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On some nested sums involving q-Fibonacci numbers

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Abstract In the present paper, we study the nested sums involving the q-Fibonacci numbers of the form

$$\sum_{h_n=1}^{\ell_n}\sum_{h_{n-1}=1}^{\ell_{n-1}}\cdots\sum_{h_1=1}^{\ell_1}F_{q,\beta+\sum_{i=1}^{n}k_ih_i},$$

where $F_{q,n}$ is the n^{th} q-Fibonacci number. We establish explicit formulas for these sums. As application, we derive closed formulas for $F_{q,\sum_{i=1}^{n}k_i}$ and hence we answer to the conjectures presented by Bilgici and Şentürk in [2].

1 Introduction

For a positive integer q and for all $n \geq 2$, the q-Fibonacci numbers satisfy the following recurrence relation; see, for example [4],

$$F_{q,n} = qF_{q,n-1} + F_{q,n-2}, \tag{1.1}$$

with initial conditions, $F_{q,0} = 0$ and $F_{q,1} = 1$. Clearly, for $q = 1$, the classical Fibonacci numbers are obtained, $F_{1,n} = F_n$ and for $q = 2$, the classical Pell numbers are obtained, $F_{2,n} = P_n$. Further details about the Fibonacci, Pell, and q-Fibonacci sequences could be found in [4, 5, 6].

The nested sums have attracted a lot of attention recently; for instance, Butler and Karasik [3] derived closed formulas for the nested sums involving binomial coefficients. They showed also how to write the Stirling numbers of the second kind as a nested sum.

In 2014, Belbachir and Harik [1] provided explicit formulas for $F_{\sum_{i=1}^{n}k_i+1}$ using aspects from graph theory. In a recent work, Bilgici and Şentürk [2] gave the explicit formulas of $F_{\sum_{i=1}^{n}k_i}$, $P_{\sum_{i=1}^{n}k_i}$ and $J_{\sum_{i=1}^{n}k_i}$ for the small values of n, where F_n, P_n, and J_n are the n^{th} Fibonacci, Pell and Jacobsthal numbers, respectively. They conjectured that for any integer $n \geq 2$ the following identities hold:

Conjecture 1.1. [2] Let $n \geq 3$ be an integer. For all positive integers k_i $(1 \leq i \leq n)$, we have

$$\begin{aligned}
F_{\sum_{i=1}^{n}k_i} &= \prod_{i=1}^{n}F_{k_i+1} - \prod_{i=1}^{n}F_{k_i-1} + (F_n-1)\prod_{i=1}^{n}F_{k_i} + (F_{n-1}-1)\sum_{j=1}^{n}F_{k_j-1}\prod_{\substack{i=1\\i\neq j}}^{n}F_{k_i}\\
&+ (F_{n-2}-1)\sum_{j_1=1}^{n-1}\sum_{j_2=j_1+1}^{n}F_{k_{j_1}-1}F_{k_{j_2}-1}\prod_{\substack{i=1\\i\neq j_1,j_2}}^{n}F_{k_i}\\
&+\cdots\\
&+ (F_3-1)\sum_{j_1=1}^{4}\sum_{j_2=j_1+1}^{5}\sum_{j_3=j_2+1}^{6}\cdots\sum_{j_{n-3}=j_{n-4}+1}^{n}F_{k_{j_1}-1}F_{k_{j_2}-1}\cdots F_{k_{j_{n-3}}-1}\prod_{\substack{i=1\\i\neq j_1,\ldots,j_{n-3}}}^{n}F_{k_i}.
\end{aligned}$$

Conjecture 1.2. [2] Let $n \geq 3$ be an integer. For all positive integers k_i $(1 \leq i \leq n)$, we have

$$\begin{aligned}
P_{\sum_{i=1}^n k_i} &= \frac{1}{2}\prod_{i=1}^{n} P_{k_i+1} - \frac{1}{2}\prod_{i=1}^{n} P_{k_i-1} + A_{n-2}\prod_{i=1}^{n} P_{k_i} + A_{n-3}\sum_{j=1}^{n} P_{k_j-1}\prod_{\substack{i=1\\ i\neq j}}^{n} P_{k_i} \\
&+ A_{n-4}\sum_{j_1=1}^{n-1}\sum_{j_2=j_1+1}^{n} P_{k_{j_1}-1}P_{k_{j_2}-1}\prod_{\substack{i=1\\ i\neq j_1,j_2}}^{n} P_{k_i} \\
&+ \cdots \\
&+ A_1\sum_{j_1=1}^{4}\sum_{j_2=j_1+1}^{5}\sum_{j_3=j_2+1}^{6}\cdots\sum_{j_{n-3}=j_{n-4}+1}^{n} P_{k_{j_1}-1}P_{k_{j_2}-1}\cdots P_{k_{j_{n-3}}-1}\prod_{\substack{i=1\\ i\neq j_1,\ldots,j_{n-3}}}^{n} P_{k_i},
\end{aligned}$$

where $A_n = \frac{3^n-1}{2}$.

The aim of this paper is to give an explicit formula for $F_{q,\sum_{i=1}^n k_i}$. Hence, we prove that Conjecture 1.1 is true but Conjecture 1.2 is wrong. Furthermore, we give an explicit formula for $P_{\sum_{i=1}^n k_i}$.

We organize our paper as follows: In Section 2, we give some useful lemmas needed in the next sections. In Section 3, we study the nested sum:

$$\sum_{h_n=1}^{\ell_n}\sum_{h_{n-1}=1}^{\ell_{n-1}}\cdots\sum_{h_1=1}^{\ell_1} F_{q,\beta+\sum_{i=1}^n k_i h_i}.$$

We also deduce an explicit formula for $F_{q,\sum_{i=1}^n k_i}$. Finally, in Section 4, we answer to the Conjectures 1.1 and 1.2.

