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Commutative rings with Artinian spectrum

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Abstract The rings considered in this article are commutative with identity $1 \neq 0$. In this paper, we introduce and study commutative ring with Artinian spectrum, that is commutative ring with descending chain condition on radical ideals. A characterization via prime ideals of such commutative ring is given. As part of the study, we introduce a topology τ_c on Spec$R$ for which the open subsets are exactly the Zariski closed subsets. For instance, it is shown that for a commutative ring R with Artinian spectrum, $(\text{Spec}R, \tau_c)$ is a spectral space if and only if R has Noetherian spectrum.

1 Introduction

Throughout this paper, all rings are commutative with unit $1 \neq 0$.

Recall that a commutative ring R is said to have Noetherian spectrum if it satisfies the ascending chain condition on radical ideals. Many authors studied commutative ring with Noetherian spectrum. It is well known that a commutative ring R has Noetherian spectrum if and only if R satisfies the ascending chain condition on prime ideals and each ideal has only finitely many prime ideals minimal over it. This is a characterization of rings with Noetherian spectrum via chain of prime ideals. Motivated by this notion, we study its dual notion by introducing the notion of ring with Artinian spectrum.

A commutative ring R is said to have Artinian spectrum if it satisfies the descending chain condition on radical ideals. This is equivalent to the condition that Spec$R$ with its Zariski topology satisfies the ascending chain condition (respectively, descending chain condition) on closed subsets (respectively, open subsets). Note that if R is an Artinian ring then it has Artinian spectrum and the converse is false, as in example 4.9. In this paper, we study rings with Artinian spectrum. A characterization via chain of prime ideals is given, precisely a commutative ring R has Artinian spectrum if and only if R satisfies descending chain condition for prime ideals and all antichains of prime ideals are finite. In particular ring with Artinian spectrum has FC property (i.e. every ideal of R has a finite number of minimal prime divisors). On the other hand, if R is a commutative ring with Artinian spectrum, then the intersection of any collection of a Zariski open subsets of Spec$R$ is a Zariski open. This property leads us to introduce a new topology τ_c on Spec$R$ for which a subset is open if and only if it is Zariski closed. Finally, we study the spectral property of Spec$R$ equipped with the new topology τ_c.

2 Artinian topological space

In this section, we briefly discuss some facts and results in the topological setting.

Definition 2.1. A topological space X is called Artinian if it satisfies the ascending chain condition for closed subsets; for any sequence $F_1 \subseteq F_2 \subseteq \dots$ of closed subsets F_i there is an integer N such that $F_N = F_{N+1} = \dots$.

Remark 2.2. A topological space is Artinian if and only if one of the following conditions hold.

(i) X satisfies the descending chain condition for open subsets.

(ii) Any nonempty collection of closed subsets has a maximal element.

(iii) Any nonempty collection of open subsets has a minimal element.

Proposition 2.3. *Every subspace of an Artinian space is Artinian.*

Proof. Let Y be a subspace of an Artinian space X. Let $Y_1 \subseteq Y_1 \subseteq \dots$ be an ascending chain of closed subsets of Y. Write $Y_i = Y \cap F_i$ with F_i closed subset of X. For $i > 0$, consider the closed subset $F'_i = \cup_{j=1}^{i} F_j$ and it is easy to see that $F'_i \cap Y = Y_i$. We have an ascending chain of closed subsets

$$F'_1 \subseteq F'_2 \subseteq F'_3 \subseteq \dots$$

This chain stabilizes by assumption, so $F'_N = F'_{N+1} = \dots$ for some $N \in \mathbb{N}^*$. Hence $Y_N = Y_{N+1} = \dots$. □

Proposition 2.4. *Let X be an Artinian space. Any antichain of irreducible closed subsets is finite.*

Proof. Assume that X has an infinite antichain of irreducible closed subsets. Consider an infinite sequence F_n, $n \in \mathbb{N}$ of elements of this antichain. For $n \in \mathbb{N}$, set $Z_n = \cup_{i=1}^{n} F_i$. Since $Z_1 \subseteq Z_2 \subseteq \dots$ is an ascending chain of closed subsets. This chain stabilizes by assumption, so $Z_N = Z_{N+1} = \dots$ for some N. It follows that $F_{N+1} \subseteq F_1 \cup \dots \cup F_N$. Since F_{N+1} is irreducible it must be contained in one of the F_i with $1 \leq i \leq N$, a contradiction. □

Corollary 2.5. *Let X be an Artinian space.*

(i) X has finitely many closed points.

