Palestine Journal of Mathematics

Vol. 12(2)(2023) , 208–216

Amalgamated ring defined by the (n, d)-perfect property

Karima Alaoui Ismaili

Communicated by John LaGrange

MSC 2010 Classifications: 16E05, 16E10, 16E30, 16E65

Keywords and phrases: (n, d)-perfect ring, S-ring, $A(n)$-ring, amalgamated algebra, amalgamated duplication.

The author would like to thank the editor-in-chief and anonymous referees for their valuable suggestions and helpful remarks.

Abstract. In [21], Jhilal and Mahdou defined a commutative unital ring R to be an (n, d)-perfect ring, if every n-presented module with flat dimension at most d, has projective dimension at most d, where n and d are nonnegative integers. In this paper, we examine the transfer of the (n, d)-perfect property to amalgamated rings in order to present illustrative examples of the commutative rings exhibiting this property.

1 Introduction

All rings considered below are assumed to be commutative with nonzero identity; all modules, ring homomorphisms, and inclusions of rings are assumed to be unital. We devote this opening paragraph to some definitions and a review of some standard reference documents. Let R be a ring and let M be an R-module. We use $pd_R(M)$ and $fd_R(M)$ to denote, respectively, the classical projective and flat dimensions of M. $gl.dim(R)$, is the classical global dimension of R and $w.gl.dimR$ the weak (or flat) global dimension of a ring R. The weak global dimension is the measure of the flatness of modules over R.

For a nonnegative integer n, an R-module M is called n-presented if there is an exact sequence of R-modules:

$$F_n \to F_{n-1} \to \dots F_1 \to F_0 \to M \to 0$$

where each F_i is a finitely generated free R-module. In particular, 0-presented and 1-presented R-modules are, respectively, finitely generated and finitely presented R-module. Consider the λ-dimension of M:

$$\lambda_R(M) = sup\ \{n \geq 0 : M\ is\ n - presented\ R - module\}$$

If M is not finitely generated we set $\lambda_R(M) = -1$.

In 1994, Costa [7] introduced a doubly filtered set of classes of rings in order to categorize the structure of non-Noetherian rings: for non-negative integers n and d, we say that a ring R is an (n, d)-ring if $pd_R(E) \leq d$ for each n-presented R-module E. An integral domain with this property will be called an (n, d)-domain.

For example, the $(n,0)$-domains are the fields, the $(0,1)$-domains are the Dedekind domains, and the $(1,1)$-domains are the Prüfer domains [7].

A ring R is perfect if every flat R-module is a projective R-module. The pioneering work on perfect rings were done by Bass [4] and most of the principal characterizations of perfect rings are contained in Theorem P from that paper.

In 2005, Enochs, Jenda, and López-Romos extended the notion of perfect rings to n-perfect rings, such that a ring is called n-perfect if every flat module has a projective dimension less or equal to n [17].

In 2009, Mahdou and Jhilal defined a commutative unital ring R to be an (n,d)-perfect ring, if every n-presented module with flat dimension at most d, has projective dimension at most d, where n and d are nonnegative integers (see [21]). For every $n > d$, R is an (n,d)-perfect ring and if R is an (n,d)-perfect ring, then R is an (n',d)-perfect ring for every $n' \geq n$. It is well known that if a flat R-module M is finitely presented, or finitely generated with R either a semilocal ring or an integral domain, then M is projective [16, Theorem 2]. Thus, if R is a domain or a semilocal ring, then R is an (n,n)-perfect ring for every $n \geq 0$ (see [21]). (n,d)-rings, perfect rings, and rings with global dimension at most d are (n,d)-perfect. If R is a Noetherian ring, then R is an (n,d)-perfect ring for every $n \geq 0$ and $d \geq 0$, and if R is a coherent ring, then R is an (n,d)-perfect ring for every $n \geq 1$ and $d \geq 0$ [21, Proposition 2.3].

In 2010, Jhilal and Mahdou defined a commutative unital ring R to be strongly n-perfect if any R-module of flat dimension less or equal than n has a projective dimension less or equal to n [20]. It is trivial to remark that every strongly n-perfect ring is an n-perfect ring and note that if $n = 0$ then the strongly 0-perfect rings are the perfect rings. Strongly d-perfect rings are (n,d)-perfect.

