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HOMOMORPHISMS BETWEEN RESTRICTED GENUS GROUPS

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Abstract For two finitely generated groups with finite commutator subgroup G_1 and G_2, homomorphisms between genera of groups $\mathcal{G}(G_1)$ and $\mathcal{G}(G_2)$ have been established in the literature, especially when G_2 is some quotient group obtained from G_1 or when G_2 is some power of G_1. These groups G_i $i = 1, 2$ are called χ_0-groups. For χ_0-groups under a given finite group F, we establish a homomorphism between the restricted genera. In the case, the homomorphism is surjective, it provides a computational method of the restricted genus..

1 Introduction

The theory of π-localization of groups, where π is a family of primes, appears to have been first discussed in [7, 8] by Mal'cev and Lazard. In their work emphasis was placed on the explicit construction of the localization and properties of the localization G_π of the nilpotent group G were deduced from the construction, utilizing nilpotent group theory. Baumslag in [1] has given a comprehensive treatment of the main properties of nilpotent groups as they relate to the problem of localization. He has explicitly shown in [2] how to construct G_π in the case of an arbitrary nilpotent group G and an arbitrary family of primes π. Thus extending the generality of Malcev's original construction. Bousfield-Kan [3] exploit this general Mal'cev construction in their study of completion and localization.

In the 1970s, Hilton and Mislin became interested, through their work on the localization of nilpotent spaces, in the localization of nilpotent groups. Mislin [10] defines the genus $\mathcal{G}(N)$ of a finitely generated nilpotent group N to be the set of isomorphism classes of finitely generated nilpotent groups M such that the localizations M_p and N_p are isomorphic at every prime p. This version of genus became known as the *Mislin genus*, and other very useful variations of this concept came into being. In [6] Hilton and Mislin define an abelian group structure on the genus set $\mathcal{G}(N)$ of a finitely generated nilpotent group N with finite commutator subgroup. For nilpotent groups which belong to class K (of semidirect products of the form $T \rtimes \mathbb{Z}^k$, where T is a finite abelian group and k is a positive integer), many computations of the genus groups appear in the literature.

The concept of cancellation is closely related to those of genus and localization of groups. When localizing non-nilpotent groups, it may happen that the kernel of the localizing homomorphism is bigger than what we would require. So, for a non-nilpotent finitely generated group G with finite commutator subgroup, rather than considering localization, the idea of the genus is generalized through non-cancellation. For a group G, the *non-cancellation set*, denoted by $\chi(G)$, is the set of isomorphism classes of groups H such that $G \times \mathbb{Z} \cong H \times \mathbb{Z}$. For a nilpotent $\mathcal{X}_0$-group H it was shown by Warfield [14] that $\chi(H) = \mathcal{G}(H)$, where $\mathcal{G}(H)$ is the *Mislin genus* of H. In O'Sullivan's paper [11, theorem 4.2] there are further equivalent definitions for the set $\chi(G)$. It is proved in [11] that for two $\mathcal{X}_0$-groups H and G, $H \times \mathbb{Z} \cong G \times \mathbb{Z}$ if and only if for every finite set π of primes, we have $H_\pi \cong G_\pi$ (the π-localizations are isomorphic). Thus for a $\mathcal{X}_0$-group H, the set $\chi(H)$ coincides with the *restricted genus* $\Gamma_f(H)$ of H.

Aspects of localization as in groups and related categories have been studied in a unified way in a categorical setting, see [12] for instance. In [9], Mba and Witbooi introduced a category Grp_F of groups under a finite group F. An object of Grp_F is a group homomorphism $h : F \to G$, and is denoted by (G, h). They defined a group structure on the restricted genus set $\Gamma_f(h) = \chi(h)$ of such object. We will $\chi(G, h)$ for the notation of this restricted genus. This genus group coincides with the non-cancellation group $\chi(G)$ of the underlying group G, in the case that F is the trivial group.

