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Bounds related to product variants of graphs

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Abstract. Operations in graph theory have a significant influence in the theoretical and application aspect of the domain. Topological indices serve as a crucial component in chemical graph theory linked with some molecular structure. Recently, the study on the new graph product variants is initiated. In the article, the computation of some bounds for atom-bond connectivity index, inverse sum indeg index, geometric-arithmetic index and sombor index of graph operations notably the corona join product, subdivision vertex join product and the subdivision vertex-edge join is carried out.

1 Introduction

For the computation of molecular descriptors, a chemical compound needs to amend itself to a molecular graph such that the atoms of the molecule correspond to vertices and the atomic links are depicted to be the edges. For a molecular graph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$, $\mathcal{V}(\mathcal{H})$ represents the vertex set and $\mathcal{E}(\mathcal{H})$ represents the edge set. p and q denote the cardinality of the vertex and edge sets of the graph, $\mathcal{H}$ respectively. Let $d(w)$ denote the degree of vertex w in $\mathcal{H}$ and $e = vw$ is the edge joining the vertex v with vertex w. The minimum and maximum degree of the graph, $\mathcal{H}$ is denoted by $\delta_{\mathcal{H}}$ and $\Delta_{\mathcal{H}}$ respectively.

Topology of a molecule is fundamentally a non numerical mathematical unit. Numerous measurable characteristics of molecule are often revealed in the form of specific numerals. For relating the molecular topology to any real chemical attribute, the conversion of the relevant details embedded into chemical structure to some numeric value becomes so vital which ultimately paves the way for the emergence of topological indices.

Wiener index is one of the earliest and extensively explored molecular descriptors [28]. The First and Second Zagreb Indices are the widely known graph invariants defined by Gutman to specify π-electron energy of the molecules[18, 17]. Theories and applications on atom-bond connectivity index(ABC) were postulated and further work has been carried out with respect to the graph invariant[11, 6, 8]. ABC index of graph $\mathcal{H}$ is defined as:

$$ABC(\mathcal{H}) = \sum_{vw\in\mathcal{E}(\mathcal{H})} \sqrt{\frac{d(v)+d(w)-2}{d(v)d(w)}}$$

Inverse sum indeg index(ISI) was presented as and when the concept of adriatic indices came into existence [26]. Later bounds and theoretical traits of the descriptor have been well-demonstrated in [23, 12]. Also, relation between inverse sum indeg index with the other invariants have also been established in [22].ISI of graph, $\mathcal{H}$ is denoted as:

$$ISI(\mathcal{H}) = \sum_{vw\in\mathcal{E}(\mathcal{H})} \frac{d(v)d(w)}{d(v)+d(w)}$$

Damir Vukičević and Boris Furtula implemented the arithmetic and geometric mean to establish another novel descriptor identified as the geometric-arithmetic index(GA) [25, 30], as specified by:

$$GA(\mathcal{H}) = \sum_{vw \in \mathcal{E}(\mathcal{H})} \frac{2\sqrt{d(v)d(w)}}{d(v)+d(w)}$$

Lately, the study on another graph invariant, sombor index(SO) was initiated by Ivan Gutman while approaching geometrically towards the notion of indices which are dependent on vertex degrees[16]. The sombor index is being explored widely by many researchers related to its theoretical properties, characterisations and utilities [7, 5].For graph $\mathcal{H}$, it is determined by:

$$SO(\mathcal{H}) = \sum_{vw \in \mathcal{E}(\mathcal{H})} \sqrt{(d(v))^2 + (d(w))^2}$$

The join $\mathcal{H}_1 + \mathcal{H}_2$ of graphs $\mathcal{H}_1(p_1, q_1)$ and $\mathcal{H}_2(p_2, q_2)$ with distinct vertex set $\mathcal{V}(\mathcal{H}_1)$ and $\mathcal{V}(\mathcal{H}_2)$ and edge sets $\mathcal{E}(\mathcal{H}_1)$ and $\mathcal{E}(H_2)$ is the graph $\mathcal{H}_1 \cup \mathcal{H}_2$ along with all the links connecting $\mathcal{V}(\mathcal{H}_1)$ and $\mathcal{V}(\mathcal{H}_2)$. $|\mathcal{V}(\mathcal{H}_1 + \mathcal{H}_2)| = p_1 + p_2$ and $|\mathcal{E}(\mathcal{H}_1 + \mathcal{H}_2)| = q_1 + q_2 + p_1p_2$. The corona product $\mathcal{H}_1 \circ \mathcal{H}_2$ of graphs $\mathcal{H}_1(p_1, q_1)$ and $\mathcal{H}_2(p_2, q_2)$ is acquired while taking a replica of $\mathcal{H}_1$ with p_1 replicas of $\mathcal{H}_2$ and by linking every node of the p_{th} replica of $\mathcal{H}_2$ to the p_{th} node of $\mathcal{H}_1$; $1 \leq p \leq p_1$. $|\mathcal{V}(\mathcal{H}_1 \circ \mathcal{H}_2)| = p_1p_2 + p_1$ and $|\mathcal{E}(\mathcal{H}_1 \circ \mathcal{H}_2)| = q_1 + p_1q_2 + p_1p_2$.
Bounds and inequalities related to some graph operation series have been well investigated for certain molecular descriptors [13, 14, 9, 21]. The subdivision graph, $S(\mathcal{H})$ is generated by replacing every link of the graph with a node of degree two, keeping the original nodes unchanged. The vertex set of $\mathcal{S}(\mathcal{H})$ have two partitions: $\mathcal{V}(\mathcal{H})$ with the initial nodes of $\mathcal{H}$ and $\mathcal{I}(\mathcal{H})$ with the inserted nodes in the original links of $\mathcal{H}$ for subdividing links of $\mathcal{H}$. Some graph operations are performed on sombor indices in [1]. The join, cartesian, corona and lexicographic products are implemented on the reformulated second zagreb index in [20]. Consequently, three relevant variations of graph operations namely the corona join product, subdivision vertex join and subdivision vertex-edge join were discussed and explored using the graph invariants[3, 24].
Corona join product:[3] Suppose $H_1 = (p_1, q_1)$ & $H_2 = (p_2, q_2)$ are simple(no loops and multiple edges), undirected, unweighted, connected graphs, then corona join graph of H_1 and H_2 as indicated by $H_1 \oplus H_2$ and acquired by taking one replica of H_1, p_1 replicas of H_2 and linking each vertex of the i_{th} copy of H_2 with all vertices of H_1.

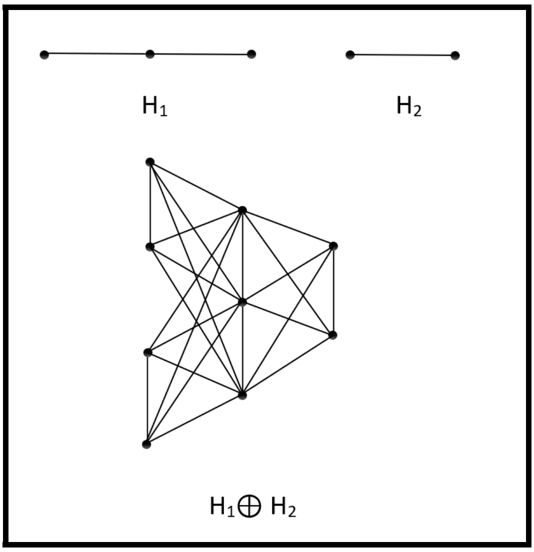

Figure 1. Corona Join Product $H_1 \oplus H_2$

Sub-division vertex join graph:[3] Suppose $H_1 = (p_1, q_1)$ & $H_2 = (p_2, q_2)$ are two simple(no loops and multiple edges), undirected, unweighted, connected graphs, then sub-division vertex join graph, $H_1 \dot{+} H_2$ is generated by linking each vertex of $J(H_1)$ to all vertices of H_2, where $J(H_1)$ is the set of new vertices of the subdivision graph $S(H_1)$.

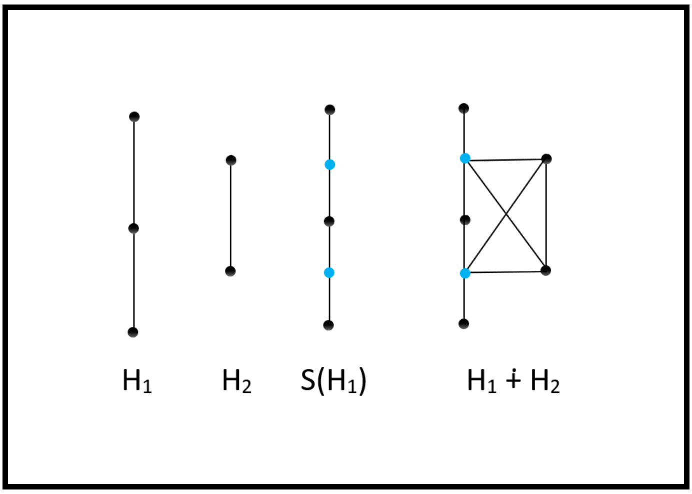

Figure 2. Subdivision Vertex Join $H_1 \dotplus H_2$

Sub-division vertex-edge join : [27, 29, 19, 2] Suppose $H_1 = (p_1, q_1)$, $H_2 = (p_2, q_2)$ & $H_3 = (p_3, q_3)$ are simple(no loops and multiple edges), undirected, unweighted, connected graphs, then sub-division vertex edge join graph, $H_1^{\mathcal{S}} \triangleright (H_2^{\mathcal{V}} \cup H_3^{\mathcal{I}})$ formed by $S(H_1)$, H_2 and H_3, all possessing distinct vertices is generated by linking the i_{th} node of $\mathcal{V}(H_1)$ to every node in $\mathcal{V}(H_2)$ and the i_{th} node of $J(H_1)$ to every node in $\mathcal{V}(H_3)$.

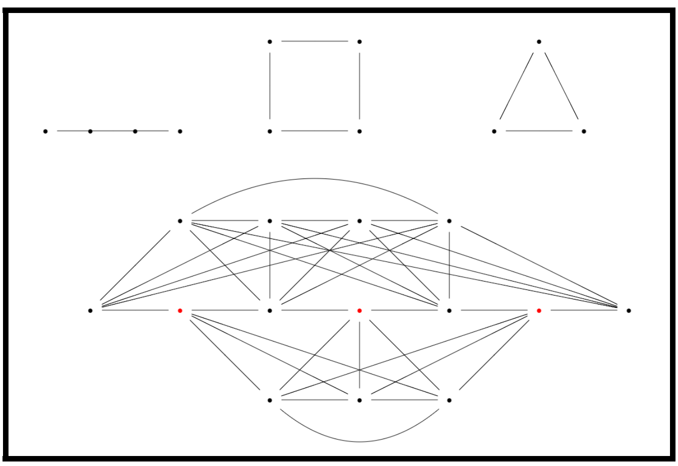

Figure 3. Subdivision Vertex Edge Join $P_4^{\mathcal{S}} \triangleright (C_4^{\mathcal{V}} \cup K_3^{\mathcal{I}})$

We carry the work forward in this paper by exploring the bounds and inequalities for the same graph operation variants for some significant descriptors.

2 Methodology

This segment reports and discusses our key findings on the relevant graph product versions. The arguments over the below stated postulates have been directly obtained from the descriptions.