A ring is called an S-ring if every finitely generated flat R-module is projective (see [26]). The notion of (n,d)-perfect rings is in some way a generalization of the notion of S-rings. Then R is an S-ring if and only if R is an $(0,0)$-perfect ring.

Let n be a nonnegative integer. A ring R is said to be an $A(n)-$ ring if given any exact sequence $0 \to M \to E_1 \to \ldots \to E_n$ of finitely generated R-modules with M flat and E_i free for each i, then M is projective (see [8, page 139]). A ring R is an $A(n)$- ring if and only if R is an (n,n)-perfect ring [21, Theorem 3.2]. Noetherian rings are $A(n)$-rings for every $n \geq 0$ and coherent rings are $A(n)$- rings for every $n \geq 1$.

Let A be a ring, E be an A-module, and $R := A \propto E$ be the set of pairs (a,e) with pairwise addition and multiplication given by $(a,e)(b,f) = (ab, af + be)$. R is called the trivial ring extension of A by E (also called the idealization of E over A). Considerable work, part of it summarized in Glaz [18] and Huckaba [19], has been concerned with trivial ring extensions. These have proven to be useful in solving many open problems and conjectures for various contexts in (commutative and noncommutative) ring theory. See for instance [18, 19, 24, 23, 26].

In 2006, M. D'Anna and M. Fontana [12] introduced a new construction, called amalgamated duplication of a ring A along an $A-$submodule E of $Q(A)$ (the total ring of fractions of A) such that $E^2 \subseteq E$. When $E^2 = \{0\}$, this construction coincides with the trivial ring extension of A by E. Motivations and more applications of the amalgamated duplication $A \bowtie E$ of A along an $A-$submodule E of $Q(A)$ are discussed in more details, especially in the particular case where E is an ideal of A, in recent papers, for instance, see [13, 10, 9, 11, 12].

In 2010, D'Anna, Finocchiaro, and Fontana [10] extended the notion of amalgamated duplication construction $A \bowtie I$ of a ring A along an ideal I of A to the general context of ring homomorphism extensions as follows: Let A and B be two rings with identity elements, J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we consider the following subring of $A \times B$; $A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$ called the amalgamation of A and B along J with respect to f. Moreover, other classical constructions (such as the $A + XB[X], A + XB[[X]]$, and the $D + M$ constructions) can be studied as particular cases of the amalgamation [10, Examples 2.5 & 2.6] and other classical constructions such as Nagata's idealization and the CPI extensions (in the sense of Boisen and Sheldon [5]) strictly related to it (see [10, Example 2.7 & Remark 2.8]).

In this paper, we examine the transfer of the (n, d)-perfect property to amalgamated rings in order to present illustrative examples of the commutative rings exhibiting this property.

2 Transfer of the (n, d)-perfect property

The main result (Theorem 2.1) examines the property of (n, d)-perfect that the amalgamation algebra $A \bowtie^f J$ might inherit from the ring A for some classes of ideals J and homomorphisms f, and hence generates new families of $(0, 1)$-perfect rings and $(1, 0)$-perfect rings which are not $(0, 0)$-perfect rings, and new examples of $(1, 1)$-perfect domains which are not $(0, 1)$-perfect domains.

Theorem 2.1. *Let $f : A \to B$ be a ring homomorphism and J be a proper ideal of B. Then:*

(i) If $A \bowtie^f J$ is an (n, d)-perfect ring, then A is an (n, d)-perfect ring.

(ii) Assume that $f^{-1}(J)$ and J are pure ideals of A and $f(A) + J$ respectively and $\lambda_A(f^{-1}(J)) \geq n - 1$. If $A \bowtie^f J$ is an (n, d)-perfect ring, then $f(A) + J$ is an (n, d)-perfect ring.

(iii) a. If A and $f(A) + J$ are S-rings, then so is $A \bowtie^f J$.

b. Assume that $f^{-1}(J)$ and J are pure ideals of A and $f(A) + J$ respectively. Then:

i. If A and $f(A)+J$ are (n, d)-perfect rings, then $A \bowtie^f J$ is an (n, d)-perfect ring (in particular, If A and $f(A) + J$ are $A(n)$-rings, then so is $A \bowtie^f J$).

ii. Assume that $\lambda_A(f^{-1}(J)) \geq n - 1$. Then $A \bowtie^f J$ is an (n, d)-perfect ring if and only if A and $f(A) + J$ are (n, d)-perfect rings (in particular, $A \bowtie^f J$ is an $A(n)$-ring if and only if A and $f(A)+J$ are $A(n)$-rings).