Constructing a map that preserves algebraic structure is a natural exercise when dealing with objects having interesting algebraic structure, and presents computations advantages. For example, for a semidirect product $H = \mathbb{Z}_m \rtimes_\omega \mathbb{Z}$, the authors in [5] showed that there is a well-defined surjective homomorphism $\Gamma : \chi(H) \to \chi(H^r)$ given by $[K] \to [K \times H^{r-1}]$ where K is a group such that $K \times \mathbb{Z} \cong H \times \mathbb{Z}$ and r is a natural number. Thus, in order to compute the group $\chi(H^r)$ one needs only to compute the kernel of the homomorphism Γ. For a fixed morphism $h : F \to G$, the restricted genus $\chi(G, h)$[1] is the set of isomorphism classes of morphisms $F \to H$, which are π-equivalent to h at every finite set of primes π. For a well-defined integer n depending on G, in [9] an epimorphism $\zeta : (\mathbb{Z}/n)^*/\pm 1 \to \chi(G, h)$ is established and it is shown that there exist natural epimorphisms $\chi(G, h) \to \chi(G/h(F))$ (provided $h(F)$ is normal in G) and $\chi(G, h) \to \chi(G, h \circ i))$ (provided $i : F_0 \to F$ is a morphism), which are compatible with the various involved maps ζ.

Having such homomorphisms is not always given. In [9], computation methods of $\chi(G, h)$ in the special case G is a semidirect product $T \rtimes_\omega \mathbb{Z}^k$ are used in a very particular example to provide a concrete computation of $\chi(G, h)$. It is used to show that there doesn't exist any natural morphism $\chi(G) \to \chi(G/h(F))$ under $\chi(G, h)$.

In this paper, we focus on the class $\mathcal{X}_0$ of all finitely generated groups with finite commutator subgroup. Given two such groups G_1 and G_2 for which $n_1 = n(G_1)$ and $n_2 = n(G_2)$ are relatively prime, we aim at establishing a homomorphism between localization genera of such groups under a given finite group F.

The remainder of this paper is organized as follows:

Section 2 presents some preliminaries and notations. In Section 3, we establish a homomorphism $\chi(G_1, h) \to \chi(G_2, h)$ where $(n_1, n_2) = 1$ and provide the conditions under which it is an epimorphism. Under this condition, it follows from the first Isomorphism Theorem that $\chi(G_1, h_1)/\mathrm{Ker}\varphi \cong \chi(G_2, h_2)$. This gives a computational method to obtain $\chi(G_i, h_i)$ from $\chi(G_1, h_1)$ when n_1 and n_i are relatively prime.

2 Preliminaries and Notation

Let us fix a finite group F (and throughout this chapter F will denote some finite group). Let Grp_F be the category of groups under F. Here we mean that the objects of Grp_F are group homomorphisms $\varphi : F \to G$. Given another object $\varphi_1 : F \to G_1$, a morphism in Grp_F corresponds to a group homomorphism $\alpha : G \to G_1$ such that $\alpha \circ \varphi = \varphi_1$.

For a set of primes π, let us regard the π-localization of an object $\varphi : F \to G$ as being the object $\varphi_\pi : F \to G_\pi$. Then localization is an endofunctor of Grp_F. Let $\mathcal{X}_F$ be the full subcategory of $\mathcal{X}_0$-groups under F. Thus we can define the restricted genus $\Gamma_f(\varphi)$ as the set of all isomorphism classes $[\psi]$ of objects $\psi \in \mathcal{X}_F$ such that for every finite set π of primes, we have ψ_π isomorphic to φ_π. If F is the trivial group, then $\mathcal{X}_F$ can be identified with the class $\mathcal{X}_0$ of groups. In line with [15] and in analogy with $\mathcal{X}_0$-groups we shall write $\Gamma_f(\varphi) = \chi(\varphi)$.

For a specified homomorphism $\beta : E \to F$, there is a functor $\beta_* : \mathrm{Grp}_F \to \mathrm{Grp}_E$. Henceforth we shall also specify the codomain when referring to an object $\varphi : F \to G$ of $\mathcal{X}_F$, and denote the object by (G, φ).

For any $\mathcal{X}_0$-group G, the torsion elements form a characteristic subgroup of G, which we shall denote by T_G.