(ii) X has finitely many irreducible components.

Proof. (1) A closed point is an irreducible closed subset, so the set of closed points is an antichain, hence its is finite.

(2) The set of irreducible components is an antichain, so finite. □

Proposition 2.6. *Let X be an Artinian space. Then X has finitely many connected components, in particular the connected components are opens.*

Proof. Assume the converse. Let $C_n, n \in \mathbb{N}$ be an infinite sequence of connected components. Consider the ascending chain of closed subsets $Z_1 \subseteq Z_2 \subseteq \dots$ where $Z_n = C_1 \cup \dots \cup C_n$. This chain stabilizes, so there is an integer N such $Z_N = Z_{N+1} = \dots$, so $C_{N+1} \subseteq C_1 \cup \dots \cup C_N$ which is a contradiction. □

3 Rings with Artinian spectrum

We start this section with definition and basic properties of rings with Artinian spectrum.

Definition 3.1. Let R be a commutative ring. We say that R has Artinian spectrum if Spec$R$ endowed with the Zariski topology is an Artinian space.

Since there is a one to one order reversing correspondence between closed subsets of Spec$R$ and radical ideals of R, the commutative ring R has Artinian spectrum if and only if it satisfies the descending chain condition for radical ideals. It is easy to see that an Artinian ring has Artinian spectrum, but the converse is not true. A discrete valuation ring V is not Artinian and has Artinian spectrum since Spec$V$ is finite.

Remark 3.2. Let R be a commutative ring. As in the topological setting, we have the following characterization: R has Artinian spectrum if and only if any collection of radical ideals has a minimal element.

Proposition 3.3. *Let R be a commutative ring with Artinian spectrum.*

(i) If I is an ideal of R, then R/I has Artinian spectrum.

(ii) If S is a multiplicative subset of R, then $S^{-1}R$ has Artinian spectrum.

(iii) R is semi-local, that is, has a finitely many maximal ideals.

(iv) All antichains of prime ideals are finite.

(v) R has the FC property (each ideal has only finitely many prime ideals minimal over it).

Proof. (1) SpecR/I is a closed subset of Spec$R$, so it is an Artinian space. Thus R/I has Artinian spectrum.

(2) Spec$S^{-1}R$ is homeomorphic to a subset (endowed with its induced topology) of Spec$R$, so it is an Artinian space.

(3) Artinian space has finitely many closed points and the closed points of Spec$R$ are the maximal ideals of R.

(4) This is a consequence to the fact that an antichain of prime ideals of R correspond to an antichain of closed irreducible subsets of Spec$R$.

(5) Let I be an ideal of R. The set of all prime ideals minimal over I is an antichain, so finite. □

Proposition 3.4. *Let R be a commutative ring with Artinian spectrum and k be an integer. Then R has finitely many prime ideals of height less or equal to k.*

Proof. Follows from the following remark. For $i \leq k$, the collection of prime ideals of height i is an antichain, so finite. □

Corollary 3.5. *Let R be a commutative ring with finite Krull dimension. Then R has Artinian spectrum if and only if it has finitely many prime ideals.*

We begin with proving the following lemma in order to characterize rings with Artinian spectrum via chains of prime ideals.

Lemma 3.6. *Let R be a commutative ring. Assume that R satisfies descending chain condition for prime ideals and all antichains of prime ideals are finite. Then the following statements hold.*

(i) For every prime ideals $P \subsetneq Q$, there is a prime ideal Q' such that $P \subsetneq Q' \subseteq Q$ and $\text{ht}(Q'/P) = 1$.