Throughout this article, q is a positive integer, and $F_{q,n}$ is defined as in (1.1).

2 Preliminary results

In the present section, we give some notation and useful lemmas.

For any positive integers k, ℓ, and any integer $\beta \geq -1$, we denote by $\varphi_{q,\beta}^{\ell,k}$ the sum:

$$\varphi_{q,\beta}^{\ell,k} := \sum_{h=1}^{\ell} F_{q,\beta+kh}. \tag{2.1}$$

Lemma 2.1. *[4] For any non-negative integer m and any positive integer n, we have*

$$F_{q,m+n} = F_{q,n}F_{q,m+1} + F_{q,n-1}F_{q,m}. \tag{2.2}$$

From Lemma 2.1, it is clear that for any positive integers k, ℓ, and any integer $\beta \geq 0$,

$$F_{q,\beta+kh} = F_{q,kh}F_{q,\beta+1} + F_{q,kh-1}F_{q,\beta},$$

and we get the following lemma.

Lemma 2.2. *For any positive integers k, ℓ, and any integer $\beta \geq 0$, we have*

$$\varphi_{q,\beta}^{\ell,k} = F_{q,\beta+1}\varphi_{q,0}^{\ell,k} + F_{q,\beta}\varphi_{q,-1}^{\ell,k}. \tag{2.3}$$

Now, let $A(k_{j_1}, k_{j_2}, \ldots, k_{j_{m+1}})$ be as follows:

$$A(k_{j_1}, k_{j_2}, \ldots, k_{j_{m+1}}) := \varphi_{q,-1}^{\ell_{j_1},k_{j_1}}\varphi_{q,-1}^{\ell_{j_2},k_{j_2}}\cdots\varphi_{q,-1}^{\ell_{j_{m+1}},k_{j_{m+1}}}\prod_{\substack{i=1\\ i\neq j_1,\ldots,j_{m+1}}}^{n+1}\varphi_{q,0}^{\ell_i,k_i},$$

and we give the following lemma:

Lemma 2.3. *Let $n \geq 2$ be an integer. For any integer $m \in \{1,2,\dots,n-1\}$ and any positive integers k_i $(1 \leq i \leq n+1)$, we have*

$$\begin{aligned}&\sum_{j_1=1}^{n-m+1}\sum_{j_2=j_1+1}^{n-m+2}\cdots\sum_{j_{m+1}=j_m+1}^{n+1} A(k_{j_1},k_{j_2},\dots,k_{j_{m+1}})\\&=\sum_{j_1=1}^{n-m+1}\sum_{j_2=j_1+1}^{n-m+2}\cdots\sum_{j_m=j_{m-1}+1}^{n} A(k_{j_1},k_{j_2},\dots,k_{j_m},k_{n+1})\\&+\sum_{j_1=1}^{n-m}\sum_{j_2=j_1+1}^{n-m+1}\cdots\sum_{j_{m+1}=j_m+1}^{n} A(k_{j_1},k_{j_2},\dots,k_{j_{m+1}}),\end{aligned}$$

or equivalently,

$$\begin{aligned}&\sum_{j_1=1}^{n-m+1}\sum_{j_2=j_1+1}^{n-m+2}\cdots\sum_{j_{m+1}=j_m+1}^{n+1} \varphi_{q,-1}^{\ell_{j_1},k_{j_1}}\varphi_{q,-1}^{\ell_{j_2},k_{j_2}}\cdots\varphi_{q,-1}^{\ell_{j_{m+1}},k_{j_{m+1}}}\prod_{\substack{i=1\\ i\neq j_1,\dots,j_{m+1}}}^{n+1}\varphi_{q,0}^{\ell_i,k_i}\\&=\varphi_{q,-1}^{\ell_{n+1},k_{n+1}}\sum_{j_1=1}^{n-m+1}\sum_{j_2=j_1+1}^{n-m+2}\cdots\sum_{j_m=j_{m-1}+1}^{n} \varphi_{q,-1}^{\ell_{j_1},k_{j_1}}\varphi_{q,-1}^{\ell_{j_2},k_{j_2}}\cdots\varphi_{q,-1}^{\ell_{j_m},k_{j_m}}\prod_{\substack{i=1\\ i\neq j_1,\dots,j_m}}^{n}\varphi_{q,0}^{\ell_i,k_i}\\&+\sum_{j_1=1}^{n-m}\sum_{j_2=j_1+1}^{n-m+1}\cdots\sum_{j_{m+1}=j_m+1}^{n} \varphi_{q,-1}^{\ell_{j_1},k_{j_1}}\varphi_{q,-1}^{\ell_{j_2},k_{j_2}}\cdots\varphi_{q,-1}^{\ell_{j_{m+1}},k_{j_{m+1}}}\prod_{\substack{i=1\\ i\neq j_1,\dots,j_{m+1}}}^{n+1}\varphi_{q,0}^{\ell_i,k_i}.\end{aligned}\tag{2.4}$$

Proof. It suffices to use the fact that the union of the sets:

$\{(j_1,j_2,\dots,j_m,n+1)|1 \leq j_1 < \cdots < j_m \leq n\} \cup \{(j_1,j_2,\dots,j_{m+1})|1 \leq j_1 < \cdots < j_{m+1} \leq n\}$,

yields simply the bigger set as follows:

$$\{(j_1,j_2,\dots,j_{m+1})|1 \leq j_1 < \cdots < j_{m+1} \leq n+1\}.$$

□

3 Main results

We present in this part our main results.