Lemma 2.1. *[3] Suppose $H_1(p_1, q_1)$ & $H_2(p_2, q_2)$ are two graphs; the respective degrees of the nodes in graph $H_1 \oplus H_2$:*

$$d_{H_1 \oplus H_2}(w) = \begin{cases} d_{H_1}(w) + p_1 p_2 & ;\text{if } w \in \mathcal{V}(H_1) \\ d_{H_2}(w) + p_1 & ;\text{if } w \in \mathcal{V}(H_2) \end{cases}$$

Lemma 2.2. *[3] Suppose $H_1(p_1, q_1)$, $H_2(p_2, q_2)$ & $S(H_1) = (p_1^{'}, q_1^{'})$ are connected simple graphs; the respective degrees of the nodes in graph $H_1 \dotplus H_2$:*

$$d_{H_1\dot{+}H_2}(w)=\begin{cases}d_{H_1}(w) & ;\textit{if } w\in\mathcal{V}(H_1)\\ 2+p_2 & ;\textit{if } w\in\mathcal{V}_s(H_1)\\ d_{H_2}(w)+q_1 & ;\textit{if } w\in\mathcal{V}(H_2)\end{cases}$$

Lemma 2.3. *[15] Suppose $H_1(p_1,q_1)$, $H_2(p_2,q_2)$ & $H_2(p_3,q_3)$ are connected simple graphs; the respective degrees of the nodes in graph $H_1^{\mathcal{S}}\rhd(H_2^{\mathcal{V}}\cup H_3^{\mathcal{I}})$:*

$$d_{H_1^{\mathcal{S}}\rhd(H_2^{\mathcal{V}}\cup H_3^{\mathcal{I}})}(w)=\begin{cases}d_{H_1}(w)+p_2 & ;\textit{if } w\in\mathcal{V}(H_1)\\ 2+p_3 & ;\textit{if } w\in\mathcal{I}(H_1)\\ d_{H_2}(w)+p_1 & ;\textit{if } w\in\mathcal{V}(H_2)\\ d_{H_3}(w)+q_1 & ;\textit{if } w\in\mathcal{V}(H_3)\end{cases}$$

2.1 Bounds on the corona join product of certain topological indices

Theorem 2.4. *Assume H_1 and H_2 represent arbitrary graphs;*

$$ABC(H_1\oplus H_2)\leq\frac{\Delta_{H_1}}{(\delta_{H_1}+p_1p_2)}ABC(H_1)+q_1\frac{\sqrt{2p_1p_2}}{(\delta_{H_1}+p_1p_2)}+p_1\frac{\Delta_{H_2}}{(\delta_{H_2}+p_1)}ABC(H_2)$$
$$+p_1q_2\frac{\sqrt{2p_1}}{(\delta_{H_2}+p_1)}+p_1^2p_2\sqrt{\frac{\Delta_{H_1}+p_1p_2+\Delta_{H_2}+p_1-2}{(\delta_{H_1}+p_1p_2)(\delta_{H_2}+p_1)}}$$

Proof.

$$\begin{aligned}ABC(H_1\oplus H_2) &= \sum_{vw\in\mathcal{E}(H_1\oplus H_2)}\sqrt{\frac{d(v)+d(w)-2}{d(v)d(w)}}\\
&= \sum_{vw\in\mathcal{E}(H_1)}\sqrt{\frac{d_{H_1}(v)+d_{H_1}(w)+2p_1p_2-2}{(d_{H_1}(v)+p_1p_2)(d_{H_1}(w)+p_1p_2)}}\\
&\quad+\sum_{w\in\mathcal{V}(H_1)}\sum_{vw\in\mathcal{E}(H_2)}\sqrt{\frac{d_{H_2}(v)+d_{H_2}(w)+2p_1-2}{(d_{H_2}(v)+p_1)(d_{H_2}(w)+p_1)}}\\
&\quad+\sum_{w\in\mathcal{V}(H_1)}\sum_{\substack{vw\in\mathcal{E}(H_1\oplus H_2)\\ v\in\mathcal{V}(H_1)\\ w\in\mathcal{V}(H_2)}}\sqrt{\frac{d_{H_1}(v)+p_1p_2+d_{H_2}(w)+p_1-2}{(d_{H_1}(v)+p_1p_2)(d_{H_2}(w)+p_1)}}\\
&= \sum_{vw\in\mathcal{E}(H_1)}\sqrt{\frac{d_{H_1}(v)+d_{H_1}(w)+2p_1p_2-2}{(d_{H_1}(v)+p_1p_2)(d_{H_1}(w)+p_1p_2)}}\\
&\quad+p_1\sum_{vw\in\mathcal{E}(H_2)}\sqrt{\frac{d_{H_2}(v)+d_{H_2}(w)+2p_1-2}{(d_{H_2}(v)+p_1)(d_{H_2}(w)+p_1)}}\\
&\quad+p_1\sum_{\substack{vw\in\mathcal{E}(H_1\oplus H_2)\\ v\in\mathcal{V}(H_1)\\ w\in\mathcal{V}(H_2)}}\sqrt{\frac{d_{H_1}(v)+p_1p_2+d_{H_2}(w)+p_1-2}{(d_{H_1}(v)+p_1p_2)(d_{H_2}(w)+p_1)}}\\
&= \sum f+\sum g+\sum h\end{aligned}$$

where,

$$\begin{aligned}\sum f &= \sum_{vw\in\mathcal{E}(H_1)} \sqrt{\frac{d_{H_1}(v)+d_{H_1}(w)+2p_1p_2-2}{(d_{H_1}(v)+p_1p_2)(d_{H_1}(w)+p_1p_2)}}\\ \sum g &= p_1 \sum_{vw\in\mathcal{E}(H_2)} \sqrt{\frac{d_{H_2}(v)+d_{H_2}(w)+2p_1-2}{(d_{H_2}(v)+p_1)(d_{H_2}(w)+p_1)}}\\ \sum h &= p_1 \sum_{\substack{vw\in\mathcal{E}(H_1\oplus H_2)\\ v\in\mathcal{V}(H_1)\\ w\in\mathcal{V}(H_2)}} \sqrt{\frac{d_{H_1}(v)+p_1p_2+d_{H_2}(w)+p_1-2}{(d_{H_1}(v)+p_1p_2)(d_{H_2}(w)+p_1)}}\end{aligned}$$

Now,

$$\begin{aligned}\frac{d_{H_1}(v)+d_{H_1}(w)+2p_1p_2-2}{(d_{H_1}(v)+p_1p_2)(d_{H_1}(w)+p_1p_2)} = \frac{d_{H_1}(v)+d_{H_1}(w)-2}{d_{H_1}(v)d_{H_1}(w)}.\frac{d_{H_1}(v)d_{H_1}(w)}{(d_{H_1}(v)+p_1p_2)(d_{H_1}(w)+p_1p_2)}\\ +\frac{2p_1p_2}{(d_{H_1}(v)+p_1p_2)(d_{H_1}(w)+p_1p_2)}\end{aligned}$$

Since,

$$\frac{d_{H_1}(v)d_{H_1}(w)}{(d_{H_1}(v)+p_1p_2)(d_{H_1}(w)+p_1p_2)} \leq \frac{\Delta_{H_1}^2}{(\delta_{H_1}+p_1p_2)^2}$$

and;

$$\frac{1}{(d_{H_1}(v)+p_1p_2)(d_{H_1}(w)+p_1p_2)} \leq \frac{1}{(\delta_{H_1}+p_1p_2)^2}$$

Hence for $\sum f$;

$$\sqrt{\frac{d_{H_1}(v)+d_{H_1}(w)+2p_1p_2-2}{(d_{H_1}(v)+p_1p_2)(d_{H_1}(w)+p_1p_2)}} \leq \frac{\Delta_{H_1}}{(\delta_{H_1}+p_1p_2)}\sqrt{\frac{d_{H_1}(v)+d_{H_1}(w)-2}{d_{H_1}(v)d_{H_1}(w)}}+\frac{\sqrt{2p_1p_2}}{(\delta_{H_1}+p_1p_2)}$$

Similarly for $\sum g$;

$$\sqrt{\frac{d_{H_2}(v)+d_{H_2}(w)+2p_1-2}{(d_{H_2}(v)+p_1)(d_{H_2}(w)+p_1)}} \leq \frac{\Delta_{H_2}}{(\delta_{H_2}+p_1)}\sqrt{\frac{d_{H_2}(v)+d_{H_2}(w)-2}{d_{H_2}(v)d_{H_2}(w)}}+\frac{\sqrt{2p_1}}{(\delta_{H_2}+p_1)}$$

Also for $\sum h$;

$$\sqrt{\frac{d_{H_1}(v)+p_1p_2+d_{H_2}(w)+p_1-2}{(d_{H_1}(v)+p_1p_2)(d_{H_2}(w)+p_1)}} \leq \sqrt{\frac{\Delta_{H_1}+p_1p_2+\Delta_{H_2}+p_1-2}{(\delta_{H_1}+p_1p_2)(\delta_{H_2}+p_1)}}$$

From the computations;

$$\begin{aligned}ABC(H_1\oplus H_2) \leq \frac{\Delta_{H_1}}{(\delta_{H_1}+p_1p_2)} \sum_{vw\in\mathcal{E}(H_1)} \sqrt{\frac{d_{H_1}(v)+d_{H_1}(w)-2}{d_{H_1}(v)d_{H_1}(w)}} + q_1\frac{\sqrt{2p_1p_2}}{(\delta_{H_1}+p_1p_2)}\\ + p_1\frac{\Delta_{H_2}}{(\delta_{H_2}+p_1)} \sum_{vw\in\mathcal{E}(H_2)} \sqrt{\frac{d_{H_2}(v)+d_{H_2}(w)-2}{d_{H_2}(v)d_{H_2}(w)}} + p_1q_2\frac{\sqrt{2p_1}}{(\delta_{H_2}+p_1)}\\ + p_1^2p_2\sqrt{\frac{\Delta_{H_1}+p_1p_2+\Delta_{H_2}+p_1-2}{(\delta_{H_1}+p_1p_2)(\delta_{H_2}+p_1)}}\end{aligned}$$

Thus;

$$\begin{aligned} ABC(H_1 \oplus H_2) &\leq \frac{\Delta_{H_1}}{(\delta_{H_1}+p_1p_2)}ABC(H_1) + q_1\frac{\sqrt{2p_1p_2}}{(\delta_{H_1}+p_1p_2)} + p_1\frac{\Delta_{H_2}}{(\delta_{H_2}+p_1)}ABC(H_2) \\ &+ p_1q_2\frac{\sqrt{2p_1}}{(\delta_{H_2}+p_1)} + p_1^2p_2\sqrt{\frac{\Delta_{H_1}+p_1p_2+\Delta_{H_2}+p_1-2}{(\delta_{H_1}+p_1p_2)(\delta_{H_2}+p_1)}} \end{aligned}$$

This proves the result. □

Theorem 2.5. *Assume H_1 and H_2 represent arbitrary graphs;*

$$\begin{aligned} ABC(H_1 \oplus H_2) &\geq \left|\frac{\delta_{H_1}}{(\Delta_{H_1}+p_1p_2)}ABC(H_1) - q_1\frac{\sqrt{2p_1p_2}}{(\Delta_{H_1}+p_1p_2)}\right| \\ &+ \left|p_1\frac{\delta_{H_2}}{(\Delta_{H_2}+p_1)}ABC(H_2) - p_1q_2\frac{\sqrt{2p_1}}{(\Delta_{H_2}+p_1)}\right| \\ &+ p_1^2p_2\sqrt{\frac{\delta_{H_1}+p_1p_2+\delta_{H_2}+p_1-2}{(\Delta_{H_1}+p_1p_2)(\Delta_{H_2}+p_1)}} \end{aligned}$$