(ii) For a non-maximal ideal P, there is finitely many prime ideals Q containing P such that $\text{ht}(Q/P) = 1$.

(iii) R has FC.

(iv) If I is a radical ideal, then there are finitely many radical ideals J containing strictly I, such that every radical ideal J' containing strictly I contain one of those J.

Proof. (1) Suppose the contrary. In this cases $\text{ht}(Q/P) > 1$, so there exists a prime ideal Q_1 such that $P \subsetneq Q_1 \subsetneq Q$. By assumption $\text{ht}(Q_1/P) > 1$, there exists a prime ideal Q_2 such that $P \subsetneq Q_2 \subsetneq Q_1$, we do this many times, we construct a strictly descending chain of prime ideals, a contradiction.

(2) The collection of prime ideals Q containing P such that $\text{ht}(Q/P) = 1$ is an antichain, hence finite.

(3) Let I be an ideal of R. The collection of minimal prime ideals over I is an antichain, so finite.

(4) Let I be a radical ideal of R. Let $Q_1, \ldots, Q_s$ be the minimal prime ideals over I, in particular $I = Q_1 \cap \ldots \cap Q_s$. For $1 \leq i \leq s$, set $H_i := \{P \in V(Q_i) \ / \ \text{ht}(P/Q_i) = 1\}$ and $H := \cup_{i=1}^{s} H_i$. Consider C the set of radical ideals of the form $(\cap_{P \in S} P) \cap (\cap_{P \in A} P)$ where S is a strictly subset of $\{Q_1, \ldots, Q_s\}$ and A is a non empty subset of H. It is easy to see that C is finite since $\{Q_1, \ldots, Q_s\}$ and H are finite. Let S be a strictly subset of $\{Q_1, \ldots, Q_s\}$ and A a non empty subset of H. Since every element P of A contain an ideal Q_j, we have $I \subseteq \cap_{P \in A} P$, on the other hand, it is easy to see that $I \subseteq \cap_{P \in S} P$. So $I \subseteq (\cap_{P \in S} P) \cap (\cap_{P \in A} P)$. There is j such Q_j is not is S, so Q_j is not in $S \cup A$, thus $I \neq (\cap_{P \in S} P) \cap (\cap_{P \in A} P)$. It follows that I is strictly contained in every elements of C. Now let J be a radical ideal containing strictly I. There is $1 \leq i_0 \leq s$ such that Q_{i_0} is not a minimal ideal of J. Denote D the set of minimal prime ideals over J, and write $D = (D \cap \{Q_1, \ldots, Q_s\}) \cup (D \setminus \{Q_1, \ldots, Q_s\}) = S \cup A'$ where $S = D \cap \{Q_1, \ldots, Q_s\}$

and $A' = D \setminus \{Q_1, \ldots, Q_s\}$. Note that $J = \cap_{P \in D} P = (\cap_{P \in S} P) \cap (\cap_{P \in A'} P)$. Since Q_{i_0} is not in D, S is strictly contained in $\{Q_1, \ldots, Q_s\}$. Let P be an element of A', then $I \subseteq P$, so there is a prime ideal Q_j such that $Q_j \subseteq P$, and in fact this inclusion is strict. It follows that there exists a prime ideal P' such that $Q_j \subsetneq P' \subseteq P$ with $\mathrm{ht}(P'/Q_j) = 1$. Thus, for each prime ideal P in A' there exists a prime ideal P' in H such that $P' \subseteq P$ and if we denote A the set of those prime ideals, we get, $\cap_{P' \in A} P' \subseteq \cap_{P \in A'} P$, hence $(\cap_{P \in S} P) \cap (\cap_{P \in A} P) \subseteq J$. □

Theorem 3.7. *Let R be a commutative ring. Then R has Artinian spectrum if and only if R satisfies descending chain condition for prime ideals and all antichains of prime ideals are finite.*