Remark 2.1. (a) Recall from [15, Section 2] how we assign to a $\mathcal{X}_0$-group G a natural number $n(G)$: Let n_1 be the exponent of T_G, let n_2 be the exponent of the group $\mathrm{Aut}(T_G)$, and let n_3 be the exponent of the torsion subgroup of the centre of G. Now we take $n(G) = n_1 n_2 n_3$. Then $n = n(G)$ has the property that the subgroup $G^{(n)} = \langle g^n \ : \ g \in G \rangle$ of G belongs to the centre of G and $G/G^{(n)}$ is a finite group.

[1]For the special case F is trivial ($F = *$), $\chi(G, * \to G) = \chi(G)$

(b) Let $\pi = \{p :\ p \text{ is a prime and } p|n(G)\}$. Then the short exact sequence
$1 \to G^{(n)} \to G \to G/G^{(n)} \to 1$
determines G as an extension of a π'-torsion-free finitely generated abelian group $G^{(n)}$ by a π-torsion group $G/G^{(n)}$. From [4, Proposition 3.1], it follows that the π-localization homomorphism $G \to G_\pi$ is injective.

(c) In [15] it is shown that $\chi(G)$ is a group and there is an epimorphism
$\zeta : \mathbb{Z}_n^*/\pm 1 \to \chi(G)$.

(d) Throughout the rest of Section 2, G will denote an infinite $\mathcal{X}_0$-group, $n = n(G)$, T_G a torsion subgroup of G, F a finite group and $h : F \to G$ a homomorphism.

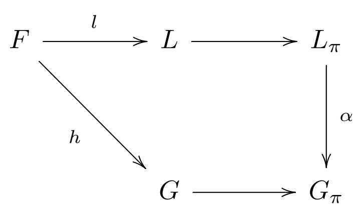
We note that [15, Theorem 4.3] can be formulated in a stronger form.

Proposition 2.2. *Let G be a $\mathcal{X}_0$-group and let $n = n(G)$. Let H and L be subgroups of finite index in G. If the index $[G : H]$ of H in G is relatively prime to n and $[G : L] \equiv \pm[G : H]$ mod n, then there is an isomorphism $\phi : L \to H$ such that $\phi(x) = x$ whenever $x \in T_G$.*

Proof. A proof of Proposition 2.2 can be obtained from the proof of [15, Theorem 4.3], noting that the latter proof starts off with the *inclusion* map of T_G into G and we consistently work *under* the inclusions of T_G into H, L and G.
Note that if $X \leq G$ and $T_G \subseteq X$, then h induces a homomorphism $h_X : F \to X$. We use this notation henceforth. □

Proposition 2.3. *[9, Theorem 2.3.] Let (L, l) be an object representing a member of $\chi(G, h)$. Then there exist a subgroup J of G with $[G : J]$ finite and $[G : J]$ relatively prime to n, such that in Grp_F the object $F \to J$ is isomorphic to (L, l).*

Proof. Suppose that (L, l) is an object representing a member of $\chi(G, h)$.
Let π be the set consisting of all the prime divisors of n and all the prime divisors of $|T_G|$. Then there is an isomorphism $\alpha : L_\pi \to G_\pi$ $[(L_\pi, l') \to (G_\pi, h')]$ such that the following diagram is commutative



We observe that G_π contains an isomorphic copy of T_G and every torsion element of G_π is contained in this subset, which we shall denote by T_G.
Now T_G is normal in G_π and G_π/T_G is a π-local torsion-free abelian group.
Let $\eta : G_\pi \to G_\pi/T_G$ be the canonical epimorphism.
We now set out to find a subgroup M of G_π which has the following properties :

(1) $M \supseteq \Im(L) \cup \Im(G)$,

(2) $[M : \Im(G)] \equiv 1 \bmod n$.