Proof.

$$\begin{aligned} ABC(H_1 \oplus H_2) &= \sum_{vw\in\mathcal{E}(H_1\oplus H_2)}\sqrt{\frac{d(v)+d(w)-2}{d(v)d(w)}} \\ &= \sum_{vw\in\mathcal{E}(H_1)}\sqrt{\frac{d_{H_1}(v)+d_{H_1}(w)+2p_1p_2-2}{(d_{H_1}(v)+p_1p_2)(d_{H_1}(w)+p_1p_2)}} \\ &+ \sum_{w\in\mathcal{V}(H_1)}\sum_{vw\in\mathcal{E}(H_2)}\sqrt{\frac{d_{H_2}(v)+d_{H_2}(w)+2p_1-2}{(d_{H_2}(v)+p_1)(d_{H_2}(w)+p_1)}} \\ &+ \sum_{w\in\mathcal{V}(H_1)}\sum_{\substack{vw\in\mathcal{E}(H_1\oplus H_2)\\ v\in\mathcal{V}(H_1)\\ w\in\mathcal{V}(H_2)}}\sqrt{\frac{d_{H_1}(v)+p_1p_2+d_{H_2}(w)+p_1-2}{(d_{H_1}(v)+p_1p_2)(d_{H_2}(w)+p_1)}} \end{aligned}$$

Now, proceeding similar to the theorem (2.4) while applying the following inequality:

$$\sqrt{s+t} \geq \left|\sqrt{s}-\sqrt{t}\right|$$

We have;

$$\begin{aligned} \frac{d_{H_1}(v)+d_{H_1}(w)+2p_1p_2-2}{(d_{H_1}(v)+p_1p_2)(d_{H_1}(w)+p_1p_2)} = \frac{d_{H_1}(v)+d_{H_1}(w)-2}{d_{H_1}(v)d_{H_1}(w)}.\frac{d_{H_1}(v)d_{H_1}(w)}{(d_{H_1}(v)+p_1p_2)(d_{H_1}(w)+p_1p_2)} \\ + \frac{2p_1p_2}{(d_{H_1}(v)+p_1p_2)(d_{H_1}(w)+p_1p_2)} \end{aligned}$$

Since,

$$\frac{d_{H_1}(v)d_{H_1}(w)}{(d_{H_1}(v)+p_1p_2)(d_{H_1}(w)+p_1p_2)} \geq \frac{\delta_{H_1}^2}{(\Delta_{H_1}+p_1p_2)^2}$$

and;

$$\frac{1}{(d_{H_1}(v)+p_1p_2)(d_{H_1}(w)+p_1p_2)} \geq \frac{1}{(\Delta_{H_1}+p_1p_2)^2}$$

Hence ;

$$\sqrt{\frac{d_{H_1}(v)+d_{H_1}(w)+2p_1p_2-2}{(d_{H_1}(v)+p_1p_2)(d_{H_1}(w)+p_1p_2)}} \geq \left|\frac{\delta_{H_1}}{(\Delta_{H_1}+p_1p_2)}\sqrt{\frac{d_{H_1}(v)+d_{H_1}(w)-2}{d_{H_1}(v)d_{H_1}(w)}}-\frac{\sqrt{2p_1p_2}}{(\Delta_{H_1}+p_1p_2)}\right|$$

Similarly;

$$\sqrt{\frac{d_{H_2}(v)+d_{H_2}(w)+2p_1-2}{(d_{H_2}(v)+p_1)(d_{H_2}(w)+p_1)}} \geq \left|\frac{\delta_{H_2}}{(\Delta_{H_2}+p_1)}\sqrt{\frac{d_{H_2}(v)+d_{H_2}(w)-2}{d_{H_2}(v)d_{H_2}(w)}}-\frac{\sqrt{2p_1}}{(\Delta_{H_2}+p_1)}\right|$$

Also;

$$\sqrt{\frac{d_{H_1}(v)+p_1p_2+d_{H_2}(w)+p_1-2}{(d_{H_1}(v)+p_1p_2)(d_{H_2}(w)+p_1)}} \geq \sqrt{\frac{\delta_{H_1}+p_1p_2+\delta_{H_2}+p_1-2}{(\Delta_{H_1}+p_1p_2)(\Delta_{H_2}+p_1)}}$$

Applying the summation and from the above computations;

$$\begin{aligned} ABC(H_1\oplus H_2) \quad \geq \quad & \left|\frac{\delta_{H_1}}{(\Delta_{H_1}+p_1p_2)}ABC(H_1)-q_1\frac{\sqrt{2p_1p_2}}{(\Delta_{H_1}+p_1p_2)}\right| \\ & + \left|p_1\frac{\delta_{H_2}}{(\Delta_{H_2}+p_1)}ABC(H_2)-p_1q_2\frac{\sqrt{2p_1}}{(\Delta_{H_2}+p_1)}\right| \\ & + p_1^2p_2\sqrt{\frac{\delta_{H_1}+p_1p_2+\delta_{H_2}+p_1-2}{(\Delta_{H_1}+p_1p_2)(\Delta_{H_2}+p_1)}} \end{aligned}$$

This proves the result. □

Theorem 2.6. *Assume H_1 and H_2 represent two arbitrary graphs. Then ;*

$$\begin{aligned} ISI(H_1\oplus H_2) \quad \leq \quad & \frac{\Delta_{H_1}}{(\delta_{H_1}+p_1p_2)}(ISI(H_1)+p_1p_2q_1)+p_1^2p_2^2q_1\frac{1}{2(\delta_{H_2}+p_1p_2)} \\ & + p_1\frac{\Delta_{H_2}}{(\delta_{H_2}+p_1)}(ISI(H_2)+p_1q_2)+p_1^3q_2\frac{1}{2(\delta_{H_2}+p_1)} \\ & + p_1^2p_2\frac{(\Delta_{H_1}+p_1p_2)(\Delta_{H_2}+p_1)}{(\delta_{H_1}+p_1(p_2+1)+\delta_{H_2})} \end{aligned}$$

Proof.

$$
\begin{aligned}
ISI(H_1 \oplus H_2) &= \sum_{vw \in \mathcal{E}(H_1 \oplus H_2)} \frac{d(v)d(w)}{d(v)+d(w)} \\
&= \sum_{vw \in \mathcal{E}(H_1)} \frac{(d_{H_1}(v)+p_1p_2)(d_{H_1}(w)+p_1p_2)}{(d_{H_1}(v)+d_{H_1}(w)+2p_1p_2)} \\
&\quad + \sum_{w \in \mathcal{V}(H_1)} \sum_{vw \in \mathcal{E}(H_2)} \frac{(d_{H_2}(v)+p_1)(d_{H_2}(w)+p_1)}{(d_{H_2}(v)+d_{H_2}(w)+2p_1)} \\
&\quad + \sum_{w \in \mathcal{V}(H_1)} \sum_{\substack{vw \in \mathcal{E}(H_1 \oplus H_2) \\ v \in \mathcal{V}(H_1) \\ w \in \mathcal{V}(H_2)}} \frac{(d_{H_1}(v)+p_1p_2)(d_{H_2}(w)+p_1)}{(d_{H_1}(v)+d_{H_2}(w)+p_1p_2+p_1)} \\
&= \sum_{vw \in \mathcal{E}(H_1)} \frac{(d_{H_1}(v)+p_1p_2)(d_{H_1}(w)+p_1p_2)}{(d_{H_1}(v)+d_{H_1}(w)+2p_1p_2)} \\
&\quad + p_1 \sum_{vw \in \mathcal{E}(H_2)} \frac{(d_{H_2}(v)+p_1)(d_{H_2}(w)+p_1)}{(d_{H_2}(v)+d_{H_2}(w)+2p_1)} \\
&\quad + p_1 \sum_{\substack{vw \in \mathcal{E}(H_1 \oplus H_2) \\ v \in \mathcal{V}(H_1) \\ w \in \mathcal{V}(H_2)}} \frac{(d_{H_1}(v)+p_1p_2)(d_{H_2}(w)+p_1)}{(d_{H_1}(v)+d_{H_2}(w)+p_1p_2+p_1)} \\
&= \sum f + \sum g + \sum h
\end{aligned}
$$

where,

$$
\begin{aligned}
\sum f &= \sum_{vw \in \mathcal{E}(H_1)} \frac{(d_{H_1}(v)+p_1p_2)(d_{H_1}(w)+p_1p_2)}{(d_{H_1}(v)+d_{H_1}(w)+2p_1p_2)} \\
\sum g &= p_1 \sum_{vw \in \mathcal{E}(H_2)} \frac{(d_{H_2}(v)+p_1)(d_{H_2}(w)+p_1)}{(d_{H_2}(v)+d_{H_2}(w)+2p_1)} \\
\sum h &= p_1 \sum_{\substack{vw \in \mathcal{E}(H_1 \oplus H_2) \\ v \in \mathcal{V}(H_1) \\ w \in \mathcal{V}(H_2)}} \frac{(d_{H_1}(v)+p_1p_2)(d_{H_2}(w)+p_1)}{(d_{H_1}(v)+d_{H_2}(w)+p_1p_2+p_1)}
\end{aligned}
$$

Now, we solve for $\sum f$;

$$
\begin{aligned}
\frac{(d_{H_1}(v)+p_1p_2)(d_{H_1}(w)+p_1p_2)}{(d_{H_1}(v)+d_{H_1}(w)+2p_1p_2)} &= \frac{d_{H_1}(v)d_{H_1}(w)+p_1p_2(d_{H_1}(v)+d_{H_1}(w))+p_1^2p_2^2}{(d_{H_1}(v)+d_{H_1}(w)+2p_1p_2)} \\
&= \frac{(d_{H_1}(v)d_{H_1}(w))}{(d_{H_1}(v)+d_{H_1}(w))} \cdot \frac{(d_{H_1}(v)+d_{H_1}(w))}{(d_{H_1}(v)+d_{H_1}(w)+2p_1p_2)} \\
&\quad + \frac{p_1p_2(d_{H_1}(v)+d_{H_1}(w))}{(d_{H_1}(v)+d_{H_1}(w)+2p_1p_2)} \\
&\quad + \frac{p_1^2p_2^2}{(d_{H_1}(v)+d_{H_1}(w)+2p_1p_2)}
\end{aligned}
$$

Since,

$$
\frac{(d_{H_1}(v)+d_{H_1}(w))}{(d_{H_1}(v)+d_{H_1}(w)+2p_1p_2)} \le \frac{\Delta_{H_1}}{(\delta_{H_1}+p_1p_2)}
$$

Hence;

$$\sum f \leq \frac{\Delta_{H_1}}{(\delta_{H_1}+p_1p_2)}ISI(H_1)+p_1p_2q_1\frac{\Delta_{H_1}}{(\delta_{H_1}+p_1p_2)}+p_1^2p_2^2q_1\frac{1}{2(\delta_{H_1}+p_1p_2)}$$

Similarly;

$$\sum g \leq p_1\frac{\Delta_{H_2}}{(\delta_{H_2}+p_1)}ISI(H_2)+p_1^2q_2\frac{\Delta_{H_2}}{(\delta_{H_2}+p_1)}+p_1^3q_2\frac{1}{2(\delta_{H_2}+p_1)}$$

Also;

$$\sum h \leq p_1^2p_2\frac{(\Delta_{H_1}+p_1p_2)(\Delta_{H_2}+p_1)}{(\delta_{H_1}+p_1(p_2+1)+\delta_{H_2})}$$