The proof of Theorem 2.1 draws on the following results.

Lemma 2.2. *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Assume that $f^{-1}(J)$ and J are pure ideals of A and $f(A) + J$ respectively. Let M be an $(A \bowtie^f J)$-module. Then:*

(i) $\lambda_{A\bowtie^f J}(M) \geq n$ if and only if $\lambda_A(M \otimes_{A\bowtie^f J} A) \geq n$ and $\lambda_{(f(A)+J)}(M \otimes_{A\bowtie^f J} (f(A) + J)) \geq n$.

(ii) $fd_{A\bowtie^f J}(M) \leq n$ if and only if $fd_A(M \otimes_{A\bowtie^f J} A) \leq n$ and $fd_{(f(A)+J)}(M \otimes_{A\bowtie^f J} (f(A) + J)) \leq n$.

(iii) $pd_{A\bowtie^f J}(M)\leq n$ *if and only if* $pd_A(M\otimes_{A\bowtie^f J}A)\leq n$ *and* $pd_{(f(A)+J)}(M\otimes_{A\bowtie^f J}(f(A)+J))\leq n$.

Proof. The result follows from [21, Lemma 2.12 & Lemma 2.13 & Lemma 4.3] since $\phi: A\bowtie^f J\hookrightarrow A\times(f(A)+J)$ is an injective flat ring homomorphism, and $\{0\}\times J$ is a pure ideal of $A\bowtie^f J$ by [1, Lemma 2.2]. □

Lemma 2.3. *Let $f: A\to B$ be a ring homomorphism and J be an ideal of B. Assume that $f^{-1}(J)$ and J are pure ideals of A and $f(A)+J$ respectively. Then:*

(i) *(a)* $\lambda_{A\bowtie^f J}(\{0\}\times J)\geq n$ *if and only if* $\lambda_{(f(A)+J)}(J)\geq n$.
(b) $pd_{A\bowtie^f J}(\{0\}\times J)\leq n$ *if and only if* $pd_{(f(A)+J)}(J)\leq n$.
(c) $fd_{A\bowtie^f J}(\{0\}\times J)\leq n$ *if and only if* $fd_{(f(A)+J)}(J)\leq n$.

(ii) *(a)* $\lambda_{A\bowtie^f J}(f^{-1}(J)\times\{0\})\geq n$ *if and only if* $\lambda_A(f^{-1}(J))\geq n$.
(b) $pd_{A\bowtie^f J}(f^{-1}(J)\times\{0\})\leq n$ *if and only if* $pd_A(f^{-1}(J))\leq n$.
(c) $fd_{A\bowtie^f J}(f^{-1}(J)\times\{0\})\leq n$ *if and only if* $fd_A(f^{-1}(J))\leq n$.

Proof. By [1, Lemma 2.2], A and $f(A)+J$ are flat $(A\bowtie^f J)$-modules. So,

(i) $(\{0\}\times J)\otimes_{A\bowtie^f J}A\cong(\{0\}\times J)A\cong 0$ and $(\{0\}\times J)\otimes_{A\bowtie^f J}(f(A)+J)\cong(\{0\}\times J)(f(A)+J)\cong J$, so the result is deduced by Lemma 2.2.

(ii) $(f^{-1}(J)\times\{0\})\otimes_{A\bowtie^f J}(f(A)+J)\cong(f^{-1}(J)\times\{0\})(f(A)+J)\cong 0$ and $(f^{-1}(J)\times\{0\})\otimes_{A\bowtie^f J}A\cong(f^{-1}(J)\times\{0\})A\cong f^{-1}(J)$, so the result is deduced by Lemma 2.2.

□

Proof of Theorem 2.1

(i) Assume that $A\bowtie^f J$ is an (n,d)-perfect ring. Since $A\bowtie^f J$ is a $\frac{A\bowtie^f J}{\{0\}\times J}$ $(\cong A)$-flat module and A is a module retract of $A\bowtie^f J$, then A is an (n,d)-perfect ring by [22, Theorem 2.1].