Theorem 3.1. *Let $n \geq 1$ be an integer. For any positive integers k_i and ℓ_i $(1 \leq i \leq n)$ and any integer $\beta \geq 0$, we have*

$$\begin{aligned}&\sum_{h_n=1}^{\ell_n}\sum_{h_{n-1}=1}^{\ell_{n-1}}\cdots\sum_{h_1=1}^{\ell_1} F_{q,\beta+\sum_{i=1}^n k_i h_i}\\&=F_{q,\beta+n}\prod_{i=1}^{n}\varphi_{q,0}^{\ell_i,k_i}+F_{q,\beta+n-1}\sum_{j=1}^{n}\varphi_{q,-1}^{\ell_j,k_j}\prod_{\substack{i=1\\ i\neq j}}^{n}\varphi_{q,0}^{\ell_i,k_i}\\&+F_{q,\beta+n-2}\sum_{j_1=1}^{n-1}\sum_{j_2=j_1+1}^{n}\varphi_{q,-1}^{\ell_{j_1},k_{j_1}}\varphi_{q,-1}^{\ell_{j_2},k_{j_2}}\prod_{\substack{i=1\\ i\neq j_1,j_2}}^{n}\varphi_{q,0}^{\ell_i,k_i}\\&+\cdots\\&+F_{q,\beta}\sum_{j_1=1}^{1}\sum_{j_2=j_1+1}^{2}\cdots\sum_{j_n=j_{n-1}+1}^{n}\varphi_{q,-1}^{\ell_{j_1},k_{j_1}}\varphi_{q,-1}^{\ell_{j_2},k_{j_2}}\cdots\varphi_{q,-1}^{\ell_{j_n},k_{j_n}}\prod_{\substack{i=1\\ i\neq j_1,\cdots,j_n}}^{n}\varphi_{q,0}^{\ell_i,k_i}.\end{aligned}\tag{3.1}$$

Proof. For the proof, we use mathematical induction on n. By Identity (2.3), Formula (3.1) is true for $n = 1$. Assume now that Formula (3.1) is true for $n \geq 1$ and let us prove it for $n+1$. Let A denote the following quantity:

$$A := \sum_{h_{n+1}=1}^{\ell_{n+1}} \sum_{h_n=1}^{\ell_n} \cdots \sum_{h_1=1}^{\ell_1} F_{q,\beta+\sum_{i=1}^{n+1} k_i h_i} = \sum_{h_{n+1}=1}^{\ell_{n+1}} \left\{ \sum_{h_n=1}^{\ell_n} \cdots \sum_{h_1=1}^{\ell_1} F_{q,(\beta+k_{n+1}h_{n+1})+\sum_{i=1}^{n} k_i h_i} \right\},$$

by the induction hypothesis,

$$A = \sum_{h_{n+1}=0}^{\ell_{n+1}} \left\{ F_{q,\beta+k_{n+1}h_{n+1}+n} \left(\prod_{i=1}^{n} \varphi_{q,0}^{\ell_i,k_i} \right) + F_{q,\beta+k_{n+1}h_{n+1}+n-1} \sum_{j=1}^{n} \varphi_{q,-1}^{\ell_j,k_j} \prod_{\substack{i=1 \\ i\neq j}}^{n} \varphi_{q,0}^{\ell_i,k_i} \right.$$

$$+ F_{q,\beta+k_{n+1}h_{n+1}+n-2} \sum_{j_1=1}^{n-1} \sum_{j_2=j_1+1}^{n} \varphi_{q,-1}^{\ell_{j_1},k_{j_1}} \varphi_{q,-1}^{\ell_{j_2},k_{j_2}} \prod_{\substack{i=1 \\ i\neq j_1,j_2}}^{n} \varphi_{q,0}^{\ell_i,k_i}$$

$$+ \cdots$$

$$\left. + F_{q,\beta+k_{n+1}h_{n+1}} \sum_{j_1=1}^{1} \sum_{j_2=j_1+1}^{2} \cdots \sum_{j_n=j_{n-1}+1}^{n} \varphi_{q,-1}^{\ell_{j_1},k_{j_1}} \cdots \varphi_{q,-1}^{\ell_n,k_{j_n}} \prod_{\substack{i=1 \\ i\neq j_1,\ldots,j_n}}^{n} \varphi_{q,0}^{\ell_i,k_i} \right\}.$$

Therefore,

$$A = \left\{ \sum_{h_{n+1}=1}^{\ell_{n+1}} F_{q,(\beta+n)+k_{n+1}h_{n+1}} \right\} \prod_{i=1}^{n} \varphi_{q,0}^{\ell_i,k_i}$$

$$+ \left\{ \sum_{h_{n+1}=1}^{\ell_{n+1}} F_{q,(\beta+n-1)+k_{n+1}h_{n+1}} \right\} \sum_{j=1}^{n} \varphi_{q,-1}^{\ell_j,k_j} \prod_{\substack{i=1 \\ i\neq j}}^{n} \varphi_{q,0}^{\ell_i,k_i}$$

$$+ \left\{ \sum_{h_{n+1}=1}^{\ell_{n+1}} F_{q,(\beta+n-2)+k_{n+1}h_{n+1}} \right\} \sum_{j_1=1}^{n-1} \sum_{j_2=j_1+1}^{n} \varphi_{q,-1}^{\ell_{j_1},k_{j_1}} \varphi_{q,-1}^{\ell_{j_2},k_{j_2}} \prod_{\substack{i=1 \\ i\neq j_1,j_2}}^{n} \varphi_{q,0}^{\ell_i,k_i}$$

$$+ \cdots$$

$$+ \left\{ \sum_{h_{n+1}=1}^{\ell_{n+1}} F_{q,\beta+k_{n+1}h_{n+1}} \right\} \sum_{j_1=1}^{1} \sum_{j_2=j_1+1}^{2} \cdots \sum_{j_n=j_{n-1}+1}^{n} \varphi_{q,-1}^{\ell_{j_1},k_{j_1}} \cdots \varphi_{q,-1}^{\ell_{j_n},k_{j_n}} \prod_{\substack{i=1 \\ i\neq j_1,\ldots,j_n}}^{n} \varphi_{q,0}^{\ell_i,k_i}.$$