First we choose $M_0 = \langle \Im(G) \cup \Im(L) \rangle < G_\pi$.
Then $r = [M_0 : G] < \infty$ and r is relatively prime to n. Choose r' such that $(r', n) = 1$ and $r'r \equiv 1 \bmod n$.
Let $M_1 = \Im[M_0 \to G_\pi/T_G] = \eta M_0$. Then M_1 is finitely generated and torsion-free. Take any $\mathbb{Z}$-basis $\delta_1, \cdots, \delta_k$ for M_1. Then since $\mathbb{Z}_\pi^k$ is π-local and $(r', q) = 1$ $\forall q \in \pi$, we can find $\delta_0 \in G_\pi/T_G$ such that $r'\delta_0 = \delta_1$.
Then $\{\delta_0, \delta_2, \delta_3, \cdots, \delta_k\}$ is linearly independent.
Let $M_2 = \langle \delta_0, \delta_2, \delta_3, \cdots, \delta_k \rangle$ and take $M = \eta^{-1}(M_2)$.
It follows that $[M : M_0] = r'$ and therefore $[M : G] = [M : M_0] \cdot [M_0 : G] = r'r \equiv 1 \bmod n$.
Then by Theorem 2.2, (M, h') is isomorphic to (G, h) (here h' is h followed by the inclusion $G \to M$). Furthermore we have a morphism $\beta : (L, l) \to (M, h')$ such that $\beta : L \to M$ is a monomorphism. The theorem follows. □

Notation Fix any $m \in \mathbb{N}$. Let $X(m) = \{u \in \mathbb{N} | \ (u, m) = 1\}$. Now consider any $G \in \mathcal{X}_0$ and let $n = n(G)$. Let $Y(G, h)$ be the set of all $u \in X(n)$ for which there exists a subgroup J of G with $[G : J] = u$ and such that the object (J, h_J) represents a member of $\chi(G, h)$. Here h_J is the induced homomorphism obtained from h by restriction of the codomain. For each $u \in Y(G, h)$, let us choose a subgroup G_u of G such that $T_G \subseteq G_u$ and $[G : G_u] = u$. Let $h_u : F \to G_u$ be the induced homomorphism defined by $h_u : x \mapsto h(x)$. Now let us denote the isomorphism class of the object h_u of $\mathcal{X}_F$ by $[G_u, h_u]$. Then we obtain a function $\xi : Y(G, h) \to \chi(G, h)$. Let $Y^*(G, h)$ denote the image of $Y(G, h)$ in $\mathbb{Z}_n^*$.

Proposition 2.4. *[9, Theorem 2.5.]*

(a) $Y^*(G, h)$ is a subgroup of $\mathbb{Z}_n^*$.

(b) If ξ is a function of the type defined in 2, then ξ induces a (well-defined) function $\zeta : Y^*(G, h)/ \pm 1 \to \chi(G, h)$.

(c) The fibre $\zeta^{-1}[G, h]$ of ζ over $[G, h]$ is a subgroup of $Y^*(G, h)/ \pm 1$.

(d) For any $[K, k] \in \chi(G, h)$, $\zeta^{-1}[K, k]$ is a coset of $\zeta^{-1}[G, h]$.

Proof. (a) Since $1 \in Y^*(G, h)$, the subset in non-empty. It suffices to prove multiplicative closure. Now given any $u, v \in Y(G, h)$, choose a subgroup G_u. We obtain a subgroup of the form G_{uv} by taking any subgroup K of G_u such that (K, h_K) represents a member of $\chi(G_u, h_u)$ and with $[G_u : K] = v$. This proves (a).
(b) This follows from Theorem 2.2
(c) We first prove that $\zeta^{-1}(G, h)$ is closed with respect to inversion. Pick any $u \in X(n)$ such that $u \in \zeta^{-1}(G, h)$, and let $v \in X(n)$ be such that $uv \equiv \pm 1 \bmod n$. Let K and L be subgroups of indices u and v in G. Since $(K, h_K) \cong (G, h)$ there is an isomorphism $\alpha : G \to K$ such that $\alpha(x) = x$ for each $x \in h(F)$.
Let $\alpha(L) = J$. Now we note that $(L, h_L) \cong (J, h_J)$. On the other hand,
$[G : J] = [G : K] \cdot [K : J] = u \cdot [G : L] = uv \equiv \pm 1 \bmod m$,
and thus by Theorem 2.2, we have $(J, h_J) \cong (G, h)$. Thus $(L, h_L) \cong (G, h)$ and it follows that $v \in \zeta^{-1}[G, h]$. So we have proved that $\zeta^{-1}(G, h)$ is closed with respect to inversion. It is easy to prove that $\zeta^{-1}(G, h)$ is multiplicatively closed.
(d) Let $u, v \in X(n)$. Suppose that $\zeta(v) = [G, h]$, and let $K < G$ be such that $[G : K] = u$. Now we prove that $\zeta(uv) = [K, h_K]$. Since $\zeta(v) = [G, h]$, there is an embedding $\alpha : G \to G$ such that $\alpha(x) = x$ for each $x \in h(F)$, and such that $[G : \alpha(G)] = v$.
Let $L = \alpha(K)$. Then $[L, h_L] = \zeta(uv)$ because
$[G : L] = [G : \alpha(G)] \cdot [\alpha(G) : \alpha(K)] = vu$. Moreover, $[L, h_L] = [K, h_K]$. Thus $v\zeta^{-1}[G, h] \subset \zeta^{-1}[K, h_K]$. The inclusion $v\zeta^{-1}[G, h] \supset \zeta^{-1}[K, h_K]$ can be proved in a similar manner. This completes the proof of (c). □