(ii) Assume that $A\bowtie^f J$ is an (n,d)-perfect ring, $f^{-1}(J)$ and J are pure ideals of A and $f(A)+J$ respectively, and $\lambda_A(f^{-1}(J))\geq n-1$. Then by [1, Lemma 2.2], $f^{-1}(J)\times\{0\}$ is a pure ideal of $A\bowtie^f J$ and by Lemma 2.3, $\lambda_{A\bowtie^f J}(f^{-1}(J)\times\{0\})\geq n-1$. Therefore, $f(A)+J\cong\frac{A\bowtie^f J}{f^{-1}(J)\times\{0\}}$ is an (n,d)-perfect ring by [21, Lemma 4.2].

$(iii)(a)$ The result follows immediately from [21, Corollary 3.3] and [21, Theorem 2.11] since $\phi: A\bowtie^f J\hookrightarrow A\times(f(A)+J)$ is an injective ring homomorphism.

(b) Assume that $f^{-1}(J)$ and J are pure ideals of A and $f(A)+J$ respectively.

$(b)(i)$ Assume that A and $f(A)+J$ are (n,d)-perfect rings, and let M be an $(A\bowtie^f J)$-module such that $\lambda_{A\bowtie^f J}(M)\geq n$ and $fd_{A\bowtie^f J}(M)\leq d$. Then by Lemma 2.2, $\lambda_A(M\otimes_{A\bowtie^f J}A)\geq n$ and $\lambda_{(f(A)+J)}(M\otimes_{A\bowtie^f J}(f(A)+J))\geq n$ also, $fd_A(M\otimes_{A\bowtie^f J}A)\leq d$ and $fd_{(f(A)+J)}(M\otimes_{A\bowtie^f J}(f(A)+J))\leq d$. Thus, $pd_A(M\otimes_{A\bowtie^f J}A)\leq d$ since A is an (n,d)-perfect ring and $pd_{(f(A)+J)}(M\otimes_{A\bowtie^f J}(f(A)+J))\leq d$ since $f(A)+J$ is an (n,d)-perfect ring. Therefore $pd_{A\bowtie^f J}(M)\leq d$ by Lemma 2.2. So, $A\bowtie^f J$ is an (n,d)-perfect ring.

$(b)(ii)$ Assume that A and $f(A)+J$ are (n,d)-perfect rings. Then $A\bowtie^f J$ is an (n,d)-perfect ring by $(iii)(b)(i)$. Conversely, assume that $A\bowtie^f J$ is an (n,d)-perfect ring. Then A is an (n,d)-perfect ring by (i), and $f(A)+J$ is an (n,d)-perfect ring by (ii). □

The Corollary below follows immediately from Theorem 2.1 which examines the case of the amalgamated duplication.

Corollary 2.4. *Let A be a ring and I be an ideal of A.*

(i) If $A \bowtie I$ is an (n,d)-perfect ring then so is A.

(ii) Assume that I is a pure ideal of A. Then $A \bowtie I$ is an (n,d)-perfect ring if and only if A is an (n,d)-perfect ring.

This result allows us to generate a new class of $(0,1)$-perfect rings and $(1,0)$-perfect rings which are not $(0,0)$-perfect rings.

Example 2.5. Let A be a hereditary Von Neumann regular ring that is a non-semisimple ring (for example, we can consider [7, Example 2.7]). Then $A \bowtie I$ is a $(0,1)$-perfect ring and $(1,0)$-perfect ring which is not a $(0,0)$-perfect ring for all ideals I of A.

Proof. Let I be an ideal of A. Then I is a pure ideal of A since A is a Von Neumann regular ring. So by Corollary 2.4, $A \bowtie I$ is a $(0,1)$-perfect and a $(1,0)$-perfect ring since A is a $(0,1)$-perfect and a $(1,0)$-perfect ring. But $A \bowtie I$ is not a $(0,0)$-perfect ring. Otherwise, every finitely generated $(A \bowtie I)$-module is projective since $A \bowtie I$ is a Von Neumann regular ring by [6, Theorem 2.1] which implies that $A \bowtie I$ is a semisimple ring. This is absurd since $A \bowtie I$ is a non-semisimple by [6, Corollary 2.3]. □

The following result generates a new class of $(1,1)$-perfect domains which are not $(0,1)$-perfect domains.