By identity (2.3), we get

$$A = \left\{ \varphi_{q,0}^{\ell_{n+1},k_{n+1}} F_{\beta+n+1} + \varphi_{q,-1}^{\ell_{n+1},k_{n+1}} F_{\beta+n} \right\} \prod_{i=1}^{n} \varphi_{q,0}^{\ell_i,k_i}$$

$$+ \left\{ \varphi_{q,0}^{\ell_{n+1},k_{n+1}} F_{\beta+n} + \varphi_{q,-1}^{\ell_{n+1},k_{n+1}} F_{\beta+n-1} \right\} \sum_{j=1}^{n} \varphi_{q,-1}^{\ell_j,k_j} \prod_{\substack{i=1 \\ i\neq j}}^{n} \varphi_{q,0}^{\ell_i,k_i}$$

$$+ \left\{ \varphi_{q,0}^{\ell_{n+1},k_{n+1}} F_{q,\beta+n-1} + \varphi_{q,-1}^{\ell_{n+1},k_{n+1}} F_{q,\beta+n-2} \right\} \sum_{j_1=1}^{n-1} \sum_{j_2=j_1+1}^{n} \varphi_{q,-1}^{k_{j_1},k_{j_1}} \varphi_{q,-1}^{k_{j_2},k_{j_2}} \prod_{\substack{i=1 \\ i\neq j_1,j_2}}^{n} \varphi_{q,0}^{\ell_i,k_i}$$

$$+ \cdots$$

$$+ \left\{ \varphi_{q,0}^{\ell_{n+1},k_{n+1}} F_{q,\beta+1} + \varphi_{q,-1}^{\ell_{n+1},k_{n+1}} F_{q,\beta} \right\} \sum_{j_1=1}^{1} \sum_{j_2=j_1+1}^{2} \cdots \sum_{j_n=j_{n-1}+1}^{n} \varphi_{q,-1}^{\ell_{j_1},kj_1} \cdots \varphi_{q,-1}^{\ell_n,k_{j_n}} \prod_{\substack{i=1 \\ i\neq j_1,\ldots,j_n}}^{n} \varphi_{q,0}^{\ell_i,k_i}.$$

Arrange the sums, we find

$$
\begin{aligned}
A &= F_{q,\beta+n+1}\prod_{i=1}^{n+1}\varphi_{q,0}^{\ell_i,k_i}\\
&+ F_{q,\beta+n}\left\{\varphi_{q,-1}^{\ell_{n+1},k_{n+1}}\prod_{i=1}^{n}\varphi_{q,0}^{\ell_i,k_i}+\sum_{j=1}^{n}\varphi_{q,-1}^{\ell_i,k_j}\prod_{\substack{i=1\\ i\neq j}}^{n+1}\varphi_{q,0}^{\ell_i,k_i}\right\}\\
&+ F_{\beta+n-1}\left\{\varphi_{q,-1}^{\ell_{n+1},k_{n+1}}\sum_{j=0}^{n}\varphi_{q,-1}^{\ell_j,k_j}\prod_{\substack{i=1\\ i\neq j}}^{n}\varphi_{q,0}^{\ell_i,k_i}+\sum_{j_1=1}^{n-1}\sum_{j_2=j_1+1}^{n}\varphi_{q,-1}^{\ell_{j_1},k_{j_1}}\varphi_{q,-1}^{\ell_{j_2},k_{j_2}}\prod_{\substack{i=1\\ i\neq j_1,j_2}}^{n+1}\varphi_{q,0}^{\ell_i,k_i}\right\}\\
&+\cdots\\
&+ F_{q,\beta+1}\left\{\varphi_{q,-1}^{\ell_{n+1},k_{n+1}}\sum_{j_1=1}^{2}\sum_{j_2=j_1+1}^{3}\cdots\sum_{j_{n-1}=j_{l-2}+1}^{n}\varphi_{q,-1}^{\ell_{j_1},k_{j_1}}\cdots\varphi_{q,-1}^{\ell_{n-1},k_{j_{n-1}}}\prod_{\substack{i=1\\ i\neq j_1,\dots,j_{n-1}}}^{n}\varphi_{q,0}^{\ell_i,k_i}\right.\\
&\qquad\left.+\sum_{j_1=1}^{1}\sum_{j_2=j_1+1}^{2}\cdots\sum_{j_n=j_{n-1}+1}^{n}\varphi_{q,-1}^{\ell_{j_1},k_{j_1}}\cdots\varphi_{q,-1}^{\ell_{j_n},k_{j_n}}\prod_{\substack{i=1\\ i\neq j_1,\dots,j_n}}^{n+1}\varphi_{q,0}^{\ell_i,k_i}\right\}\\
&+ F_{q,\beta}\left\{\varphi_{q,-1}^{\ell_{n+1},k_{n+1}}\sum_{j_1=1}^{1}\sum_{j_2=j_1+1}^{2}\cdots\sum_{j_n=j_{n-1}+1}^{n}\varphi_{q,-1}^{\ell_{j_1},k_{j_1}}\cdots\varphi_{q,-1}^{\ell_{j_n},k_{j_n}}\prod_{\substack{i=1\\ i\neq j_1,\dots,j_n}}^{n}\varphi_{q,0}^{\ell_i,k_i}\right\}.
\end{aligned}
$$