Remark 2.5. Suppose that $h : F \to G$ is a group homomorphism and $h(F) \leq K \leq G$. If $h(F) \lhd G$ and $h(F) < K < G$, then $h(F) \lhd K$. In this case, the quotient group $K/h(F)$ is defined and we shall denote it by $\tilde{K}$.

3 Homomorphisms between localization genera

Let G_1 and G_2 be $\mathcal{X}_0$- groups and let $h_1 : F \to G_1$ and $h_2 : F \to G_2$ be fixed morphisms for a given finite group F.
Consider the non-cancellation groups $\chi(G_1, h_1)$ and $\chi(G_2, h_2)$. Let $n = n(G)$ be as defined above. In the following, we assume that $n_1 = n(G_1)$ and $n_2 = n(G_2)$ are relatively prime. We have the following proposition which is a consequence of Euclid's Lemma.

Proposition 3.1. *Given two integers n, m such that $(n, m) = 1$, it follows that for any u, $k \in \mathbb{N}$, $(n^u, m^k) = 1$, that is any powers of n and m are also relatively prime.*

Notation Let $m \in \mathbb{N}$ and $X(m) = \{u \in \mathbb{N} \ : \ (u,m) = 1\}$. Let G_i be a X_0-group and $n_i = n(G_i)$. Let $Y(G_i, h_i)$ be the set of all $u_i \in X(n_i)$ for which there exist a subgroup J_i of G_i with $[G_i : J_i] = u_i$ and such that (J_i, h_{J_i}) represents a member of $\chi(G_i, h_i)$, and h_{J_i} is the induced homomorphism obtained from h_i by restricting the codomain. Let $Y^*(G_i, h_i)$ be the image of $Y(G_i, h_i)$ in $\mathbb{Z}_n^*$, $i \in \{1, 2\}$.

Definition 3.2. For $x \in \mathbb{R}$, two functions f and g can be associated to x such that $x = f(x) + g(x)$, where f is the floor function defined by $f(x) = x$ (It is a function that returns the largest integer less than or equal to x) and g known as the fraction part of x is given by $g(x) = x \pmod 1$.

Lemma 3.3. *Let x be the floor function. Then: $x + y \leq x + y$ The equality holds: $x + y = x + y$ if and only if: x mod $1 + y$ mod $1 < 1$*

Proof. From the definition of the modulo operation, we have that: $x = x + (x \bmod 1)$, from which we obtain: $x + y = x + (x \bmod 1) + y + (y \bmod 1) = x + y + (x \bmod 1) + (y \bmod 1)$. Hence the inequality. The equality holds if and only if: $(x \bmod 1) + (y \bmod 1) = 0$. That is, if and only if: $x \bmod 1 + y \bmod 1 < 1$ □

Remark 3.4. Note that the description of u_i given in Notation 3 guarantees that u_i is always positive. We state the following result which is a consequence of Proposition 3.1.