Proof. Assume that R satisfies descending chain condition for prime ideals and all antichains of prime ideals are finite. Let $\ldots \subseteq I_2 \subseteq I_1$ be a descending chain of radical ideals and set $I = \cap_{n \geq 1} I_n$. If there is an integer N such that $I_N = I$ then the chain stabilize. Now assume that for all n, $I \subsetneq I_n$. Let $J_1, \ldots, J_m$ be a radical ideals containing strictly I such that every radical ideal containing strictly I contain one J_i. For $1 \leq i \leq m$, consider $N_i = \{n \geq 1 / \ J_i \subseteq I_n\}$. It is easy to see that $\cup_{i=1}^m N_i = \mathbb{N}^*$, so one of the sets N_i is infinite, say N_{i_0}. Now, write $N_{i_0} = \{n_k \ \ /k \geq 1\}$ with $n_k < n_{k+1}$. Then $J_{i_0} \subseteq \cap_k I_{n_k} = I \subsetneq J_{i_0}$, a contradiction. The converse is immediate from Proposition 3.3 and the fact that prime ideal is radical. □

Theorem 3.8. *Let R be a commutative ring. Then R has Artinian spectrum if and only if any non empty set of prime ideals has a minimal element and its set of minimal elements if finite.*

Proof. Since prime ideals are radical, every non empty set of prime ideals has a minimal element. The set of minimal elements is an antichain hence finite. Conversely, let P_n be a descending chain of prime ideals. The set of all prime ideals P_n has a minimal element, say P_N. Then $P_N = P_{N+1} = \ldots$. All elements of an antichain are minimal, so all antichains are finite. □

Corollary 3.9. *Let R be a commutative ring with Artinian spectrum. If P is a prime ideal of R and not the only maximal ideal, then $\cap_{Q \not\subseteq P} Q \not\subseteq P$.*

Proof. Let P be a prime ideal of R which is not the only maximal ideal. Let X be the set of all prime ideals Q such that $Q \not\subseteq P$. X is a non empty subset of prime ideals, so it has a minimal elements. The collection of minimal elements of X is an antichain, hence finite, say $Q_1, \ldots, Q_r$. Clearly $\cap_{Q \not\subseteq P} Q = Q_1 \cap \ldots \cap Q_r \not\subseteq P$ since for all i, Q_i is not contained in P. □

The previous corollary is the avoidance-type property relative to the intersection, that is, for a commutative ring with Artinian spectrum, if P_i is any family of prime ideals, then for every prime ideal P, $\cap_i P \subseteq P$ implies that $P_i \subseteq P$ for some i.

Now, we close this section with the following corollary regarding the topological aspect of minimal prime ideals.

Corollary 3.10. *Let R be a commutative ring with Artinian spectrum. Every generic point of* Spec$R$ *is open.*

Proof. Let R be a commutative ring with Artinian spectrum and P be a minimal prime ideal of R. Since $\cap_{Q \not\subseteq P} Q \not\subseteq P$, there exists $f \in \cap_{Q \not\subseteq P} Q$ such that $f \notin P$, so $\{P\} = \mathrm{Spec}R - V(f)$ is open. □

4 Spectral property for $\mathrm{Spec}_c R$

We start this section with the following result in order to define a new topology on Spec$R$.

For a prime ideal P, we denote O_P the set $O_P := \{Q \in \mathrm{Spec}R \ / \ Q \subseteq P\}$.

Proposition 4.1. *Let R be a commutative ring with Artinian spectrum and $X =$ Spec$R$.*

(i) Let $P \in X$. Then O_P is an open subset of X.

(ii) If U is an open subset of X containing P, then $O_P \subseteq U$.

(iii) Any intersection of open subsets of X is open.