Using (2.4), we get

$$
\begin{aligned}
&\sum_{h_{n+1}=1}^{\ell_{n+1}}\sum_{h_n=1}^{\ell_n}\cdots\sum_{h_1=1}^{\ell_1}F_{q,\sum_{i=1}^{n+1}k_ih_i+j}\\
&=F_{q,\beta+n+1}\prod_{i=1}^{n+1}\varphi_{q,0}^{\ell_i,k_i}+F_{q,\beta+n}\sum_{j=1}^{n+1}\varphi_{q,-1}^{\ell_j,k_j}\prod_{\substack{i=1\\ i\neq j}}^{n+1}\varphi_{q,0}^{\ell_i,k_i}\\
&+F_{q,\beta+n-1}\sum_{j_1=1}^{n}\sum_{j_2=j_1+1}^{n+1}\varphi_{q,-1}^{\ell_{j_1},k_{j_1}}\varphi_{q,-1}^{\ell_{j_2},k_{j_2}}\prod_{\substack{i=1\\ i\neq j_1,j_2}}^{n+1}\varphi_{q,0}^{\ell_i,k_i}\\
&+\cdots\\
&+F_{q,\beta}\sum_{j_1=1}^{1}\sum_{j_2=j_1+1}^{2}\cdots\sum_{j_{n+1}=j_n+1}^{n+1}\varphi_{q,-1}^{\ell_{j_1},k_{j_1}}\varphi_{q,-1}^{\ell_{j_2},k_{j_2}}\cdots\varphi_{q,-1}^{\ell_{n+1},k_{n+1}}\prod_{\substack{i=1\\ i\neq j_1,\dots,j_{n+1}}}^{n+1}\varphi_{q,0}^{\ell_i,k_i}.
\end{aligned}
$$

Thus Formula (3.1) holds for all integer $n \geq 1$ and the proof is complete. □

From Theorem 3.1, we deduce the following important corollary.

Corollary 3.2. *Let $n \geq 2$ be an integer. For any positive integers k_i $(1 \leq i \leq n)$, we have*

$$
\begin{aligned}
F_{q,\sum_{i=1}^{n} k_i} =& F_{q,n} \prod_{i=1}^{n} F_{q,k_i} + F_{q,n-1} \sum_{j=1}^{n} F_{q,k_j-1} \prod_{\substack{i=1 \\ i\neq j}}^{n} F_{q,k_i} \\
&+ F_{q,n-2} \sum_{j_1=1}^{n-1} \sum_{j_2=j_1+1}^{n} F_{q,k_{j_1}-1} F_{q,k_{j_2}-1} \prod_{\substack{i=1 \\ i\neq j_1,j_2}}^{n} F_{q,k_i} + \cdots \\
&+ F_{q,1} \sum_{j_1=1}^{2} \sum_{j_2=j_1+1}^{3} \cdots \sum_{j_{n-1}=j_{n-2}+1}^{n} F_{q,k_{j_1}-1} F_{q,k_{j_2}-1} \cdots F_{q,k_{j_{n-1}}-1} \prod_{\substack{i=1 \\ i\neq j_1,\ldots,j_{n-1}}}^{n} F_{q,k_i}.
\end{aligned}
$$

Proof. From Theorem 3.1, if we let $\beta = 0$ and $\ell_i = 1$ $(1 \leq i \leq n)$, we obtain $\varphi_{q,0}^{1,k_i} = F_{q,k_i}$ and $\varphi_{q,-1}^{1,k_i} = F_{q,k_i-1}$ $(1 \leq i \leq n)$, and the result is immediately seen. □

4 Answers to some conjectures

In this part, we discuss Conjectures 1.1 and 1.2 presented in [2].

4.1 Conjecture 1.1 is true

In this part, we study Conjecture 1.1 involving classical Fibonacci numbers, and we answer the conjecture in the positive.

Using our notation, we have $\varphi_{1,0}^{1,k_i} = F_{k_i}$ and $\varphi_{1,-1}^{1,k_i} = F_{k_i-1}$ where F_n is the n^{th} classical Fibonacci number.
Now, we let $q = 1$ and $\ell_i = 1$ $(1 \leq i \leq n)$, for any positive integers $n \geq 2$, $m \in \{1, 2, \ldots, n-1\}$ and k_i $(1 \leq i \leq n+1)$, Lemma 2.3 became as follows:

$$
\begin{aligned}
&\sum_{j_1=1}^{n-m+1} \sum_{j_2=j_1+1}^{n-m+2} \cdots \sum_{j_{m+1}=j_m+1}^{n+1} F_{k_{j_1}-1} F_{k_{j_2}-1} \cdots F_{k_{j_{m+1}}-1} \prod_{\substack{i=1 \\ i\neq j_1,\ldots,j_{m+1}}}^{n+1} F_{k_i} \qquad (4.1) \\
&= \sum_{j_1=1}^{n-m+1} \sum_{j_2=j_1+1}^{n-m+2} \cdots \sum_{j_m=j_{m-1}+1}^{n} F_{k_{j_1}-1} F_{k_{j_2}-1} \cdots F_{k_{j_m}-1} \left(\prod_{\substack{i=1 \\ i\neq j_1,\ldots,j_m}}^{n} F_{k_i} \right) F_{k_{n+1}-1} \\
&+ \sum_{j_1=1}^{n-m} \sum_{j_2=j_1+1}^{n-m+1} \cdots \sum_{j_{m+1}=j_m+1}^{n} F_{k_{j_1}-1} F_{k_{j_2}-1} \cdots F_{k_{j_{m+1}}-1} \prod_{\substack{i=1 \\ i\neq j_1,\ldots,j_{m+1}}}^{n+1} F_{k_i}.
\end{aligned}
$$