From the computations;

$$\begin{aligned} ISI(H_1\oplus H_2) \leq\ & \frac{\Delta_{H_1}}{(\delta_{H_1}+p_1p_2)}(ISI(H_1)+p_1p_2q_1)+p_1^2p_2^2q_1\frac{1}{2(\delta_{H_2}+p_1p_2)} \\ & + p_1\frac{\Delta_{H_2}}{(\delta_{H_2}+p_1)}(ISI(H_2)+p_1q_2)+p_1^3q_2\frac{1}{2(\delta_{H_2}+p_1)} \\ & + p_1^2p_2\frac{(\Delta_{H_1}+p_1p_2)(\Delta_{H_2}+p_1)}{(\delta_{H_1}+p_1(p_2+1)+\delta_{H_2})} \end{aligned}$$

This proves the result. □

Theorem 2.7. *Assume H_1 and H_2 represent arbitrary graphs;*

$$\begin{aligned} ISI(H_1\oplus H_2) \geq\ & \frac{\delta_{H_1}}{(\Delta_{H_1}+p_1p_2)}(ISI(H_1)+p_1p_2q_1)+p_1^2p_2^2q_1\frac{1}{2(\Delta_{H_2}+p_1p_2)} \\ & + p_1\frac{\delta_{H_2}}{(\Delta_{H_2}+p_1)}(ISI(H_2)+p_1q_2)+p_1^3q_2\frac{1}{2(\Delta_{H_2}+p_1)} \\ & + p_1^2p_2\frac{(\delta_{H_1}+p_1p_2)(\delta_{H_2}+p_1)}{(\Delta_{H_1}+p_1(p_2+1)+\Delta_{H_2})} \end{aligned}$$

Theorem 2.8. *Assume H_1 and H_2 represent arbitrary graphs;*

$$\begin{aligned} GA(H_1\oplus H_2) \leq\ & \frac{\Delta_{H_1}}{(\delta_{H_1}+p_1p_2)}GA(H_1)+\frac{q_1}{(\delta_{H_1}+p_1p_2)}\left(\sqrt{2p_1p_2\Delta_{H_1}}+p_1p_2\right) \\ & + p_1\frac{\Delta_{H_2}}{(\delta_{H_2}+p_1)}GA(H_2)+\frac{p_1q_2}{(\delta_{H_2}+p_1)}\left(\sqrt{2p_1\Delta_{H_2}}+p_1\right) \\ & + 2p_1^2p_2\frac{\sqrt{(\Delta_{H_1}+p_1p_2)(\Delta_{H_2}+p_1)}}{(\delta_{H_1}+p_1(p_2+1)+\delta_{H_2})} \end{aligned}$$

Proof.

$$\begin{aligned}
GA(H_1 \oplus H_2) &= \sum_{vw\in\mathcal{E}(H_1\oplus H_2)} \frac{2\sqrt{d(v)d(w)}}{d(v)+d(w)} \\
&= \sum_{vw\in\mathcal{E}(H_1)} \frac{2\sqrt{(d_{H_1}(v)+p_1p_2)(d_{H_1}(w)+p_1p_2)}}{(d_{H_1}(v)+d_{H_1}(w)+2p_1p_2)} \\
&\quad + \sum_{w\in\mathcal{V}(H_1)} \sum_{vw\in\mathcal{E}(H_2)} \frac{2\sqrt{(d_{H_2}(v)+p_1)(d_{H_2}(w)+p_1)}}{(d_{H_2}(v)+d_{H_2}(w)+2p_1)} \\
&\quad + \sum_{w\in\mathcal{V}(H_1)} \sum_{\substack{vw\in\mathcal{E}(H_1\oplus H_2)\\ v\in\mathcal{V}(H_1)\\ w\in\mathcal{V}(H_2)}} \frac{2\sqrt{(d_{H_1}(v)+p_1p_2)(d_{H_2}(w)+p_1)}}{(d_{H_1}(v)+d_{H_2}(w)+p_1p_2+p_1)} \\
&= \sum_{vw\in\mathcal{E}(H_1)} \frac{2\sqrt{(d_{H_1}(v)+p_1p_2)(d_{H_1}(w)+p_1p_2)}}{(d_{H_1}(v)+d_{H_1}(w)+2p_1p_2)} \\
&\quad + p_1 \sum_{vw\in\mathcal{E}(H_2)} \frac{2\sqrt{(d_{H_2}(v)+p_1)(d_{H_2}(w)+p_1)}}{(d_{H_2}(v)+d_{H_2}(w)+2p_1)} \\
&\quad + p_1 \sum_{\substack{vw\in\mathcal{E}(H_1\oplus H_2)\\ v\in\mathcal{V}(H_1)\\ w\in\mathcal{V}(H_2)}} \frac{2\sqrt{(d_{H_1}(v)+p_1p_2)(d_{H_2}(w)+p_1)}}{(d_{H_1}(v)+d_{H_2}(w)+p_1p_2+p_1)} \\
&= \sum f + \sum g + \sum h
\end{aligned}$$

where,

$$\begin{aligned}
\sum f &= \sum_{vw\in\mathcal{E}(H_1)} \frac{2\sqrt{(d_{H_1}(v)+p_1p_2)(d_{H_1}(w)+p_1p_2)}}{(d_{H_1}(v)+d_{H_1}(w)+2p_1p_2)} \\
\sum g &= p_1 \sum_{vw\in\mathcal{E}(H_2)} \frac{2\sqrt{(d_{H_2}(v)+p_1)(d_{H_2}(w)+p_1)}}{(d_{H_2}(v)+d_{H_2}(w)+2p_1)} \\
\sum h &= p_1 \sum_{\substack{vw\in\mathcal{E}(H_1\oplus H_2)\\ v\in\mathcal{V}(H_1)\\ w\in\mathcal{V}(H_2)}} \frac{2\sqrt{(d_{H_1}(v)+p_1p_2)(d_{H_2}(w)+p_1)}}{(d_{H_1}(v)+d_{H_2}(w)+p_1p_2+p_1)}
\end{aligned}$$

Now ;

$$\begin{aligned}
\frac{2\sqrt{(d_{H_1}(v)+p_1p_2)(d_{H_1}(w)+p_1p_2)}}{(d_{H_1}(v)+d_{H_1}(w)+2p_1p_2)} &= \frac{2\sqrt{d_{H_1}(v)d_{H_1}(w)+p_1p_2(d_{H_1}(v)+d_{H_1}(w))+p_1^2p_2^2}}{(d_{H_1}(v)+2p_1p_2+d_{H_1}(w))} \\
&\leq \frac{2\sqrt{(d_{H_1}(v)d_{H_1}(w))}}{(d_{H_1}(v)+2p_1p_2+d_{H_1}(w))} + \frac{2\sqrt{p_1p_2(d_{H_1}(v)+d_{H_1}(w))}}{(d_{H_1}(v)+2p_1p_2+d_{H_1}(w))} \\
&\quad + \frac{2p_1p_2}{(d_{H_1}(v)+2p_1p_2+d_{H_1}(w))} \\
&\leq \frac{2\sqrt{(d_{H_1}(v)d_{H_1}(w))}}{(d_{H_1}(v)+d_{H_1}(w))} . \frac{(d_{H_1}(v)+d_{H_1}(w))}{(d_{H_1}(v)+2p_1p_2+d_{H_1}(w))} \\
&\quad + \frac{2\sqrt{p_1p_2(d_{H_1}(v)+d_{H_1}(w))}}{(d_{H_1}(v)+2p_1p_2+d_{H_1}(w))} + \frac{2p_1p_2}{(d_{H_1}(v)+2p_1p_2+d_{H_1}(w))}
\end{aligned}$$

Hence;

$$\sum f \leq \frac{\Delta_{H_1}}{(\delta_{H_1}+p_1p_2)}GA(H_1)+q_1\frac{\sqrt{2p_1p_2\Delta_{H_1}}}{(\delta_{H_1}+p_1p_2)}+p_1p_2q_1\frac{1}{(\delta_{H_1}+p_1p_2)}$$

Similarly;

$$\sum g \leq p_1\frac{\Delta_{H_2}}{(\delta_{H_2}+p_1)}GA(H_2)+p_1q_2\frac{\sqrt{2p_1\Delta_{H_2}}}{(\delta_{H_2}+p_1)}+p_1^2q_2\frac{1}{(\delta_{H_2}+p_1)}$$

Also;

$$\sum h \leq 2p_1^2p_2\frac{(\Delta_{H_1}+p_1p_2)(\Delta_{H_2}+p_1)}{(\delta_{H_1}+p_1(p_2+1)+\delta_{H_2})}$$

From the computations;

$$\begin{aligned} GA(H_1\oplus H_2) \leq\ & \frac{\Delta_{H_1}}{(\delta_{H_1}+p_1p_2)}GA(H_1)+\frac{q_1}{(\delta_{H_1}+p_1p_2)}\left(\sqrt{2p_1p_2\Delta_{H_1}}+p_1p_2\right) \\ & +p_1\frac{\Delta_{H_2}}{(\delta_{H_2}+p_1)}GA(H_2)+\frac{p_1q_2}{(\delta_{H_2}+p_1)}\left(\sqrt{2p_1\Delta_{H_2}}+p_1\right) \\ & +2p_1^2p_2\frac{\sqrt{(\Delta_{H_1}+p_1p_2)(\Delta_{H_2}+p_1)}}{(\delta_{H_1}+p_1(p_2+1)+\delta_{H_2})} \end{aligned}$$

This proves the result. □

Theorem 2.9. *Assume H_1 and H_2 represent arbitrary graphs;*

$$\begin{aligned} SO(H_1\oplus H_2) \leq\ & SO(H_1)+p_1SO(H_2)+q_1\sqrt{2p_1p_2(p_1p_2+2\Delta_{H_1})} \\ & +p_1q_2\sqrt{2p_1(p_1+2\Delta_{H_2})}+p_1^2p_2\left[(\Delta_{H_1}+p_1p_2)+(\Delta_{H_2}+p_1)\right] \end{aligned}$$

Proof.

$$\begin{aligned} SO(H_1\oplus H_2) &= \sum_{vw\in\mathcal{E}(H_1\oplus H_2)}\sqrt{d(v)^2+d(w)^2} \\ &= \sum_{vw\in\mathcal{E}(H_1)}\sqrt{(d_{H_1}(v)+p_1p_2)^2+(d_{H_1}(w)+p_1p_2)^2} \\ &+ \sum_{w\in\mathcal{V}(H_1)}\sum_{vw\in\mathcal{E}(H_2)}\sqrt{(d_{H_2}(v)+p_1)^2+(d_{H_2}(w)+p_1)^2} \\ &+ \sum_{w\in\mathcal{V}(H_1)}\sum_{\substack{vw\in\mathcal{E}(H_1\oplus H_2)\\ v\in\mathcal{V}(H_1)\\ w\in\mathcal{V}(H_2)}}\sqrt{(d_{H_1}(v)+p_1p_2)^2+(d_{H_2}(w)+p_1)^2} \end{aligned}$$

$$\begin{aligned} &= \sum_{vw\in\mathcal{E}(H_1)}\sqrt{(d_{H_1}(v)+p_1p_2)^2+(d_{H_1}(w)+p_1p_2)^2} \\ &+p_1\sum_{vw\in\mathcal{E}(H_2)}\sqrt{(d_{H_2}(v)+p_1)^2+(d_{H_2}(w)+p_1)^2} \\ &+p_1\sum_{\substack{vw\in\mathcal{E}(H_1\oplus H_2)\\ v\in\mathcal{V}(H_1)\\ w\in\mathcal{V}(H_2)}}\sqrt{(d_{H_1}(v)+p_1p_2)^2+(d_{H_2}(w)+p_1)^2} \\ &= \sum f+\sum g+\sum h \end{aligned}$$

where,

$$\begin{aligned}\sum f &= \sum_{vw\in\mathcal{E}(H_1)} \sqrt{(d_{H_1}(v)+p_1p_2)^2+(d_{H_1}(w)+p_1p_2)^2}\\ \sum g &= p_1 \sum_{vw\in\mathcal{E}(H_2)} \sqrt{(d_{H_2}(v)+p_1)^2+(d_{H_2}(w)+p_1)^2}\\ \sum h &= p_1 \sum_{\substack{vw\in\mathcal{E}(H_1\oplus H_2)\\ v\in\mathcal{V}(H_1)\\ w\in\mathcal{V}(H_2)}} \sqrt{(d_{H_1}(v)+p_1p_2)^2+(d_{H_2}(w)+p_1)^2}\end{aligned}$$