Proposition 3.5. *For each $u_1 \in Y(G_1, h_1)$, let $u_2 \in Y(G_2, h_2)$ be such that $u_2 = n_1^{ln(u_1)}$. Then the association $u_1 \to u_2$ defines a mapping $\gamma_{12} : Y(G_1, h_1) \to Y(G_2, h_2)$ such that $\gamma_{12}(u_1) = n_1^{ln(u_1)}$ with the property that for w in the domain of γ_{12},*

$$\gamma_{12}(u_1 w) = \begin{cases} n_1^{ln(u_1 w)}, & if \ ln(u_1) \ mod \ 1 + ln(w) \ mod \ 1 < 1; \\ n_1^{ln(u_1 w) - 1}, & otherwise. \end{cases}$$

and such that we have the following diagram

$$\begin{array}{ccc} Y(G_1, h_1) & \xrightarrow{\gamma_{12}} & Y(G_2, h_2) \\ \downarrow \gamma_1 & & \downarrow \gamma_2 \\ Y^*(G_1, h_1)/\pm 1 & \xrightarrow[\alpha]{} & Y^*(G_2, h_2)/\pm 1 \end{array}$$

and $\alpha \circ \gamma_1 = \gamma_2 \circ \gamma_{12}$

Proof. The existence and well-definedness of γ_{12} follows directly from the proof of Proposition 3.1. □

Proposition 3.6. *Suppose that we have groups A, B and C together with a homomorphism $\beta : A \to C$ and a surjective group homomorphism $\gamma : A \to B$. If $\alpha : B \to C$ is a function (between sets) such that $\alpha \circ \gamma = \beta$, then α is a homomorphism. Moreover, if β is surjective, then α is also surjective.*

Remark 3.7. Consider the following diagram:

$$\begin{array}{ccc} Y(G_1, h_1) & \xrightarrow{\gamma_{12}} & Y(G_2, h_2) \\ \downarrow \gamma_1 & & \downarrow \gamma_2 \\ Y^*(G_1, h_1)/\pm 1 & \xrightarrow[\alpha]{} & Y^*(G_2, h_2)/\pm 1 \\ \downarrow \theta & & \downarrow \gamma \\ \chi(G_1, h_1) & \xrightarrow[\varphi]{} & \chi(G_2, h_2) \end{array}$$

Notice that if $u_i \neq \pm 1$, $u_i \in Y(G_i, h_i)$, and since the units of $\mathbb{Z}^*_{n_i}$ are relatively prime, we have that $\gamma_i(u_i) = u_i$. Therefore $(\alpha \circ \gamma_1)(u_1) = \alpha(u_1)$ and $(\gamma_2 \circ \gamma_{12})(u_1) = \gamma_2(\gamma_{12}(u_1)) = \gamma_2(n_1^{ln(u_1)}) = n_1^{ln(u_1)}$. This implies that $\alpha \circ \gamma_1 = \gamma \circ \gamma_{12}$, that is the diagram is commutative. we can therefore conclude that the map α can be defined as $\alpha : u \mapsto n_1^{ln(u)}$.

Now we state and proof our main result.

Theorem 3.8. *Let $\alpha : Y^*(G_1, h_1)/\pm 1 \to Y^*(G_2, h_2)/\pm 1$ be as defined above. Then we have the following:*

(i) α is a well-defined homomorphism for which we have the following diagram, with $\alpha \circ \gamma_1 = \gamma_2 \circ \gamma_{12}$

$$\begin{array}{ccc} Y(G_1,h_1) & \xrightarrow{\gamma_{12}} & Y(G_2,h_2) \\ \Big\downarrow{\scriptstyle \gamma_1} & & \Big\downarrow{\scriptstyle \gamma_2} \\ Y^*(G_1,h_1)/\pm 1 & \xrightarrow[\alpha]{} & Y^*(G_2,h_2)/\pm 1 \end{array}$$

(ii) α is the homomorphism such that there exist a unique homomorphism $\varphi : \chi(G_1, h_1) \to \chi(G_2, h_2)$ such that the following diagram is commutative.

$$\begin{array}{ccc} Y^*(G_1,h_1)/\pm 1 & \xrightarrow[\alpha]{} & Y^*(G_2,h_2)/\pm 1 \\ \Big\downarrow{\scriptstyle \theta} & & \Big\downarrow{\scriptstyle \gamma} \\ \chi(G_1,h_1) & \xrightarrow[\varphi]{} & \chi(G_2,h_2) \end{array}$$

Proof.