Example 2.6. Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Assume that $f(A)+J$ is a Prüfer domain and $f^{-1}(J)=\{0\}$ and A or $f(A)+J$ is not a Noetherian ring. Then $A \bowtie^f J$ is a $(1,1)$-perfect domain and $(1,0)$-perfect domain which is not a $(0,1)$-perfect domain.

Proof. $A \bowtie^f J$ is an (n,n)-perfect domain for every $n \geq 0$. So it is in particular an $(1,0)$-perfect and $(1,1)$-perfect domain. We show that $A \bowtie^f J$ is not a $(0,1)$-perfect domain. By [10, Proposition 5.6], $A \bowtie^f J$ is not a Noetherian ring. Let K be a not finitely generated ideal of $A \bowtie^f J$, then K is not a projective ideal since $A \bowtie^f J$ is a domain. Since $A \bowtie^f J$ is a Prüfer domain, then $w.g.dim(A \bowtie^f J) \leq 1$, so K is a flat ideal of $(A \bowtie^f J)$. Therefore $\frac{A \bowtie^f J}{K}$ is a 0-presented $(A \bowtie^f J)$-module and $fd_{(A \bowtie^f J)}(\frac{A \bowtie^f J}{K}) \leq 1$ but $pd_{(A \bowtie^f J)}(\frac{A \bowtie^f J}{K}) \geq 2$, as desired. □

The following Propositions generate a new class of non-Noetherian (n,d)-perfect rings and a new class of non-semisimple (n,d)-perfect rings.

Proposition 2.7. *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B such that J and $f^{-1}(J)$ are finitely generated ideals of $f(A)+J$ and A respectively. Assume that A and $f(A)+J$ are coherent rings and A or $f(A)+J$ is a not a Noetherian ring. Then is a non-Noetherian (n,d)-perfect ring for every $n \geq 1$ and $d \geq 0$.*

Proof. The result follows immediately from [3, Theorem 2.2], [21, Proposition 2.3], and [10, Proposition 5.6]. □

Example 2.8. $(\mathbb{Z}+X\mathbb{Q}[X]) \bowtie X\mathbb{Q}[X]$ is a non-Noetherian (n,d)-perfect ring for every $n \geq 1$ and $d \geq 0$, where $\mathbb{Z}$ is the ring of integers, and $\mathbb{Q}$ is the field of rational numbers.

Proof. The result follows immediately from Proposition 2.7 since $\mathbb{Z}+X\mathbb{Q}[X]$ is a non-Noetherian coherent ring by [18, Corollary 5.2.5 & Corollary 5.2.9]. □

Example 2.9. Let (A, M) be any local non-Noetherian ring (for example $K[(X_n)_{n\in\mathbb{N}^\star}]$), and let I be an ideal of A. Then $R := A \bowtie I$ is a non-Notherian (n,d)-perfect ring for every $n \geq d$. (In particular, R is an $A(n)$-ring for every integer n.)

Proposition 2.10. *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Assume that A is a Noetherian ring, A or $f(A)+J$ is a non-semisimple, and at least one of the following conditions holds:*

(i) $f(A)+J$ is a Noetherian ring.

(ii) J is a finitely generated A-module (with the structure naturally induced by f).

(iii) J is a Noetherian A-module (with the structure naturally induced by f).

(iv) f is a finite homomorphism.

Then $A \bowtie^f J$ is a non-semisimple (n,d)-perfect ring for every $n \geq 0$ and $d \geq 0$.

Proof. Let $n \geq 0$ and $d \geq 0$, by [10, Proposition 5.6 & 5.7] and [21, Proposition 2.3], $A \bowtie^f J$ is an (n,d)-perfect ring. Now, assume that $A \bowtie^f J$ is a semisimple ring then so is $f(A)+J \cong \frac{A\bowtie^f J}{f^{-1}(J)\times\{0\}}$, which is absurd. □

Example 2.11. Let $f : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ be the canonical surjection, where $\mathbb{Z}$ is the ring of integers and $n > 1$. Then $\mathbb{Z} \bowtie^f (k\mathbb{Z}/n\mathbb{Z})$ is a non-semisimple (n,d)-perfect ring for every $n,d \geq 0$ and k/n.