Now, since;

$$\begin{aligned}(d_{H_1}(v)+p_1p_2)^2+(d_{H_1}(w)+p_1p_2)^2 &= (d_{H_1}(v)^2+d_{H_1}(w)^2)+2p_1p_2(d_{H_1}(v)+d_{H_1}(w))\\ &\quad + 2p_1^2p_2^2\\ &\leq (d_{H_1}(v)^2+d_{H_1}(w)^2)+4p_1p_2\Delta_{H_1}+2p_1^2p_2^2\end{aligned}$$

Hence;

$$\sum f \quad \leq \quad SO(H_1)+q_1\sqrt{2p_1p_2(p_1p_2+2\Delta_{H_1})}$$

Similarly;

$$\sum g \quad \leq \quad p_1SO(H_2)+p_1q_2\sqrt{2p_1(p_1+2\Delta_{H_2})}$$

Also;

$$\sum h \quad \leq \quad p_1^2p_2\left[(\Delta_{H_1}+p_1p_2)+(\Delta_{H_2}+p_1)\right]$$

From the computations;

$$\begin{aligned}SO(H_1\oplus H_2) \quad &\leq \quad SO(H_1)+p_1SO(H_2)+q_1\sqrt{2p_1p_2(p_1p_2+2\Delta_{H_1})}\\ &+p_1q_2\sqrt{2p_1(p_1+2\Delta_{H_2})}+p_1^2p_2\left[(\Delta_{H_1}+p_1p_2)+(\Delta_{H_2}+p_1)\right]\end{aligned}$$

This proves the result. □

Theorem 2.10. *Assume H_1 and H_2 represent arbitrary graphs;*

$$\begin{aligned}SO(H_1\oplus H_2) \geq &\left|SO(H_1)-q_1\sqrt{2p_1p_2(p_1p_2+2\delta_{H_1})}\right|+\left|p_1SO(H_2)-p_1q_2\sqrt{2p_1(p_1+2\delta_{H_2})}\right|\\ &+p_1^2p_2\left[(\delta_{H_1}+p_1p_2)-(\delta_{H_2}+p_1)\right]\end{aligned}$$

2.2 Bounds on the subdivision vertex join of certain topological indices

Theorem 2.11. *Assume H_1 and H_2 represent arbitrary graphs;*

$$\begin{aligned}ABC(H_1\dot{+}H_2) \quad &\leq \quad \frac{q_1^{'}}{\sqrt{2+p_2}}\left(1+\sqrt{\frac{p_2}{\delta_{H_1}}}\right)+ABC(H_2)\frac{\Delta_{H_2}}{\delta_{H_2}+q_1}\\ &+q_2\frac{\sqrt{2q_1}}{\delta_{H_2}+q_1}+\frac{q_1p_2}{\sqrt{2+p_2}}\left(1+\sqrt{\frac{p_2}{q_1+\delta_{H_2}}}\right)\end{aligned}$$

Proof.

$$
\begin{aligned}
ABC(H_1 \dotplus H_2) &= \sum_{vw\in\mathcal{E}(H_1\dotplus H_2)} \sqrt{\frac{d(v)+d(w)-2}{d(v)d(w)}}\\
&= \sum_{\substack{vw\in\mathcal{E}(\mathcal{S}(H_1))\\ v\in\mathcal{V}(H_1)\\ w\in V_s(H_1)}} \sqrt{\frac{d_{H_1}(v)+p_2}{d_{H_1}(v)(2+p_2)}} + \sum_{vw\in\mathcal{E}(H_2)} \sqrt{\frac{d_{H_2}(v)+d_{H_2}(w)+2q_1-2}{(d_{H_2}(v)+q_1)(d_{H_2}(w)+q_1)}}\\
&\quad + \sum_{\substack{vw\in\mathcal{E}(H_1\dotplus H_2)\\ v\in V_s(H_1)\\ w\in\mathcal{V}(H_2)}} \sqrt{\frac{d_{H_2}(w)+p_2+q_1}{(2+p_2)(d_{H_2}(w)+q_1)}}\\
&= \sum f + \sum g + \sum h
\end{aligned}
$$

where,

$$
\begin{aligned}
\sum f &= \sum_{\substack{vw\in\mathcal{E}(\mathcal{S}(H_1))\\ v\in\mathcal{V}(H_1)\\ w\in V_s(H_1)}} \sqrt{\frac{d_{H_1}(v)+p_2}{d_{H_1}(v)(2+p_2)}}\\
\sum g &= \sum_{vw\in\mathcal{E}(H_2)} \sqrt{\frac{d_{H_2}(v)+d_{H_2}(w)+2q_1-2}{(d_{H_2}(v)+q_1)(d_{H_2}(w)+q_1)}}\\
\sum h &= \sum_{\substack{vw\in\mathcal{E}(H_1\dotplus H_2)\\ v\in V_s(H_1)\\ w\in\mathcal{V}(H_2)}} \sqrt{\frac{d_{H_2}(w)+p_2+q_1}{(2+p_2)(d_{H_2}(w)+q_1)}}
\end{aligned}
$$

For the computation of $\sum f$;

$$
\begin{aligned}
\sqrt{\frac{d_{H_1}(v)+p_2}{d_{H_1}(v)(2+p_2)}} &= \sqrt{\frac{1}{2+p_2}+\frac{p_2}{d_{H_1}(v)(2+p_2)}}\\
\Rightarrow \sum f &\leq \frac{q_1^{'}}{\sqrt{2+p_2}}\left(1+\sqrt{\frac{p_2}{\delta_{H_1}}}\right)
\end{aligned}
$$

For $\sum g$;

$$
\begin{aligned}
\frac{d_{H_2}(v)+d_{H_2}(w)+2q_1-2}{(d_{H_2}(v)+q_1)(d_{H_2}(w)+q_1)} &= \frac{d_{H_2}(v)+d_{H_2}(w)-2}{d_{H_2}(v)d_{H_2}(w)}.\frac{d_{H_2}(v)d_{H_2}(w)}{(d_{H_2}(v)+q_1)(d_{H_2}(w)+q_1)}\\
&\quad + \frac{2q_1}{(d_{H_2}(v)+q_1)(d_{H_2}(w)+q_1)}\\
\Rightarrow \sum g &\leq \frac{\Delta_{H_2}}{\delta_{H_2}+q_1}ABC(H_2) + q_2\frac{\sqrt{2q_1}}{\delta_{H_2}+q_1}
\end{aligned}
$$

Also;

$$
\sum h \leq \frac{q_1p_2}{\sqrt{2+p_2}}\left(1+\sqrt{\frac{p_2}{\delta_{H_2}+q_1}}\right)
$$

From the computations;

$$\begin{aligned} ABC(H_1 \dotplus H_2) \quad &\leq \quad \frac{q_1^{'}}{\sqrt{2+p_2}}\left(1+\sqrt{\frac{p_2}{\delta_{H_1}}}\right) + ABC(H_2)\frac{\Delta_{H_2}}{\delta_{H_2}+q_1} + q_2\frac{\sqrt{2q_1}}{\delta_{H_2}+q_1} \\ &+ \frac{q_1 p_2}{\sqrt{2+p_2}}\left(1+\sqrt{\frac{p_2}{q_1+\delta_{H_2}}}\right) \end{aligned}$$

This proves the result. □

Theorem 2.12. *Assume H_1 and H_2 represent arbitrary graphs;*

$$\begin{aligned} ABC(H_1 \dotplus H_2) \geq q_1^{'}\left|\frac{1}{\sqrt{2+p_2}} - \sqrt{\frac{p_2}{\Delta_{H_1}(p_2+2)}}\right| + \left|ABC(H_2)\frac{\delta_{H_2}}{\Delta_{H_2}+q_1} - q_2\frac{\sqrt{2q_1}}{(\Delta_{H_2}+q_1)}\right| \\ + q_1 p_2\left|\frac{1}{\sqrt{2+p_2}} - \sqrt{\frac{p_2}{(\Delta_{H_2}+q_1)(p_2+2)}}\right| \end{aligned}$$

Proof.

$$\begin{aligned} ABC(H_1 \dotplus H_2) \quad &= \sum_{vw\in\mathcal{E}(H_1\dotplus H_2)} \sqrt{\frac{d(v)+d(w)-2}{d(v)d(w)}} \\ &= \sum_{\substack{vw\in\mathcal{E}(\mathcal{S}(H_1)) \\ v\in\mathcal{V}(H_1) \\ w\in V_s(H_1)}} \sqrt{\frac{d_{H_1}(v)+p_2}{d_{H_1}(v)(2+p_2)}} + \sum_{vw\in\mathcal{E}(H_2)} \sqrt{\frac{d_{H_2}(v)+d_{H_2}(w)+2q_1-2}{(d_{H_2}(v)+q_1)(d_{H_2}(w)+q_1)}} \\ &+ \sum_{\substack{vw\in\mathcal{E}(H_1\dotplus H_2) \\ v\in V_s(H_1) \\ w\in\mathcal{V}(H_2)}} \sqrt{\frac{d_{H_2}(w)+p_2+q_1}{(2+p_2)(d_{H_2}(w)+q_1)}} \end{aligned}$$

Now , proceeding similar to the theorem (2.11) while applying the following inequality:

$$\sqrt{s+t} \geq \left|\sqrt{s} - \sqrt{t}\right|$$

We have;

$$\sum_{\substack{vw\in\mathcal{E}(\mathcal{S}(H_1)) \\ v\in\mathcal{V}(H_1) \\ w\in V_s(H_1)}} \sqrt{\frac{d_{H_1}(v)+p_2}{d_{H_1}(v)(2+p_2)}} \geq q_1^{'}\left|\frac{1}{\sqrt{2+p_2}} - \sqrt{\frac{p_2}{\Delta_{H_1}(p_2+2)}}\right|$$

and;

$$\sum_{vw\in\mathcal{E}(H_2)} \sqrt{\frac{d_{H_2}(v)+d_{H_2}(w)+2q_1-2}{(d_{H_2}(v)+q_1)(d_{H_2}(w)+q_1)}} \geq \left|ABC(H_2)\frac{\delta_{H_2}}{\Delta_{H_2}+q_1} - q_2\frac{\sqrt{2q_1}}{(\Delta_{H_2}+q_1)}\right|$$

Also;

$$\sum_{\substack{vw\in\mathcal{E}(H_1\dotplus H_2) \\ v\in V_s(H_1) \\ w\in\mathcal{V}(H_2)}} \sqrt{\frac{d_{H_2}(w)+p_2+q_1}{(2+p_2)(d_{H_2}(w)+q_1)}} \geq q_1 p_2\left|\frac{1}{\sqrt{2+p_2}} - \sqrt{\frac{p_2}{(\Delta_{H_2}+q_1)(p_2+2)}}\right|$$