Proof. (1) If P is the only maximal ideal of R then $O_P = X$ is open. For else, let $P_1, \dots, P_r$ be the sets of minimal prime ideals over $\cap_{Q \not\subseteq P} Q$. Note that $P_1 \cap \dots \cap P_r = \cap_{Q \not\subseteq P} Q \not\subseteq P$, so each ideal P_i is not contained in P. Thus $X \setminus O_P = V(P_1 \cap \dots \cap P_r)$. In fact, let $Q \in X \setminus O_P$, then $Q \not\subseteq P$, so $P_1 \cap \dots \cap P_r \subseteq Q$, thus $Q \in V(P_1 \cap \dots \cap P_r)$. Conversely, let $Q \in V(P_1 \cap \dots, P_r)$, so $P_1 \cap \dots \cap P_r \subseteq Q$, this implies that $P_i \subseteq Q$ for some i. Hence $Q \not\subseteq P$ since $P_i \not\subseteq P$.
(2) Let $U = X \setminus V(I)$ be an open subset of X containing P. If $Q \subseteq P$ then $I \not\subseteq Q$, so $Q \in U$.
(3) Let $(U_t)_{t \in T}$ be a collection of open subsets of X. If $\cap_{t \in T} U_t$ is empty, then it is open. If $\cap_{t \in T} U_t$ is not empty and $P \in \cap_{t \in T} U_t$, then for every $t \in T$, $P \in U_t$, so $O_P \subseteq U_t$. Thus $O_P \subseteq \cap_{t \in T} U_t$. □

Remark 4.2. Let R be a commutative ring with Artinian spectrum and $P \in \text{Spec}R$. Since $\cap_{Q \not\subseteq P} Q \not\subseteq P$, there exists $f \in \cap_{Q \not\subseteq P} Q$ such that $f \notin P$. We get a morphism $i : R_f \to R_P$, $a/f^n \mapsto i(a/f^n) = a/f^n$. Then i is an isomorphism. In particular, $\text{Spec}R_P$ and $\text{Spec}R_f$ are homoemorphic.

Proof. If $a/f^n = 0$ in R_P then there exists $s \notin P$ such that $sa = 0$. If Q' is a prime ideal containing s then $Q' \not\subseteq P$, so $\cap_{Q \not\subseteq P} Q \subseteq Q'$. Thus, $f \in Q'$. It follows that $f \in \sqrt{(s)}$. Hence $f^m = bs$ for some $m \in \mathbb{N}$ and $b \in R$. As a consequence $f^m a = bsa = 0$, so i is injective. Let $a/s \in R_P$, since $s \notin P$, as in the previous steep, there exists $m \in \mathbb{N}$ and $b \in R$ such that $f^m = bs$, so $a/s = (ba)/(bs) = i(ba/f^m)$. □

Let R be a commutative ring with Artinian spectrum. By the previous result any union of closed subsets of X is a closed subset of X, so there exists a unique topology τ_c on X whose open subsets are the the Zariski closed subsets of X. We denote $\text{Spec}_c R$ to indicate that $\text{Spec}R$ endowed with this topology, and $\text{Spec}_z R$ to indicate that $\text{Spec}R$ endowed with its Zariski topology.

Proposition 4.3. *Let R be a commutative ring with Artinian spectrum. Let $P \in \text{Spec}R$.*

(i) $\overline{\{P\}}^c = O_P$ *where* $\overline{\{P\}}^c$ *is the closer of P in* $\text{Spec}_c R$.

(ii) P is a closed point of $\text{Spec}_c R$ *if and only if it is a generic point of* $\text{Spec}_z R$.

Proof. (1) Clearly O_P is a closed subset of $\text{Spec}_c R$ containing P, and if U is any closed subset of $\text{Spec}_c R$ containing P, then U is a Zariski open subset containing P. Thus $O_p \subseteq U$. It follows that $\overline{\{P\}} = O_P$.
(2) P is a closed point in $\text{Spec}_c R$ if and only if $\{P\} = O_P$ if and only if P is a minimal prime ideal of R. □

In the following Theorem, we describe the irreducible closed subsets of $\text{Spec}_c R$.