Now, we present a class of (n,d)-perfect rings which are non-coherent.

Example 2.12. Let A be a domain that is not a field, $K = qf(A)$, and let $R := (A \propto K) \bowtie (0 \propto A)$. Then R is a non-coherent (n,d)-perfect ring for every $n > d$.

Proof. The result follows immediately from [23, Theorem 2.8] and [3, Corollary 2.8]. □

Now, we exhibit classes of (n,d)-perfect rings that are not (n,d)-rings, and classes of (n,d)-perfect rings that are not d-perfect (so, not strongly d-perfect rings).

Proposition 2.13. *Let A be a ring such that $gl.dim(A) > d$, where d is a positive integer. Then:*

(i) A is an (n,d)-perfect ring for any integer $n > d$ that is not a $(0,d)$-ring.

(ii) Assume that A is a Noetherian ring. Then A is an (n,d)-perfect ring that is not an (n,d)-ring for any integer $n \geq 0$. In particular, A is an $A(d)$-ring that is not a (d,d)-ring.

Proof. (i) The result follows immediately from [7, Theorm 1.3].

(ii) The result follows immediately from [7, Theorm 1.3] and [21, Proposition 2.3].

□

Example 2.14. Let $\mathbb{Z}$ be the ring of integers. Then $\mathbb{Z}[X_1, \dots, X_m]$ is an (n, d)-perfect ring that is not an (n, d)-ring for any integers $n \geq 0$ and $m \geq d$. In particular, A is an $A(m)$-ring that is not an (m, m)-ring.

Example 2.15. (i) Let R be a non-Prüfer domain. Then R is an $(n, 1)$-perfect domain for any integers $n \geq 1$ that is not an $(1, 1)$-domain.

(ii) Let R be a domain that is not a field. Then R is an (n, d)-perfect domain for any integers $n \geq d$ that is not an $(n, 0)$-domain for any integers n.

Proof. The result follows immediately from [7, Theorm 1.3]. □

Example 2.16. Let A be a Von Neumann regular ring which is a non-semisimple (e.g., the infinite direct product of fields), I be an ideal of A, and let $P_A : A \bowtie I \to A$ be a ring epimorphism. Consider $R := (A \bowtie I) \bowtie^{P_A} (I \times 0)$. Then:

(i) R is a Von Neumann regular ring that is a non-semisimple.

(ii) R is an (n, d)-perfect ring which is not perfect for any integer $n > d$. In particular, R is an $(n, 0)$-perfect ring which is not a 0-perfect for any integer $n > 0$.

(iii) R is $(1, 0)$-perfect ring that is not a $(0, 0)$-perfect ring.

Proof. (i) The result follows immediately from [6, Theorem 2.1 & Corollary 2.3].

(ii) Assume that R is a perfect ring. Then every R-module is projective since R is a Von Neumann regular ring according to (1), this would contradict the fact that R is a non-semisimple ring.

(iii) Assume that R is a $(0, 0)$-perfect ring. Then every finitely generated R-module is projective since R is a Von Neumann regular ring, which contradicts the fact that R is a non-semisimple ring.

□

Example 2.17. Let R be a non-Noetherian Prüfer domain. Then R is an $(0, 0)$-perfect domain which is not a 0-perfect domain (i.e., not a perfect domain).

Proof. Since R is a domain then, R is $(0, 0)$-perfect ring, and since R is a non-Noetherian ring, then, there exists I; a not finitely generated ideal of R which is not projective since R is a domain. However, I is a flat ideal of R. Therefore R is not a 0-perfect domain. □

References

[1] K. Alaoui Ismaili, N. Mahdou and M. A. Moutui, *Commutative ring extensions defined by perfect-like conditions*, Ukrainian Mathematical Journal, DOI: https://doi.org/10.37863/umzh.v75i3.6878.

[2] K. Alaoui Ismaili and N. Mahdou, *On* (n, d)*- property in amalgamated algebra*, Asian-Eur. J. Math, 6 (1) (2016), 1650014 (13 pages), World Scientific Publishing Company.

[3] K. Alaoui Ismaili and N. Mahdou, *Coherence in amalgamated algebra along an ideal*, Bull. Iran. Math. Soc, 41 (3) (2015), 1–9.

[4] H. Bass, *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc, 95 (1960), 466–488.