(i) We first prove that α is well-defined.
Let u_1 and u_2 be in the domain of α such that $u_1 = u_2$. Then $\alpha(u_1) = n_1^{ln(u_1)} = n_1^{ln(u_2)} = \alpha(u_2)$. Next we show that α is a homomorphism, that is , for any u_1, u_2 in the domain of α, $\alpha(u_1u_2) = \alpha(u_1)\alpha(u_2)$.
Case 1: $ln(u_1) \bmod 1 + ln(u_2) \bmod 1 < 1$. $\alpha(u_1u_2) = n_1^{\ln(u_1u_2)} = n_1^{\ln(u_1)+\ln(u_2)} = n_1^{\ln(u_1)+\ln(u_2)} = n_1^{\ln(u_1)}.n_1^{\ln(u_2)} = \alpha(u_1)\alpha(u_2)$. Hence α is a homomorphism.
Case 2: $\ln(u_1) \bmod 1 + \ln(u_2) \bmod 1 \geq 1$
$\alpha(u_1u_2) = n_1^{\ln(u_1u_2)-1} = n_1^{\ln(u_1)+\ln(u_2)-1} = n_1^{(\ln(u_1)+\ln(u_2)+1)-1} = n_1^{\ln(u_1)} \cdot n_1^{\ln(u_2)} = \alpha(u_1)\alpha(u_2)$.
Hence α is a homomorphism.

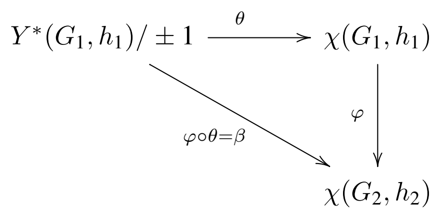
(ii) Let $u \in Y^*(G_1, h_1)/\pm 1$. Then, $\theta(u) = [J_u, h_u]$ such that $[G : J_u] = u$ and $\alpha(u) = n_1^{\ln(u)} \in Y^*(G_2, h_2)$
$\gamma(n_1^{ln(u)}) = [J_{n_1^{\ln(u)}}, h_{n_1^{\ln(u)}}]$ such that $[G : J_{n_1^{\ln(u)}}] = n_1^{\ln(u)}$.
Therefore $\varphi : \chi(G_1, h_1) \to \chi(G_2, h_2)$ is such that $\varphi([J_u, h_u]) = [J_{n_1^{\ln(u)}}, h_{n_1^{\ln(u)}}]$
Thus $(\varphi \circ \theta)(u) = \varphi(\theta(u)) = \varphi([J_u, h_u]) = [J_{n_1^{\ln(u)}}, h_{n_1^{\ln(u)}}]$ and $(\gamma \circ \alpha)(u) = \gamma(\alpha(u)) = \gamma(n_1^{\ln(u)}) = [J_{n_1^{\ln(u)}}, h_{n_1^{\ln(u)}}]$ Lastly, we prove that φ is a homomorphism. Consider the following diagram which is the top half of the above diagram.

$$\begin{array}{ccc} Y^*(G_1,h_1)/\pm 1 & \xrightarrow{\theta} & \chi(G_1,h_1) \\ & \searrow{\scriptstyle \varphi\circ\theta=\beta} & \Big\downarrow{\scriptstyle \varphi} \\ & & \chi(G_2,h_2) \end{array}$$

since $\varphi \circ \theta = \beta$, it follows from Proposition 3.6 that φ is a homomorphism.

□

Remark 3.9. Given the following diagram,



it follows from Proposition 3.6 that if β is surjective then φ is surjective. Hence, from the first Isomorphism Theorem, $\chi(G_1, h_1)/\mathrm{Ker}\varphi \cong \chi(G_2, h_2)$. This gives a computational method to obtain $\chi(G_2, h_2)$ from $\chi(G_1, h_1)$ when n_1 and n_2 are relatively prime.

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