Palestine Journal of Mathematics

Vol. 12(2)(2023) , 217–224 © Palestine Polytechnic University-PPU 2023

ON r-CLEAN IDEALS

Dian Ariesta Yuwaningsih, Indah Emilia Wijayanti and Budi Surodjo

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 16D25; Secondary 16D70, 16E50, 17C27.

Keywords and phrases: clean ideal, r-clean ring, r-clean ideal, regular element, idempotent element.

The first author would like to thank the Center for Education Financial Services and the Indonesia Endowment Funds for Education for their doctoral scholarships. In addition, all authors are grateful to Universitas Gadjah Mada for the 2022 "Rekognisi Tugas Akhir" research grant and the reviewers' valuable suggestions and comments to improve this paper.

Abstract Let R be an associative ring with identity. Ring R is considered clean if each element is the sum of an idempotent element and a unit element. A clean ring is generalized to an r-clean ring by generalizing the unit elements to regular. A ring R is said to be r-clean if each element is the sum of an idempotent element and a regular element. We introduce the notion of the r-clean ideal, a natural generalization of the clean ideal. Furthermore, we present some properties of the r-clean ideal and offer several sufficient and necessary conditions for a clean ideal to be r-clean.

1 Introduction

Let R be an associative ring with identity. According to [1], the element $a \in R$ is said to be clean if it is the sum of an idempotent element and a unit element. Then, the ring R is said to be clean if each element is clean. Referring to [2], an element $r \in R$ is considered regular if there is an element $s \in R$ such that it satisfies $r = rsr$. We know that each unit element in R is regular. By generalizing the unit element to a regular element, [3] generalizes the definition of a clean element to an r-clean element. An element $a \in R$ is said to be r-clean if it is the sum of an idempotent element and a regular element of R. Therefore, a ring R is called an r-clean ring if each element is r-clean. Thus, the r-clean ring is the generalization of the clean ring.

On the other hand, the set of all r-clean elements does not necessarily form an ideal. For example, if given a ring $\mathbb{Z}$ then the set of all r-clean elements in $\mathbb{Z}$ is $K = \{-1, 0, 1, 2\}$. Notice that K is not ideal in $\mathbb{Z}$. This phenomenon is the background to the emergence of the definition of the r-clean ideal. This article defines the r-clean ideal, a generalization of the clean ideal introduced by [4]. In addition, some properties of the r-clean ideal are presented. At the end of this article, we present several sufficient and necessary conditions for a clean ideal to form an r-clean ideal.

Throughout this article, R is an associative ring with identity unless stated otherwise. Furthermore, $Id(R)$ is the set of all idempotent elements in R, $Reg(R)$ is the set of all regular elements in R, $U(R)$ is the set of all unit elements in R, and Λ is the index set.

2 Definition of r-Clean Ideals

We begin with the definition of r-clean ideal as follows.

Definition 2.1. An ideal I of R is called an r-clean ideal if every element of I is a sum of an idempotent and a regular element of R.

Every ideal of r-clean rings is clean. However, non r-clean rings contain some r-clean ideals.

Example 2.2. Let R_1 be an R-clean ring and R_2, not an r-clean ring. Set $R = R_1 \times R_2$. Then, R is not an r-clean ring. Set an ideal $P = R_1 \times \{0_{R_2}\}$ of R. We can show that P is an r-clean ideal of R. Given any $(x, 0_{R_2}) \in P$. Then, $x_1 \in R_1$. Since R_1 is an r-clean ring, we have an idempotent element $e \in R_1$ and a regular element $a \in R_1$ such that $x = e + a$. Hence, $(x, 0_{R_2}) = (e + a, 0_{R_2}) = (e, 0_{R_2}) + (a, 0_{R_2})$. Clearly, $(e, 0_{R_2})$ is an idempotent element of R. Moreover, since $a \in Reg(R_1)$, there exist $d \in R$ such that $a = ada$. So, we obtain $(a, 0_{R_2}) = (ada, 0_{R_2}) = (a, 0_{R_2})(d, 0_{R_2})(a, 0_{R_2})$, for an $(d, 0_{R_2}) \in R$. So, $(a, 0_{R_2}) \in Reg(R)$. Thus, $(x, 0_{R_2})$ is an r-clean element of R. Therefore, P is an r-clean ideal of R.

Example 2.3. Let $\{I_i\}_{i \in \Lambda}$ be the family of ideals of R and R an r-clean ring. Then,

(i) $\bigcap_{i \in \Lambda} I_i$ is an r-clean ideal of R.

(ii) $\sum_{i \in \Lambda} I_i$ is an r-clean ideal of R.