[5] M. B. Boisen and P. B. Sheldon, *CPI-extension: Over rings of integral domains with special prime spectrum*, Canad. J. Math, 29 (1977), 722–737.

[6] M. Chhiti and N. Mahdou, *Some homological properties of amalgamated duplication of a ring along an ideal*, Bull. Iran. Math. Soc, 38 (2) (2012), 507–515.

[7] D. Costa, *Parameterizing families of non-Noetherian rings*, Commun. Algebra, 22 (10) (1994), 3997–4011.

[8] S. H. Cox and R. L. Pendleton, *Rings for which certain flat modules are projective*, Trans. Amer. Math. Soc., 150 (1) (1970), 139–156.

[9] M. D'Anna, C. A. Finocchiaro and M. Fontana, *Properties of chains of prime ideals in amalgamated algebras along an ideal*, J. Pure Appl. Algebra, 214 (9) (2010), 1633–1641.

[10] M. D'Anna, C. A. Finocchiaro and M. Fontana, *Amalgamated algebras along an ideal*, Commutative Algebra and Applications, Proceedings of the Fifth International Fez Conference on Commutative Algebra and Applications, Fez, Morocco, 2008, W. de Gruyter Publisher Berlin (2009), 155–172.

[11] M. D'Anna and M. Fontana, *Amalgamated duplication of a ring along a multiplicative-canonical ideal*, Ark. Mat, 45 (2) (2007), 241–252.

[12] M. D'Anna and M. Fontana, *An amalgamated duplication of a ring along an ideal: the basic properties*, J. Algebra Appl, 6 (3) (2007), 443–459.

[13] M. D'Anna, *A construction of Gorenstein rings*, J. Algebra, 306 (2) (2006), 507–519.

[14] D. E. Dobbs, S. Kabbaj, N. Mahdou and M. Sobrani, *When is $D + M$ n-coherent and an (n, d)-domain?*, Lecture Notes in Pure and Appl. Math. Dekker, 205 (1999), 257–270.

[15] A. El Khalfi, H. Kim and N. Mahdou, *Amalgamation extension in commutative ring theory*, a surey, Moroccan Journal of Algebra and Geometry with Applications, 1 (1) (2022), 139–182.

[16] S. Endo, *On flat modules over commutative rings*, J. Math. Soc. Japan, 14 (1962), 285—291.

[17] E. Enochs and O. M. G. Jenda, *Relative Homological Algebra*, Walter de Gruyter, Berlin, New York, 30 (2000).

[18] S. Glaz, *Commutative coherent rings*, Lecture Notes in Mathematics (1371), Springer-Verlag, Berlin (1989).

[19] J. A. Huckaba, *Commutative rings with zero divisors*, Marcel Dekker, New York-Basel (1988).

[20] A. Jhilal and N. Mahdou, *On strong n-perfect rings*, Commun. Algebra, 38 (3) (2010), 1057–1065.

[21] A. Jhilal and N. Mahdou, *On (n, d)-perfect rings*, Afr. Diaspora J. Math., 9 (1) (2010), 1–7.

[22] A. Jhilal and N. Mahdou, *On strong n-perfect and (n, d)-perfect rings*, Afr. Diaspora J. Math., 9 (1) (2010), 1–7.

[23] S. Kabbaj and N. Mahdou, *Trivial extensions defined by coherent-like conditions*, Commun. Algebra, 32 (10) (2004), 3937–3953.

[24] S. Kabbaj and N. Mahdou, *Trivial extensions of local rings and a conjecture of Costa*, Lecture Notes in Pure and Appl. Math., Dekker, 231 (2003), 301–311.

[25] N. Mahdou, *On costa's conjecture*, Commun. Algebra, 29 (7) (2001), 2775–2785.

[26] G. Puninski and P. Rothmaler, *When every finitely generated flat module is projective*, J. Algebra, 277 (2) (2004), 542-–558.

Author information

Karima Alaoui Ismaili, Laboratory of Mathematics, Computing and Applications-Information Security (LabMia-SI), Department of Mathematics, Faculty of Sciences, B.P. 1014, Mohammed V University in Rabat., Morocco.
E-mail: a.ismaili@um5r.ac.ma

Received: 2022-10-13.
Accepted: 2023-03-15.