We also prove the following useful lemma:

Lemma 4.1. *Let $n \geq 2$ be an integer. For any positive integers k_i $(1 \leq i \leq n)$, we have*

$$
\begin{aligned}
\prod_{i=1}^{n} F_{k_i+1} - \prod_{i=1}^{n} F_{k_i-1} =& \prod_{i=1}^{n} F_{k_i} + \sum_{j=1}^{n} F_{k_j-1} \prod_{\substack{i=1 \\ i\neq j}}^{n} F_i \qquad (4.2) \\
&+ \sum_{j_1=1}^{n-1} \sum_{j_2=j_1+1}^{n} F_{k_{j_1}-1} F_{k_{j_2}-1} \prod_{\substack{i=1 \\ i\neq j_1,j_2}}^{n} F_{k_i} + \cdots \\
&+ \sum_{j_1=1}^{2} \sum_{j_2=j_1+1}^{3} \cdots \sum_{j_{n-1}=j_{n-2}+1}^{n} F_{k_{j_1}-1} F_{k_{j_2}-1} \cdots F_{k_{j_{n-1}}-1} \prod_{\substack{i=1 \\ i\neq j_1,\ldots,j_{n-1}}}^{n} F_{k_i}.
\end{aligned}
$$

Proof. By induction on n. For $n = 2$,

$$\begin{aligned}\prod_{i=1}^{2} F_{k_i+1} - \prod_{i=1}^{2} F_{k_i-1} &= (F_{k_1} + F_{k_1-1})(F_{k_2} + F_{k_2-1}) - F_{k_1-1}F_{k_2-1} \\ &= F_{k_1}F_{k_2} + (F_{k_1}F_{k_2-1} + F_{k_1-1}F_{k_2}) \\ &= \prod_{i=1}^{2} F_{k_i} + \sum_{j=1}^{2} F_{k_j-1} \prod_{\substack{i=1 \\ i\neq j}}^{2} F_{k_i}.\end{aligned}$$

Assume now that (4.2) is true for $n \geq 2$. We have then,

$$\begin{aligned}\prod_{i=1}^{n+1} F_{k_i+1} - \prod_{i=1}^{n+1} F_{k_i-1} =& F_{k_{n+1}+1} \prod_{i=1}^{n} F_{k_i+1} - F_{k_{n+1}-1} \prod_{i=1}^{n} F_{k_i-1} \\ =& (F_{k_{n+1}} + F_{k_{n+1}-1}) \prod_{i=1}^{n} F_{k_i+1} - F_{k_{n+1}-1} \prod_{i=1}^{n} F_{k_i-1} \\ =& F_{k_{n+1}} \prod_{i=1}^{n} F_{k_i+1} + F_{k_{n+1}-1} \left\{ \prod_{i=1}^{n} F_{k_i+1} - \prod_{i=1}^{n} F_{k_i-1} \right\}.\end{aligned}$$

By the induction hypothesis, we get the following:

$$\begin{aligned}& F_{k_{n+1}} \prod_{i=1}^{n} F_{k_i+1} \\ &= \prod_{i=1}^{n+1} F_{k_i} + \sum_{j=1}^{n} F_{k_j-1} \prod_{\substack{i=1 \\ i\neq j}}^{n+1} F_{k_i} \\ &+ \sum_{j_1=1}^{n-1} \sum_{j_2=j_1+1}^{n} F_{k_{j_1}-1} F_{k_{j_2}-1} \prod_{\substack{i=1 \\ i\neq j_1,j_2}}^{n+1} F_{k_i} + \cdots \\ &+ \sum_{j_1=1}^{2} \sum_{j_2=j_1+1}^{3} \cdots \sum_{j_{n-1}=j_{n-2}+1}^{n} F_{j_1-1} F_{j_2-1} \cdots F_{j_{n-1}-1} \prod_{\substack{i=1 \\ i\neq j_1,\ldots,j_{n-1}}}^{n+1} F_{k_i} + F_{k_{n+1}} \prod_{i=1}^{n} F_{k_i-1},\end{aligned}$$

and,

$$\begin{aligned}& F_{k_{n+1}-1} \left\{ \prod_{i=1}^{n} F_{k_i+1} - \prod_{i=1}^{n} F_{k_i-1} \right\} = \left(\prod_{i=1}^{n} F_{k_i} \right) F_{k_{n+1}-1} + \sum_{j=1}^{n} F_{k_j-1} \left(\prod_{\substack{i=1 \\ i\neq j}}^{n} F_i \right) F_{k_{n+1}-1} \\ &+ \sum_{j_1=1}^{n-1} \sum_{j_2=j_1+1}^{n} F_{k_{j_1}-1} F_{k_{j_2}-1} \left(\prod_{\substack{i=1 \\ i\neq j_1,j_2}}^{n} F_{k_i} \right) F_{k_{n+1}-1} \\ &+ \cdots \\ &+ \sum_{j_1=1}^{2} \sum_{j_2=j_1+1}^{3} \cdots \sum_{j_{n-1}=j_{n-2}+1}^{n} F_{k_{j_1}-1} F_{k_{j_2}-1} \cdots F_{k_{j_{n-1}}-1} \left(\prod_{\substack{i=1 \\ i\neq j_1,\ldots,j_{n-1}}}^{n} F_{k_i} \right) F_{k_{n+1}-1}.\end{aligned}$$