Proposition 4.4. *Let R be a commutative ring with Artinian spectrum and U be a closed subset of* $\text{Spec}_c R$.

(i) If U is a Zariski quasi-compact open. Then U is an irreducible closed subset of $\text{Spec}_c R$ *if and only if $U = O_P$ for some prime P.*

(ii) The irreducible components of $\text{Spec}_c R$ *are O_M with M maximal ideal of R.*

Proof. (1) We have $U = \cup_{P \in U} O_P$, since U is quasi-compact, it follows that $U = O_{P_1} \cup \dots \cup O_{P_r}$ with $P_1, \dots, P_r \in U$. But U is irreducible, which implies that $U = O_{P_i}$ form some $1 \leq i \leq r$. The converse is immediate from the fact that O_P is the closer of P in $\text{Spec}_c R$.
(2) Let $M_1, \dots, M_s$ be the maximal ideals of R. It is easy to see that $\text{Spec}R = O_{M_1} \cup \dots \cup O_{M_s}$. Let F be an irreducible component of $\text{Spec}_c R$, then $F = \cup_{i=1}^{s} O_{M_i} \cap F$, as F is irreducible we get $F = O_{M_i} \cap F$ for some i, that is $F \subseteq O_{M_i}$, by maximality it follows that $F = O_{M_i}$. Let $1 \leq j \leq s$, since O_{M_j} is an irreducible closed subset there is an irreducible component O_{M_i} such that $O_{M_j} \subseteq O_{M_i}$, in particular $M_j \in O_{M_i}$, so $M_j \subseteq M_i$, by maximality we get $M_j = M_i$. Thus $O_{M_j} = O_{M_i}$ is an irreducible component. □

Definition 4.5. A topological space is called sober if every irreducible closed subset has a unique generic point.

Remark 4.6. Let X be a sober space and let $x \neq y$ in X. Since $\overline{\{x\}} \neq \overline{\{y\}}$ we have $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$. So sober space is a T_0 space.

Definition 4.7. A topological space X is *spectral* if:

(i) X is quasi-compact,

(ii) X has a basis of sets which are quasi-compact and open,

(iii) The quasi-compact open sets of X are closed under finite intersections,

(iv) X is sober.

Proposition 4.8. *Let R be a commutative ring with Artinian spectrum.*

(i) $\mathrm{Spec}_c R$ *is a Noetherian space, in particular it is quasi-compact.*

(ii) $\mathrm{Spec}_c R$ *has a basis of sets which are quasi-compact and open.*

(iii) The quasi-compact open sets of $\mathrm{Spec}_c R$ *are closed under finite intersections.*

Proof. (1) An descending chain of closed subsets of $\mathrm{Spec}_c R$ is a descending chain of open subsets of $\mathrm{Spec}_z R$. The result follows from the fact that $\mathrm{Spec}_z R$ is an Artinian space.
(2) Since $\mathrm{Spec}_c R$ is Noetherian every open subset is quasi-compact.
(3) Finite intersection of a quasi-compact opens is open so quasi-compact. □

Next, for commutative R with Artinian spectrum, we characterize when $\mathrm{Spec}_c R$ is a spectral space. We star by an example which illustrate that fact that $\mathrm{Spec}_c R$ is not necessarily sober.

Example 4.9. Let V be a valuation ring with prime ideals $0 = P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_n \subsetneq P_{n+1} \ldots \subsetneq M$, where M is the maximal ideal and no prime ideal is contained strictly between P_i and P_{i+1}. For the construction of a such valuation ring see [9] (exercise 3.3.26). Then V has Artinian spectrum and $\mathrm{Spec}_c V$ is not sober.