From the computations;

$$ABC(H_1 \dot{+} H_2) \geq q_1^{'} \left| \frac{1}{\sqrt{2+p_2}} - \sqrt{\frac{p_2}{\Delta_{H_1}(p_2+2)}} \right| + \left| ABC(H_2) \frac{\delta_{H_2}}{\Delta_{H_2}+q_1} - q_2 \frac{\sqrt{2q_1}}{(\Delta_{H_2}+q_1)} \right| + q_1 p_2 \left| \frac{1}{\sqrt{2+p_2}} - \sqrt{\frac{p_2}{(\Delta_{H_2}+q_1)(p_2+2)}} \right|$$

This proves the result. □

Theorem 2.13. *Assume H_1 and H_2 represent arbitrary graphs;*

$$ISI(H_1 \dot{+} H_2) \leq (p_2+2)q_1^{'} \frac{\Delta_{H_1}}{(\delta_{H_1}+p_2+2)} + \frac{\Delta_{H_2}}{(\delta_{H_2}+q_1)} ISI(H_2) + \frac{q_1 q_2}{\delta_{H_2}+q_1}\left(\Delta_{H_2} + \frac{q_1}{2}\right) + p_2 q_1 (p_2+2) \frac{\Delta_{H_2}+q_1}{(\delta_{H_2}+(p_2+q_1)+2)}$$

Proof.

$$\begin{aligned} ISI(H_1 \dot{+} H_2) &= \sum_{vw \in \mathcal{E}(H_1 \dot{+} H_2)} \frac{d(v)d(w)}{d(v)+d(w)} \\ &= \sum_{\substack{vw \in \mathcal{E}(\mathcal{S}(H_1)) \\ v \in \mathcal{V}(H_1) \\ w \in V_s(H_1)}} \frac{d_{H_1}(v)(p_2+2)}{(d_{H_1}(v)+p_2+2)} + \sum_{vw \in \mathcal{E}(H_2)} \frac{(d_{H_2}(v)+q_1)(d_{H_2}(w)+q_1)}{(d_{H_2}(v)+d_{H_2}(w)+2q_1)} \\ &+ \sum_{\substack{vw \in \mathcal{E}(H_1 \dot{+} H_2) \\ v \in V_s(H_1) \\ w \in \mathcal{V}(H_2)}} \frac{(p_2+2)(d_{H_2}(w)+q_1)}{(p_2+2+d_{H_2}(w)+q_1)} \\ &= \sum f + \sum g + \sum h \end{aligned}$$

where,

$$\begin{aligned} \sum f &= \sum_{\substack{vw \in \mathcal{E}(\mathcal{S}(H_1)) \\ v \in \mathcal{V}(H_1) \\ w \in V_s(H_1)}} \frac{d_{H_1}(v)(p_2+2)}{d_{H_1}(v)+p_2+2)} \\ \sum g &= \sum_{vw \in \mathcal{E}(H_2)} \frac{(d_{H_2}(v)+q_1)(d_{H_2}(w)+q_1)}{d_{H_2}(v)+d_{H_2}(w)+2q_1)} \\ \sum h &= \sum_{\substack{vw \in \mathcal{E}(H_1 \dot{+} H_2) \\ v \in V_s(H_1) \\ w \in \mathcal{V}(H_2)}} \frac{(p_2+2)(d_{H_2}(w)+q_1)}{(p_2+2+d_{H_2}(w)+q_1)} \end{aligned}$$

Now;

$$\sum f \leq (p_2+2)q_1^{'} \frac{\Delta_{H_1}}{(\delta_{H_1}+p_2+2)}$$

For $\sum g$;

$$\begin{aligned}\frac{(d_{H_2}(v)+q_1)(d_{H_2}(w)+q_1)}{d_{H_2}(v)+d_{H_2}(w)+2q_1} &= \frac{d_{H_2}(v)d_{H_2}(w)+q_1(d_{H_2}(v)+d_{H_2}(w))+q_1^2}{d_{H_2}(v)+d_{H_2}(w)+2q_1}\\ &= \frac{d_{H_2}(v)d_{H_2}(w)}{d_{H_2}(v)+d_{H_2}(w)}.\frac{d_{H_2}(v)+d_{H_2}(w)}{d_{H_2}(v)+d_{H_2}(w)+2q_1}\\ &+ q_1\frac{d_{H_2}(v)+d_{H_2}(w)}{d_{H_2}(v)+d_{H_2}(w)+2q_1}+q_1^2\frac{1}{d_{H_2}(v)+d_{H_2}(w)+2q_1}\\ \Rightarrow \sum g &\leq \frac{\Delta_{H_2}}{\delta_{H_2}+q_1}ISI(H_2)+q_1q_2\frac{1}{\delta_{H_2}+q_1}\left(\Delta_{H_2}+\frac{q_1}{2}\right)\end{aligned}$$

Also;

$$\sum h \leq p_2q_1(p_2+2)\frac{\Delta_{H_2}+q_1}{(\delta_{H_2}+(p_2+q_1)+2)}$$

From the computations;

$$\begin{aligned}ISI(H_1\dot{+}H_2) &\leq (p_2+2)q_1'\frac{\Delta_{H_1}}{(\delta_{H_1}+p_2+2)}+\frac{\Delta_{H_2}}{(\delta_{H_2}+q_1)}ISI(H_2)+\frac{q_1q_2}{\delta_{H_2}+q_1}\left(\Delta_{H_2}+\frac{q_1}{2}\right)\\ &+ p_2q_1(p_2+2)\frac{\Delta_{H_2}+q_1}{(\delta_{H_2}+(p_2+q_1)+2)}\end{aligned}$$

This proves the result. □

Theorem 2.14. *Assume H_1 and H_2 represent arbitrary graphs;*

$$\begin{aligned}ISI(H_1\dot{+}H_2) &\geq (p_2+2)q_1'\frac{\delta_{H_1}}{(\Delta_{H_1}+p_2+2)}+\frac{\delta_{H_2}}{(\Delta_{H_2}+q_1)}ISI(H_2)+\frac{q_1q_2}{\Delta_{H_2}+q_1}\left(\delta_{H_2}+\frac{q_1}{2}\right)\\ &+ p_2q_1(p_2+2)\frac{\delta_{H_2}+q_1}{(\Delta_{H_2}+(p_2+q_1)+2)}\end{aligned}$$

Theorem 2.15. *Assume H_1 and H_2 represent arbitrary graphs;*

$$\begin{aligned}GA(H_1\dot{+}H_2) &\leq 2q_1'\frac{\sqrt{(p_2+2)\Delta_{H_1}}}{(\delta_{H_1}+p_2+2)}+\frac{2\Delta_{H_2}}{(\delta_{H_2}+q_1)}GA(H_2)+\frac{q_2}{\delta_{H_2}+q_1}\left(\sqrt{2q_1\Delta_{H_2}}+q_1\right)\\ &+ 2p_2q_1\frac{\sqrt{(p_2+2)(\Delta_{H_2}+q_1)}}{(\delta_{H_2}+(p_2+q_1)+2)}\end{aligned}$$

Proof.

$$\begin{aligned}GA(H_1\dot{+}H_2) &= \sum_{vw\in\mathcal{E}(H_1\dot{+}H_2)}\frac{2\sqrt{d(v)d(w)}}{d(v)+d(w)}\\ &= \sum_{\substack{vw\in\mathcal{E}(\mathcal{S}(H_1))\\ v\in\mathcal{V}(H_1)\\ w\in V_s(H_1)}}2\frac{\sqrt{d_{H_1}(v)(p_2+2)}}{d_{H_1}(v)+p_2+2}+\sum_{vw\in\mathcal{E}(H_2)}2\frac{\sqrt{(d_{H_2}(v)+q_1)(d_{H_2}(w)+q_1)}}{d_{H_2}(v)+d_{H_2}(w)+2q_1}\\ &+ \sum_{\substack{vw\in\mathcal{E}(H_1\dot{+}H_2)\\ v\in V_s(H_1)\\ w\in\mathcal{V}(H_2)}}2\frac{\sqrt{(p_2+2)(d_{H_2}(w)+q_1)}}{(p_2+2+d_{H_2}(w)+q_1)}\\ &= \sum f+\sum g+\sum h\end{aligned}$$

where,

$$\sum f = \sum_{\substack{vw\in\mathcal{E}(\mathcal{S}(H_1))\\ v\in\mathcal{V}(H_1)\\ w\in V_s(H_1)}} 2\frac{\sqrt{d_{H_1}(v)(p_2+2)}}{d_{H_1}(v)+p_2+2)}$$

$$\sum g = \sum_{vw\in\mathcal{E}(H_2)} 2\frac{\sqrt{(d_{H_2}(v)+q_1)(d_{H_2}(w)+q_1)}}{d_{H_2}(v)+d_{H_2}(w)+2q_1)}$$

$$\sum h = \sum_{\substack{vw\in\mathcal{E}(H_1\dotplus H_2)\\ v\in V_s(H_1)\\ w\in\mathcal{V}(H_2)}} 2\frac{\sqrt{(p_2+2)(d_{H_2}(w)+q_1)}}{(p_2+2+d_{H_2}(w)+q_1)}$$

Now;

$$\sum f \leq 2q_1^{'}\frac{\sqrt{(p_2+2)\Delta_{H_1}}}{(\delta_{H_1}+p_2+2)}$$

For $\sum g$;

$$\begin{aligned}
\frac{2\sqrt{(d_{H_2}(v)+q_1)(d_{H_2}(w)+q_1)}}{d_{H_2}(v)+d_{H_2}(w)+2q_1)} &= \frac{2\sqrt{d_{H_2}(v)d_{H_2}(w)+q_1(d_{H_2}(v)+d_{H_2}(w))+q_1^2}}{d_{H_2}(v)+d_{H_2}(w)+2q_1} \\
&\leq 2\left(\frac{\sqrt{d_{H_2}(v)d_{H_2}(w)}}{d_{H_2}(v)+d_{H_2}(w)}.\frac{d_{H_2}(v)+d_{H_2}(w)}{d_{H_2}(v)+d_{H_2}(w)+2q_1}\right) \\
&\quad+\frac{2\sqrt{q_1(d_{H_2}(v)+d_{H_2}(w))}}{d_{H_2}(v)+d_{H_2}(w)+2q_1}+2q_1\frac{1}{d_{H_2}(v)+d_{H_2}(w)+2q_1} \\
\Rightarrow \quad \sum g &\leq \frac{2\Delta_{H_2}}{\delta_{H_2}+q_1}GA(H_2)+q_2\frac{\sqrt{2q_1\Delta_{H_2}}}{\delta_{H_2}+q_1}+q_1q_2\frac{1}{\delta_{H_2}+q_1}
\end{aligned}$$

Also;

$$\sum h \leq 2p_2q_1\frac{\sqrt{(\Delta_{H_2}+q_1)(p_2+2)}}{(\delta_{H_2}+(p_2+q_1)+2)}$$

From the computations;

$$\begin{aligned}
GA(H_1\dotplus H_2) \leq 2q_1^{'}\frac{\sqrt{(p_2+2)\Delta_{H_1}}}{(\delta_{H_1}+p_2+2)}+\frac{2\Delta_{H_2}}{(\delta_{H_2}+q_1)}GA(H_2)+\frac{q_2}{\delta_{H_2}+q_1}\left(\sqrt{2q_1\Delta_{H_2}}+q_1\right) \\
+2p_2q_1\frac{\sqrt{(p_2+2)(\Delta_{H_2}+q_1)}}{(\delta_{H_2}+(p_2+q_1)+2)}
\end{aligned}$$