Thus,

$$\prod_{i=1}^{n+1} F_{k_i+1} - \prod_{i=1}^{n+1} F_{k_i-1} = \prod_{i=1}^{n+1} F_{k_i} + \left\{ \left(\prod_{i=1}^{n} F_{k_i} \right) F_{k_{n+1}-1} + \sum_{j=1}^{n} F_{k_j-1} \prod_{\substack{i=1 \\ i \neq j}}^{n+1} F_i \right\}$$

$$+ \left\{ \sum_{j=1}^{n} F_{k_j-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n} F_i \right) F_{k_{n+1}-1} + \sum_{j_1=1}^{n-1} \sum_{j_2=j_1+1}^{n} F_{j_1-1} F_{j_2-1} \prod_{\substack{i=1 \\ i \neq j_1, j_2}}^{n+1} F_i \right\}$$

$$+ \cdots$$

$$+ \left\{ \sum_{j_1=1}^{2} \sum_{j_2=j_1+1}^{3} \cdots \sum_{j_{n-1}=j_{n-2}+1}^{n} F_{j_1-1} F_{j_2-1} \cdots F_{j_{n-1}-1} \left(\prod_{\substack{i=1 \\ i \neq j_1, \ldots, j_{n-1}}}^{n} F_i \right) F_{k_{n+1}-1} + \left(\prod_{i=1}^{n} F_{k_i-1} \right) F_{k_{n+1}} \right\}.$$

Using identity (4.1), we obtain

$$\begin{aligned} \prod_{i=1}^{n+1} F_{k_i+1} - \prod_{i=1}^{n+1} F_{k_i-1} = & \prod_{i=1}^{n+1} F_{k_i} \\ & + \sum_{j=1}^{n+1} F_{k_j-1} \prod_{\substack{i=1 \\ i \neq j}}^{n+1} F_i \\ & + \sum_{j_1=1}^{n} \sum_{j_2=j_1+1}^{n+1} F_{k_{j_1}-1} F_{k_{j_2}-1} \prod_{\substack{i=1 \\ i \neq j_1, j_2}}^{n+1} F_{k_i} + \cdots \\ & + \sum_{j_1=1}^{2} \sum_{j_2=j_1+1}^{3} \cdots \sum_{j_n=j_{n-1}+1}^{n+1} F_{k_{j_1}-1} F_{k_{j_2}-1} \cdots F_{k_{j_n}-1} \prod_{\substack{i=1 \\ i \neq j_1, \ldots, j_n}}^{n+1} F_{k_i}. \end{aligned}$$

Finally, by the principle of mathematical induction, the statement is true for all integer $n \geq 2$. □

The main result of the present section is the following Theorem:

Theorem 4.2. *The conjecture 1.1 is true.*

Proof. In Corollary 3.2, if $q = 1$, we obtain

$$\begin{aligned} F_{\sum_{i=1}^{n} k_i} = & F_n \prod_{i=1}^{n} F_{k_i} + F_{n-1} \sum_{j=0}^{n} F_{k_j-1} \prod_{\substack{i=1 \\ i \neq j}}^{n} F_i + F_{n-2} \sum_{j_1=1}^{n-1} \sum_{j_2=j_1+1}^{n} F_{k_{j_1}-1} F_{k_{j_2}-1} \prod_{\substack{i=1 \\ i \neq j_1, j_2}}^{n} F_{k_i} \\ & + \cdots \\ & + F_1 \sum_{j_1=1}^{2} \sum_{j_2=j_1+1}^{3} \cdots \sum_{j_{n-1}=j_{n-2}+1}^{n} F_{k_{j_1}-1} F_{k_{j_2}-1} \cdots F_{k_{j_{n-1}}-1} \prod_{\substack{i=1 \\ i \neq j_1, \ldots, j_{n-1}}}^{n} F_{k_i}. \end{aligned}$$

We rewrite this formula as follows:

$$
\begin{aligned}
F_{\sum_{i=1}^{n} k_i} = {} & (F_n - 1)\prod_{i=1}^{n} F_{k_i} + (F_{n-1} - 1)\sum_{j=0}^{n} F_{k_j - 1}\prod_{\substack{i=1\\ i\neq j}}^{n} F_{k_i} \\
& + (F_{n-2} - 1)\sum_{j_1=1}^{n-1}\sum_{j_2=j_1+1}^{n} F_{k_{j_1}-1}F_{k_{j_2}-1}\prod_{\substack{i=1\\ i\neq j_1,j_2}}^{n} F_{k_i} \\
& + \cdots \\
& + (F_3 - 1)\sum_{j_1=1}^{4}\sum_{j_2=j_1+1}^{5}\cdots\sum_{j_{n-3}=j_{n-4}+1}^{n} F_{k_{j_1}-1}F_{k_{j_2}-1}\cdots F_{k_{j_{n-3}}-1}\prod_{\substack{i=1\\ i\neq j_1,\ldots,j_{l-3}}}^{n} F_{k_i} \\
& + \Bigg\{ \prod_{i=1}^{n} F_{k_i} + \sum_{j=1}^{n} F_{k_j-1}\prod_{\substack{i=1\\ i\neq j}}^{n} F_{k_i} + \sum_{j_1=1}^{n-1}\sum_{j_2=j_1+1}^{n} F_{k_{j_1}-1}F_{k_{j_2}-1}\prod_{\substack{i=1\\ i\neq j_1,j_2}}^{n} F_{k_i} + \cdots \\
& + \sum_{j_1=1}^{4}\sum_{j_2=j_1+1}^{5}\cdots\sum_{j_{n-3}=j_{n-4}+1}^{n} F_{k_{j_1}-1}F_{k_{j_2}-1}\cdots F_{k_{j_{l-3}}-1}\prod_{\substack{i=1\\ i\neq j_1,\ldots,j_{n-3}}}^{n} F_{k_i} \\
& + F_2\sum_{j_1=1}^{3}\sum_{j_2=j_1+1}^{4}\cdots\sum_{j_{n-2}=j_{n-3}+1}^{n} F_{k_{j_1}-1}F_{k_{j_2}-1}\cdots F_{k_{j_{n-2}}-1}\prod_{\substack{i=1\\ i\neq j_1,\ldots,j_{n-2}}}^{n} F_{i} \\
& + F_1\sum_{j_1=1}^{2}\sum_{j_2=j_1+1}^{3}\cdots\sum_{j_{n-1}=j_{n-2}+1}^{n} F_{k_{j_1}-1}F_{k_{j_2}-1}\cdots F_{k_{j_{n-1}}-1}\prod_{\substack{i=1\\ i\neq j_1,\ldots,j_{n-1}}}^{n} F_{k_i} \Bigg\}.
\end{aligned}
$$

