{n}\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\frac{k^2(1-\alpha_n)^2}{n}\\
&+\frac{1}{n^2}\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}k^2(1-\alpha_n)^2\\
&=\left[\frac{1}{n^2(n-1)}\right]\left[\frac{nx^2}{(n-1)}+\frac{x}{(n-1)(1-\alpha_n)}\right]\\
&=\frac{x^2}{(n-1)}+\frac{x}{n(n-1)(1-\alpha_n)},
\end{aligned}
$$

from which, it follows

$$
\lim_{n\to\infty}\mathcal{A}_{n,\beta}^{\alpha_n}\left(\left(\frac{k(1-\alpha_n)}{xs}-\frac{k(1-\alpha_n)}{n}\right)^2;x\right)=0.
$$

Thus, we have

$$
\begin{aligned}
|\mathcal{A}_{n,\beta}^{\alpha_n}(f;x)-\mathcal{B}_{n,\beta}^{\alpha_n}(f;x)| &\leq \omega(f;\delta)+\frac{1}{\delta}\omega(f;\delta)\sqrt{\frac{x^2}{(n-1)}+\frac{x}{n(n-1)(1-\alpha_n)}}\\
&\leq \omega(f;\delta)\theta(x).
\end{aligned}
$$

□

6 Voronovskaya type theorem

Lemma 6.1. *If the operators $\mathcal{A}_{n,\beta}^{\alpha_n}(f;x)$ defined in (2.2) is linear and positive, we get:*

(i) $\mathcal{A}_{n,\beta}^{\alpha_n}(t-x;x)=0;$

(ii) $\mathcal{A}_{n,\beta}^{\alpha_n}((t-x)^2;x)=\frac{1}{(n-1)}x^2+\frac{x}{(n-1)(1-\alpha_n)};$

(iii) $\mathcal{A}_{n,\beta}^{\alpha_n}((t-x)^3;x)=\frac{4}{(n-1)(n-2)}x^3+\frac{6}{(n-1)(n-2)(1-\alpha_n)}x^2+\frac{(2\alpha_n+1)}{(n-1)(n-2)(1-\alpha_n)^2}x;$

(iv)

$$\begin{aligned}\mathcal{A}_{n,\beta}^{\alpha_n}((t-x)^4;x)=&\frac{-8n^2+3n+18}{(n-1)(n-2)(n-3)}x^4+\frac{6(n+6)}{(n-1)(n-2)(n-3)(1-\alpha_n)}x^3\\&+\frac{8\alpha_n(-2n+3)-11n+12}{(n-1)(n-2)(n-3)(1-\alpha_n)^2}x^2+\frac{(6\alpha_n^2+8\alpha_n+1)}{(n-1)(n-2)(n-3)(1-\alpha_n)^3}x.\end{aligned}$$

The proof can be easily done by using the linearity of operators $\mathcal{A}_{n,\beta}^{\alpha_n}$ and lemma (3.1).

Theorem 6.2. *Let $0\le\beta_n\le\alpha_n,\ x\ge 0$ and $n\in\mathbb{N}$. For $f\in\mathcal{C}^2[0,\infty)$ and bounded, we have*

$$\lim_{n\to\infty} n[\mathcal{A}_{n,\beta}^{\alpha_n}(f;x)-f(x)]=\frac{(x+x^2)}{2}f''(x).$$

Proof. Let $f\in\mathcal{C}^2(0,\infty),\ x,t\in[0,\infty)$. By Taylor's formula for f, we have

$$f(t)=f(x)+(t-x)f'(x)+\frac{(t-x)^2}{2!}f''(x)+(t-x)^2\theta(t;x), \tag{6.1}$$

where $\theta(t;x)\in D[0,\infty)$ and $\lim_{t\to x}\theta(t;x)=0$. By applying the operator $\mathcal{A}_{n,\beta}^{\alpha_n}$ to the both sides of (6.1), we have

$$\mathcal{A}_{n,\beta}^{\alpha_n}f(t)=f(x)\mathcal{A}_{n,\beta}^{\alpha_n}(1;x)+f'(x)\mathcal{A}_{n,\beta}^{\alpha_n}(t-x;x)+\frac{f''(x)}{2!}\mathcal{A}_{n,\beta}^{\alpha_n}((t-x)^2;x)+\mathcal{A}_{n,\beta}^{\alpha_n}((t-x)^2;x)\theta(t;x);x),$$

where

$$\mathcal{A}_{n,\beta}^{\alpha_n}((t-x)^2\theta(t;x);x)=\sum_{k=1}^{\infty}\frac{nx(nx+k\alpha_n)^{k-1}}{k!}e^{-(nx+k\alpha_n)}\frac{x^{n+1}}{n!}\int_0^\infty e^{-xs}s^n f\left(\frac{k(1-\alpha_n)}{xs}-x\right)^2\theta(t;x)ds.$$

Applying the Cauchy-Schwarz inequality to the right side twice in succession, we get

$$n\mathcal{A}_{n,\beta}^{\alpha_n}((t-x)^2\theta(t;x);x)\le\sqrt{n^2\mathcal{A}_{n,\beta}^{\alpha_n}((t-x)^4;x)}\sqrt{\mathcal{A}_{n,\beta}^{\alpha_n}(\theta(t;x);x)}.$$

From Lemma (6.1), we obtain $\mathcal{A}_{n,\beta}^{\alpha_n}((t-x)^4=\mathcal{O}(n^-2)$. Since $\theta(t;x)\in D[0,\infty)$ and $\lim_{n\to\infty}\theta(t;x)=0$, we get

$$\lim_{n\to\infty} n\mathcal{A}_{n,\beta}^{\alpha_n}((t-x)^2\theta(t;x);x)=0.$$

If (2.1) and Lemma (6.1) are used, the desired outcome can be achieved. □

7 Conclusion

The main motive of our research work is to construct a new sequence of Jain-Gamma operators which preserve linear function. Furthermore, we study the rate of convergence of these operators with the aid of modulus of continuity. Also discussed Voronovskaya type asymptotic theorem and Korovkin type theorem are discussed.

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Received: 2022-02-11
Accepted: 2022-04-12