Proof. Clearly, every antichain has one element, so finite. Let Q_m, $m \in \mathbb{N}$, be a descending chain of prime ideals. If for all m, $Q_m = M$ then the chain is stationary. So we can assume that there exists $m_0 \in \mathbb{N}$ such that $Q_{m_0} \neq M$. In this case $Q_{m_0} = P_N$ form some $N \in \mathbb{N}$. It is easy to see that $Q_m \in \{P_0, \ldots, P_N\}$ for all $m \geq m_0$. Thus the chain stabilize. Let $U = \{P_n \ / \ n \in \mathbb{N}\} = \mathrm{Spec}V \setminus \{M\}$. Then U is an irreducible closed subset of $\mathrm{Spec}_c V$. For every prime ideal P, $U \neq \overline{\{P\}}^c$, so U dose not have generic point. □

Remark 4.10. Let V as in the previous example. Then V has Artinian spectrum and $\mathrm{Spec}V$ is not a Noetherian spectrum.

Theorem 4.11. *Let R be a commutative ring with Artinian spectrum. The following statements are equivalent:*

(i) $\mathrm{Spec}_c R$ *is a spectral space*

(ii) R has Noetherian spectrum.

Proof. 1. $\Rightarrow$ 2. Assume that the sets of a non quasi-compact Zariski opens of $\mathrm{Spec}_z R$ is not empty, so it has a minimal element, say U' . Let U_1, U_2 be a Zariski opens such that $U' = U_1 \cup U_2$. Since U' is not quasi-compact, U_1 or U_2 is not quasi-compact. It follows by minimality that $U_1 = U'$ or $U_2 = U'$. Thus U' is an irreducible closed subset of $\mathrm{Spec}_c R$. Let P be a generic point of U', then $U' = O_P$, a contradiction with the fact that O_P is a quasi-compact Zariski open.
2. $\Rightarrow$ 1. Follows from Proposition 4.4. □

Theorem 4.12. *Let R be a commutative ring with Artinian spectrum. The following statements are equivalent.*

(i) $\mathrm{Spec}_c R$ *is a spectral space.*

(ii) R has finite Krull dimension.

Proof. 1. $\Rightarrow$ 2. For $n \in \mathbb{N}$, set $D_n = \{P \in \text{Spec}R \ / \text{ht}P \leq n\}$ and $D = \cup_{n\in\mathbb{N}} D_n$. For $n \in \mathbb{N}$ and $P \in D_n$, it is easy to see that $O_P \subseteq D_n$, so D_n is a Zariski open. Since $\text{Spec}_c R$ is spectral, D is a quasi-compact open. Thus $D = D_r$ for some $r \in \mathbb{N}$. Set $F = \text{Spec}R \setminus D$. We will show that $F = \emptyset$. So assume that F is not empty. Since R has Artinian spectrum, F has a minimal element, say, Q. Let $Q_0 \subsetneq \dots Q_{s-1} \subsetneq Q_s = Q$ be a chain of prime ideals. By minimality of Q, $Q_{s-1} \in D$. So $s - 1 \leq r$, that is $s \leq r + 1$. It follows that $\text{ht}Q \leq r + 1$, thus $Q \in D$, a contradiction. As a consequence $\text{Spec}R = D = D_r$, hence $\dim A \leq r$.
2. $\Rightarrow$ 1. If R has finite Krull dimension, then R has finitely many prime ideals, so it has Noetherian spectrum. Thus $\text{Spec}_c R$ is spectral. □

Remark 4.13. If R is a commutative ring with Artinian spectrum, by the Corollary 3.5 we have $\text{Spec}_c R$ is spectral if and only if $\text{Spec}R$ is finite.

In [8], Hochster showed that a space is spectral if and only if it is homeomorphicto the prime spectrum $\text{Spec}R$ of some commutative ring R with identity (endowed with the Zariski topology). According to this result, we close this section with the following corollary.

Corollary 4.14. *Let R be a commutative ring with Artinian spectrum. The following statements are equivalent.*

(i) $\text{Spec}_c R$ *is a spectral space.*

(ii) *There is a commutative ring R' with notherian spectrum such that* $\text{Spec}_c R$ *is homeomorphic to* $\text{Spec}_z R'$.

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