This proves the result. □

Theorem 2.16. *Assume H_1 and H_2 represent arbitrary graphs;*

$$\begin{aligned}
SO(H_1\dotplus H_2) \leq\ & q_1^{'}\left[\Delta_{H_1}+(p_2+2)\right]+SO(H_2)+q_2(2\sqrt{q_1\Delta_{H_2}}+\sqrt{2}q_1) \\
&+p_2q_1\left[(\Delta_{H_2}+q_1)+(p_2+2)\right]
\end{aligned}$$

Proof.

$$\begin{aligned}
SO(H_1 \dotplus H_2) &= \sum_{vw \in \mathcal{E}(H_1 \dotplus H_2)} \sqrt{(d(v))^2 + (d(w))^2} \\
&= \sum_{\substack{vw \in \mathcal{E}(\mathcal{S}(H_1)) \\ v \in \mathcal{V}(H_1) \\ w \in V_s(H_1)}} \sqrt{(d_{H_1}(v))^2 + (p_2+2)^2} \\
&+ \sum_{vw \in \mathcal{E}(H_2)} \sqrt{(d_{H_2}(v)+q_1)^2 + (d_{H_2}(w)+q_1)^2} \\
&+ \sum_{\substack{vw \in \mathcal{E}(H_1 \dotplus H_2) \\ v \in V_s(H_1) \\ w \in \mathcal{V}(H_2)}} \sqrt{(p_2+2)^2 + (d_{H_2}(w)+q_1)^2} \\
&= \sum f + \sum g + \sum h
\end{aligned}$$

where,

$$\begin{aligned}
\sum f &= \sum_{\substack{vw \in \mathcal{E}(\mathcal{S}(H_1)) \\ v \in \mathcal{V}(H_1) \\ w \in V_s(H_1)}} \sqrt{(d_{H_1}(v))^2 + (p_2+2)^2} \\
\sum g &= \sum_{vw \in \mathcal{E}(H_2)} \sqrt{(d_{H_2}(v)+q_1)^2 + (d_{H_2}(w)+q_1)^2} \\
\sum h &= \sum_{\substack{vw \in \mathcal{E}(H_1 \dotplus H_2) \\ v \in V_s(H_1) \\ w \in \mathcal{V}(H_2)}} \sqrt{(p_2+2)^2 + (d_{H_2}(w)+q_1)^2}
\end{aligned}$$

Now;

$$\sum f \leq q_1^{'} \left[\Delta_{H_1} + (p_2+2)\right]$$

For $\sum g$;

$$\begin{aligned}
(d_{H_2}(v)+q_1)^2 + (d_{H_2}(w)+q_1)^2 &= (d_{H_2}(v)^2 + d_{H_2}(w)^2) + 2q_1(d_{H_2}(v) + d_{H_2}(w)) + 2q_1^2 \\
&\leq (d_{H_2}(v)^2 + d_{H_2}(w)^2) + 4q_1\Delta_{H_2} + 2q_1^2 \\
\Rightarrow \sum g &\leq SO(H_2) + q_2(2\sqrt{q_1\Delta_{H_2}} + \sqrt{2}q_1)
\end{aligned}$$

Also;

$$\sum h \leq p_2 q_1 \left[(\Delta_{H_2} + q_1) + (p_2+2)\right]$$

From the computations;

$$\begin{aligned}
SO(H_1 \dotplus H_2) \leq\ & q_1^{'} \left[\Delta_{H_1} + (p_2+2)\right] + SO(H_2) + q_2(2\sqrt{q_1\Delta_{H_2}} + \sqrt{2}q_1) \\
&+ p_2 q_1 \left[(\Delta_{H_2} + q_1) + (p_2+2)\right]
\end{aligned}$$

This proves the result. □

Theorem 2.17. *Assume H_1 and H_2 represent arbitrary graphs;*

$$\begin{aligned}
SO(H_1 \dotplus H_2) \geq\ & q_1^{'} \left|\delta_{H_1} - (p_2+2)\right| + \left|SO(H_2) - q_2(\sqrt{4q_1\delta_{H_2} + 2q_1^2})\right| \\
&+ p_2 q_1 \left|(p_2+2) - (\delta_{H_2} + q_1)\right|
\end{aligned}$$

2.3 Bounds on the subdivision vertex-edge join of certain topological indices

The bounds for the subdivision vertex-edge join of graphs for the atom bond connectivity index and the geometric arithmetic index have been determined in [15]. Here, we carry the work forward and evaluate the bounds of the inverse sum indeg index and sombor index defined on the subdivision vertex-edge join of three graphs.

Theorem 2.18. *Assume H_1, H_2 and H_3 represent arbitrary graphs;*

$$\begin{aligned} SO(H_1^{\mathcal{S}} \triangleright (H_2^{\mathcal{V}} \cup H_3^{\mathcal{I}})) \leq\ & SO(H_2) + SO(H_3) + q_2\sqrt{2p_1(2\Delta_{H_2} + p_1)} + q_3\sqrt{2q_1(2\Delta_{H_3} + q_1)} \\ & + q_1\left[p_3\sqrt{(\Delta_{H_3} + q_1)^2 + (p_3 + 2)^2} + 2\sqrt{(\Delta_{H_1} + p_2)^2 + (p_3 + 2)^2}\right] \\ & + p_1p_2\sqrt{\Delta_{H_1}(\Delta_{H_1} + 2p_2) + \Delta_{H_2}(\Delta_{H_2} + 2p_1) + (p_1^2 + p_2^2)} \end{aligned}$$

Proof.

$$SO(H_1^{\mathcal{S}} \triangleright (H_2^{\mathcal{V}} \cup H_3^{\mathcal{I}})) = \sum_{z_1z_2 \in \mathcal{E}(H_1^{\mathcal{S}} \triangleright (H_2^{\mathcal{V}} \cup H_3^{\mathcal{I}}))} \sqrt{d_{H_1^{\mathcal{S}} \triangleright (H_2^{\mathcal{V}} \cup H_3^{\mathcal{I}})}(z_1)^2 + d_{H_1^{\mathcal{S}} \triangleright (H_2^{\mathcal{V}} \cup H_3^{\mathcal{I}})}(z_2)^2}$$

$$\begin{aligned} SO(H_1^{\mathcal{S}} \triangleright (H_2^{\mathcal{V}} \cup H_3^{\mathcal{I}})) =\ & \sum_{z_1z_2 \in \mathcal{E}(H_2)} \sqrt{(d_{H_2}(z_1) + p_1)^2 + (d_{H_2}(z_2) + p_1)^2} \\ & + \sum_{z_1z_2 \in \mathcal{E}(H_3)} \sqrt{(d_{H_3}(z_1) + q_1)^2 + (d_{H_3}(z_2) + q_1)^2} \\ & + \sum_{z_1 \in \mathcal{V}(H_1)} \sum_{z_2 \in \mathcal{V}(H_2)} \sqrt{(d_{H_1}(z_1) + p_2)^2 + (d_{H_2}(z_2) + p_1)^2} \\ & + \sum_{z_1 \in \mathcal{I}(H_1)} \sum_{z_2 \in \mathcal{V}(H_3)} \sqrt{(d_{H_3}(z_2) + q_1)^2 + (p_3 + 2)^2} \\ & + \sum_{\substack{z_1z_2 \in \mathcal{E}(\mathcal{S}(H_1)) \\ z_1 \in \mathcal{V}(H_1) \\ z_2 \in \mathcal{I}(H_1)}} \sqrt{(d_{H_1}(z_1) + p_2)^2 + (p_3 + 2)^2} \\ =\ & \sum f + \sum g + \sum h + \sum i + \sum j \end{aligned}$$

where,

$$\sum f = \sum_{z_1z_2 \in \mathcal{E}(H_2)} \sqrt{(d_{H_2}(z_1) + p_1)^2 + (d_{H_2}(z_2) + p_1)^2}$$

$$\sum g = \sum_{z_1z_2 \in \mathcal{E}(H_3)} \sqrt{(d_{H_3}(z_1) + q_1)^2 + (d_{H_3}(z_2) + q_1)^2}$$

$$\sum h = \sum_{z_1 \in \mathcal{V}(H_1)} \sum_{z_2 \in \mathcal{V}(H_2)} \sqrt{(d_{H_1}(z_1) + p_2)^2 + (d_{H_2}(z_2) + p_1)^2}$$

$$\sum i = \sum_{z_1 \in \mathcal{I}(H_1)} \sum_{z_2 \in \mathcal{V}(H_3)} \sqrt{(d_{H_3}(z_2) + q_1)^2 + (p_3 + 2)^2}$$

$$\sum j = \sum_{\substack{z_1z_2 \in \mathcal{E}(\mathcal{S}(H_1)) \\ z_1 \in \mathcal{V}(H_1) \\ z_2 \in \mathcal{I}(H_1)}} \sqrt{(d_{H_1}(z_1) + p_2)^2 + (p_3 + 2)^2}$$

Now,

$$\sqrt{(d_{H_2}(z_1) + p_1)^2 + (d_{H_2}(z_2) + p_1)^2} = \sqrt{d_{H_2}(z_1)^2 + d_{H_2}(z_2)^2 + 2p_1(d_{H_2}(z_1) + d_{H_2}(z_2) + p_1)}$$

Hence for the computation of $\sum f$,

$$\sum f \leq SO(H_2) + q_2\sqrt{2p_1(2\Delta_{H_2} + p_1)}$$

Similarly for $\sum g$,

$$\sum g \leq SO(H_3) + q_3\sqrt{2q_1(2\Delta_{H_3} + q_1)}$$

Also,

$$\sum h \leq p_1p_2\sqrt{\Delta_{H_1}(\Delta_{H_1} + 2p_2) + \Delta_{H_2}(\Delta_{H_2} + 2p_1) + (p_1^2 + p_2^2)}$$

$$\sum i \leq q_1p_3\sqrt{(\Delta_{H_3} + q_1)^2 + (p_3 + 2)^2}$$

and,

$$\sum j \leq 2q_1\sqrt{(\Delta_{H_1} + p_2)^2 + (p_3 + 2)^2}$$

From the computations,

$$\begin{aligned} SO(H_1^{\mathcal{S}} \rhd (H_2^{\mathcal{V}} \cup H_3^{\mathcal{I}})) \leq{}& SO(H_2) + SO(H_3) + q_2\sqrt{2p_1(2\Delta_{H_2} + p_1)} + q_3\sqrt{2q_1(2\Delta_{H_3} + q_1)} \\ &+ q_1\left[p_3\sqrt{(\Delta_{H_3} + q_1)^2 + (p_3 + 2)^2} + 2\sqrt{(\Delta_{H_1} + p_2)^2 + (p_3 + 2)^2}\right] \\ &+ p_1p_2\sqrt{\Delta_{H_1}(\Delta_{H_1} + 2p_2) + \Delta_{H_2}(\Delta_{H_2} + 2p_1) + (p_1^2 + p_2^2)} \end{aligned}$$

This proves the result. □

Theorem 2.19.