Using Lemma 4.1, we obtain

$$
\begin{aligned}
F_{\sum_{i=1}^{n} k_i} = {} & (F_n - 1)\prod_{i=1}^{n} F_{k_i} + (F_{n-1} - 1)\sum_{j=0}^{n} F_{k_j - 1}\prod_{\substack{i=1\\ i\neq j}}^{n} F_{k_i} \\
& + (F_{n-2} - 1)\sum_{j_1=1}^{n-1}\sum_{j_2=j_1+1}^{n} F_{k_{j_1}-1}F_{k_{j_2}-1}\prod_{\substack{i=1\\ i\neq j_1,j_2}}^{n} F_{k_i} \\
& + \cdots \\
& + (F_3 - 1)\sum_{j_1=1}^{4}\sum_{j_2=j_1+1}^{5}\cdots\sum_{j_{n-3}=j_{n-4}+1}^{n} F_{k_{j_1}-1}F_{k_{j_2}-1}\cdots F_{k_{j_{n-3}}-1}\prod_{\substack{i=1\\ i\neq j_1,\ldots,j_{l-3}}}^{n} F_{k_i} \\
& + \prod_{i=1}^{n} F_{k_i+1} - \prod_{i=1}^{n} F_{k_i-1}.
\end{aligned}
$$

Finally, we deduce that Conjecture 1.1 is true. □

4.2 Conjecture 1.2 is false

In this part, we answer in the negative Conjecture 1.2, which involves classical Pell numbers. To do so, we construct a counterexample.

Let $n = 6$, $k_i = 2$ $(1 \leq i \leq 6)$ and replace in Conjecture 1.2, we get:

$$\begin{aligned} P_{12} &= \frac{1}{2}\prod_{i=1}^{6} P_3 - \frac{1}{2}\prod_{i=1}^{6} P_1 + A_4 \prod_{i=1}^{6} P_2 + A_3 \sum_{j=1}^{6} P_1 \prod_{\substack{i=1\\ i\neq j}}^{n} P_2 \\ &\quad + A_2 \sum_{j_1=1}^{5} \sum_{j_2=j_1+1}^{6} P_1 P_1 \prod_{\substack{i=1\\ i\neq j_1,j_2}}^{6} P_2 + A_1 \sum_{j_1=1}^{4} \sum_{j_2=j_1+1}^{5} \sum_{j_3=j_2+1}^{6} P_1 P_1 P_1 \prod_{\substack{i=1\\ i\neq j_m}}^{6} P_2, \end{aligned}$$

with $A_1 = 1$, $A_2 = 4$, $A_3 = 13$ and $A_4 = 40$, hence the numerical values are:

$$\begin{aligned} P_{12} &= \frac{15625}{2} - \frac{1}{2} + 2560 + 2496 + 960 + 160 \\ &= 13988. \end{aligned}$$

Finally, by Conjecture 1.2, $P_{12} = 13988$. But the real value of P_{12} is 13860 and Conjecture 1.2 is false.

Using Corollary 3.2, we give an explicit formula for $P_{\sum_{i=1}^{n} k_i}$.

Theorem 4.3. *Let $n \geq 2$ be an integer. For any positive integers k_i $(1 \leq i \leq n)$, we have*

$$\begin{aligned} P_{\sum_{i=1}^{n} k_i} &= P_n \prod_{i=1}^{n} P_{k_i} + P_{n-1} \sum_{j=1}^{n} P_{k-1} \prod_{\substack{i=1\\ i\neq j}}^{n} P_{k_i} \\ &\quad + P_{n-2} \sum_{j_1=1}^{n-1} \sum_{j_2=j_1+1}^{n} P_{k_{j_1}-1} P_{k_{j_2}-1} \prod_{\substack{i=1\\ i\neq j_1,j_2}}^{n} P_{k_i} \\ &\quad + \cdots \\ &\quad + P_1 \sum_{j_1=1}^{2} \sum_{j_2=j_1+1}^{3} \cdots \sum_{j_{n-1}=j_{n-2}+1}^{n} P_{k_{j_1}-1} P_{k_{j_2}-1} \cdots P_{q,k_{j_{n-1}}-1} \prod_{\substack{i=1\\ i\neq j_1,\ldots,j_{n-1}}}^{n} P_{k_i}. \end{aligned}$$

Proof. Set $q = 2$ in Corollary 3.2, then $F_{2,n} = P_n$, and the proof is immediately seen. □

5 Conclusion

We studied some nested sums involving q-Fibonacci numbers. We gave an explicit formula for these sums.

As an application, we derived closed formulas for $F_{\sum_{i=1}^{n} k_i}$ and $P_{\sum_{i=1}^{n} k_i}$ and we answered on two conjectures presented recently in [2].

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