$$\begin{aligned} SO(H_1^{\mathcal{S}} \rhd (H_2^{\mathcal{V}} \cup H_3^{\mathcal{I}})) \geq{}& |SO(H_2) - q_2\sqrt{2p_1(2\delta_{H_2} + p_1)}| + |SO(H_3) - q_3\sqrt{2q_1(2\delta_{H_3} + q_1)}| \\ &+ q_1\left[p_3\sqrt{(\delta_{H_3} + q_1)^2 + (p_3 + 2)^2} + 2\sqrt{(\delta_{H_1} + p_2)^2 + (p_3 + 2)^2}\right] \\ &+ p_1p_2\sqrt{\delta_{H_1}(\delta_{H_1} + 2p_2) + \delta_{H_2}(\delta_{H_2} + 2p_1) + (p_1^2 + p_2^2)} \end{aligned}$$

Theorem 2.20. *Assume H_1, H_2 and H_3 represent arbitrary graphs;*

$$\begin{aligned} ISI(H_1^{\mathcal{S}} \rhd (H_2^{\mathcal{V}} \cup H_3^{\mathcal{I}})) \leq{}& ISI(H_2)\frac{\Delta_{H_2}}{\delta_{H_2} + p_1} + ISI(H_3)\frac{\Delta_{H_3}}{\delta_{H_3} + q_1} \\ &+ \frac{p_1q_2}{2(\delta_{H_2} + p_1)}(2\Delta_{H_2} + p_1) + \frac{q_1q_3}{2(\delta_{H_3} + q_1)}(2\Delta_{H_3} + q_1) \\ &+ p_1p_2\frac{(\delta_{H_1} + p_2)(\delta_{H_2} + p_1)}{\delta_{H_1} + \delta_{H_2} + (p_1 + p_2)} + q_1(p_3 + 2)\Bigg[p_3\frac{\Delta_{H_3} + q_1}{\delta_{H_3} + (q_1 + p_3 + 2)} \\ &+ 2\frac{\Delta_{H_1} + p_2}{\delta_{H_1} + (p_2 + p_3 + 2)}\Bigg] \end{aligned}$$

Proof.

$$ISI(H_1^{\mathcal{S}} \rhd (H_2^{\mathcal{V}} \cup H_3^{\mathcal{I}})) = \sum_{z_1z_2 \in \mathcal{E}(H_1^{\mathcal{S}} \rhd (H_2^{\mathcal{V}} \cup H_3^{\mathcal{I}}))} \frac{d_{H_1^{\mathcal{S}} \rhd (H_2^{\mathcal{V}} \cup H_3^{\mathcal{I}})}(z_1)\, d_{H_1^{\mathcal{S}} \rhd (H_2^{\mathcal{V}} \cup H_3^{\mathcal{I}})}(z_2)}{d_{H_1^{\mathcal{S}} \rhd (H_2^{\mathcal{V}} \cup H_3^{\mathcal{I}})}(z_1) + d_{H_1^{\mathcal{S}} \rhd (H_2^{\mathcal{V}} \cup H_3^{\mathcal{I}})}(z_2)}$$

$$\begin{aligned}
ISI(H_1^{\mathcal{S}} \rhd (H_2^{\mathcal{V}} \cup H_3^{\mathcal{I}})) = & \sum_{z_1 z_2 \in \mathcal{E}(H_2)} \frac{(d_{H_2}(z_1)+p_1)(d_{H_2}(z_2)+p_1)}{d_{H_2}(z_1)+d_{H_2}(z_2)+2p_1} \\
& + \sum_{z_1 z_2 \in \mathcal{E}(H_3)} \frac{(d_{H_3}(z_1)+q_1)(d_{H_3}(z_2)+q_1)}{d_{H_3}(z_1)+d_{H_3}(z_2)+2q_1} \\
& + \sum_{z_1 \in \mathcal{V}(H_1)} \sum_{z_2 \in \mathcal{V}(H_2)} \frac{(d_{H_1}(z_1)+p_2)(d_{H_2}(z_2)+p_1)}{d_{H_1}(z_1)+d_{H_2}(z_2)+(p_1+p_2)} \\
& + \sum_{z_1 \in \mathcal{I}(H_1)} \sum_{z_2 \in \mathcal{V}(H_3)} \frac{(p_3+2)(d_{H_3}(z_2)+q_1)}{d_{H_3}(z_2)+(q_1+p_3+2)} \\
& + \sum_{\substack{z_1 z_2 \in \mathcal{E}(\mathcal{S}(H_1)) \\ z_1 \in \mathcal{V}(H_1) \\ z_2 \in \mathcal{I}(H_1)}} \frac{(d_{H_1}(z_1)+p_2)(p_3+2)}{d_{H_1}(z_1)+(p_2+p_3+2)} \\
= & \sum f + \sum g + \sum h + \sum i + \sum j
\end{aligned}$$

where,

$$\sum f = \sum_{z_1 z_2 \in \mathcal{E}(H_2)} \frac{(d_{H_2}(z_1)+p_1)(d_{H_2}(z_2)+p_1)}{d_{H_2}(z_1)+d_{H_2}(z_2)+2p_1}$$

$$\sum g = \sum_{z_1 z_2 \in \mathcal{E}(H_3)} \frac{(d_{H_3}(z_1)+q_1)(d_{H_3}(z_2)+q_1)}{d_{H_3}(z_1)+d_{H_3}(z_2)+2q_1}$$

$$\sum h = \sum_{z_1 \in \mathcal{V}(H_1)} \sum_{z_2 \in \mathcal{V}(H_2)} \frac{(d_{H_1}(z_1)+p_2)(d_{H_2}(z_2)+p_1)}{d_{H_1}(z_1)+d_{H_2}(z_2)+(p_1+p_2)}$$

$$\sum i = \sum_{z_1 \in \mathcal{I}(H_1)} \sum_{z_2 \in \mathcal{V}(H_3)} \frac{(p_3+2)(d_{H_3}(z_2)+q_1)}{d_{H_3}(z_2)+(q_1+p_3+2)}$$

$$\sum j = \sum_{\substack{z_1 z_2 \in \mathcal{E}(\mathcal{S}(H_1)) \\ z_1 \in \mathcal{V}(H_1) \\ z_2 \in \mathcal{I}(H_1)}} \frac{(d_{H_1}(z_1)+p_2)(p_3+2)}{d_{H_1}(z_1)+(p_2+p_3+2)}$$

Now,

$$\begin{aligned}
\frac{(d_{H_2}(z_1)+p_1)(d_{H_2}(z_2)+p_1)}{d_{H_2}(z_1)+d_{H_2}(z_2)+2p_1} &= \frac{d_{H_2}(z_1)d_{H_2}(z_2)+p_1(d_{H_2}(z_1)+d_{H_2}(z_2))+p_1^2}{d_{H_2}(z_1)+d_{H_2}(z_2)+2p_1} \\
&= \frac{d_{H_2}(z_1)d_{H_2}(z_2)}{d_{H_2}(z_1)+d_{H_2}(z_2)+2p_1} . \frac{d_{H_2}(z_1)+d_{H_2}(z_2)+2p_1)}{d_{H_2}(z_1)+d_{H_2}(z_2)+2p_1)} \\
&+ p_1 \frac{d_{H_2}(z_1)+d_{H_2}(z_2)}{d_{H_2}(z_1)+d_{H_2}(z_2)+2p_1)} + p_1^2 \frac{1}{d_{H_2}(z_1)+d_{H_2}(z_2)+2p_1)}
\end{aligned}$$

Hence,

$$\sum f \quad \leq \quad ISI(H_2)\frac{\Delta_{H_2}}{\delta_{H_2}+p_1} + \frac{p_1 q_2}{2(\delta_{H_2}+p_1)}(2\Delta_{H_2}+p_1)$$

Similarly,

$$\sum g \quad \leq \quad ISI(H_3)\frac{\Delta_{H_3}}{\delta_{H_3}+q_1} + \frac{q_1 q_3}{2(\delta_{H_3}+q_1)}(2\Delta_{H_3}+q_1)$$

Also,

$$\sum h \quad \leq \quad p_1 p_2 \frac{(\delta_{H_1}+p_2)(\delta_{H_2}+p_1)}{\delta_{H_1}+\delta_{H_2}+(p_1+p_2)}$$

Also, we have

$$\sum i \quad \leq \quad q_1(p_3+2)p_3 \frac{\Delta_{H_3}+q_1}{\delta_{H_3}+(q_1+p_3+2)}$$

and,

$$\sum j \leq 2q_1(p_3+2)\frac{\Delta_{H_1}+p_2}{\delta_{H_1}+(p_2+p_3+2)}$$

From the computations,

$$\begin{aligned} ISI(H_1^{\mathcal{S}} \triangleright (H_2^{\mathcal{V}} \cup H_3^{\mathcal{I}})) \leq\ & ISI(H_2)\frac{\Delta_{H_2}}{\delta_{H_2}+p_1} + ISI(H_3)\frac{\Delta_{H_3}}{\delta_{H_3}+q_1} \\ & + \frac{p_1q_2}{2(\delta_{H_2}+p_1)}(2\Delta_{H_2}+p_1) + \frac{q_1q_3}{2(\delta_{H_3}+q_1)}(2\Delta_{H_3}+q_1) \\ & + p_1p_2\frac{(\delta_{H_1}+p_2)(\delta_{H_2}+p_1)}{\delta_{H_1}+\delta_{H_2}+(p_1+p_2)} + q_1(p_3+2)\Bigg[p_3\frac{\Delta_{H_3}+q_1}{\delta_{H_3}+(q_1+p_3+2)} \\ & + 2\frac{\Delta_{H_1}+p_2}{\delta_{H_1}+(p_2+p_3+2)}\Bigg] \end{aligned}$$

This proves the result. □

Theorem 2.21. *Assume H_1, H_2 and H_3 represent arbitrary graphs;*

$$\begin{aligned} ISI(H_1^{\mathcal{S}} \triangleright (H_2^{\mathcal{V}} \cup H_3^{\mathcal{I}})) \geq\ & ISI(H_2)\frac{\delta_{H_2}}{\Delta_{H_2}+p_1} + ISI(H_3)\frac{\delta_{H_3}}{\Delta_{H_3}+q_1} \\ & + \frac{p_1q_2}{2(\Delta_{H_2}+p_1)}(2\delta_{H_2}+p_1) + \frac{q_1q_3}{2(\Delta_{H_3}+q_1)}(2\delta_{H_3}+q_1) \\ & + p_1p_2\frac{(\Delta_{H_1}+p_2)(\Delta_{H_2}+p_1)}{\Delta_{H_1}+\Delta_{H_2}+(p_1+p_2)} + q_1(p_3+2)\Bigg[p_3\frac{\delta_{H_3}+q_1}{\Delta_{H_3}+(q_1+p_3+2)} \\ & + 2\frac{\delta_{H_1}+p_2}{\Delta_{H_1}+(p_2+p_3+2)}\Bigg] \end{aligned}$$

3 Conclusion

In chemical graph theory, every characteristic of a molecular graph is crucial to acquire advancements and the process can be facilitated by the proper analysis of topological indices.

Applicability: The potency of the descriptors is investigated by performing quantitative analysis. Topological indices have the ability to predict the theoretic physico-chemical attributes of any molecular compound. Also, some nanostructures or molecular compounds can be moulded in the form of some graphs involved in certain particular operation and the invariants can be determined utilising the properties [10, 4].

Accordingly, this paper investigates the proposed graph operation variants namely the corona join product , the subdivision vertex join of two graphs and the subdivision vertex-edge join of three graphs by exploring the bounds of atom connectivity index, inverse sum indeg index, geometric index and sombor index. The outcomes procured have favourable angles towards further research for the analysis of degree and distance dependent descriptors and series of graph operation.

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