Example 2.4. Let R and S be rings, and $f : R \to S$ is the ring homomorphism. If R is an r-clean ring, then

(i) $Ker(f)$ is an r-clean ideal of R.

(ii) $f^{-1}(S)$ is an r-clean ideal of R.

3 Some Properties of r-Clean Ideals

In this section, we present some properties of r-clean ideals. The first property shows that the infinite intersection of the r-clean ideals is an r-clean ideal.

Proposition 3.1. *Let $\{I_i\}_{i \in \Lambda}$ be the family of ideals of R and I_i an r-clean ideal for each $i \in \Lambda$. Then, $\bigcap_{i \in \Lambda} I_i$ is an r-clean ideal of R.*

Proof. Since $\{I_i\}_{i \in \Lambda}$ is a family of ideals of R, it is clear that $\bigcap_{i \in \Lambda} I_i$ is an ideal of R. Let $a \in \bigcap_{i \in \Lambda} I_i$. Then, $a \in I_i$ for each $i \in \Lambda$. Since I_i is r-clean for each $i \in \Lambda$, we have $a = e + r$ with $e \in Id(R)$ and $r \in Reg(R)$. Hence, a is an r-clean element of $\bigcap_{i \in \Lambda} I_i$. Thus, $\bigcap_{i \in \Lambda} I_i$ is an r-clean ideal of R. □

Next, we show that the homomorphic image of the r-clean ideal is also r-clean.

Proposition 3.2. *Let R and S be rings, $f : R \to S$ is the ring epimorphism, and P an r-clean ideal of R. Then, $f(P)$ is an r-clean ideal of S.*

Proof. Since P is an ideal of R and f is a ring epimorphism, $f(P)$ is an ideal of S. Let $y \in f(P)$. There exists $x \in P$ such that $y = f(x)$. As P is an r-clean ideal, we have $x = e + r$ with $e \in Id(R)$ and $r \in Reg(R)$. Then,

$$y = f(x) = f(e + r) = f(e) + f(r).$$

It is clear that $f(e) \in Id(S)$. Since $r \in Reg(R)$, $r = rtr$ for an $t \in R$. So, we obtain $f(r) = f(rtr) = f(r)f(t)f(r)$, for an $f(t) \in S$. Thus, we get $f(r) \in Reg(S)$. Which implies that y is an r-clean element of $f(P)$, so $f(P)$ is an r-clean ideal of S. □

Referring to Proposition 3.2, the following presents the sufficient condition for an ideal factor to be r-clean.

Proposition 3.3. *Let N be an ideal of R, P an ideal of R containing N, and $f : R \to R/N$ is the ring epimorphism. If P is an r-clean ideal of R, then P/N is an r-clean ideal of R/N.*

Proof. Assume that $f : R \to R/N$ is a ring epimorphism, and P is an r-clean ideal of R. Referring to Proposition 3.2, we have $f(P)$ is an r-clean ideal of R/N. Since f is a ring epimorphism, we obtain $f(P) = P/N$. Thus, P/N is an r-clean ideal of R/N. □

The converse of Proposition 3.3 is not necessarily true, as the following example shows.

Example 3.4. Let $f : \mathbb{Z} \to \mathbb{Z}_6$ be a ring epimorphism with $f(a) = a \ mod\ 6$, for every $a \in \mathbb{Z}$. Let $H = \{\bar{0}, \bar{2}, \bar{4}\}$ is an r-clean ideal of $\mathbb{Z}_6$. Clearly, $f(2\mathbb{Z}) = H$. But, $2\mathbb{Z}$ is not an r-clean ideal of $\mathbb{Z}$.

According to [5], in the following, we give the sufficient condition for an element of ring R to form a regular element.

Lemma 3.5. *Let $a \in R$, and $y \in R$ satisfy $a - aya$ is a regular element of R. Then, a is a regular element of R.*

Furthermore, [5] defines an ideal I of R as a regular ideal if every element of it is a regular element of R.

Let I be an ideal ring R. Then, according to [6], we say that idempotents lift modulo I if for each $x \in R$ such that $x - x^2 \in I$, there exists an idempotent $e \in R$ such that $e - x \in I$. Moreover, the converse of Proposition 3.3 will hold if we give the following sufficient conditions.

Proposition 3.6. *Let P and I be ideals of R satisfies $I \subseteq P$, I a regular ideal of R, and suppose that idempotents can be lifted modulo I. If P/I is an r-clean ideal of R/I, then P is an r-clean ideal of R.*

Proof. Assume that P/I is an r-clean ideal of R/I. So, for every $a \in P$ we have $a + I$ is an r-clean element of R/I. There exist $e + I \in Id(R/I)$ such that $(a - e) + I \in Reg(R/I)$. It means that there is $x \in R$ such that it satisfies $((a - e) + I)(x + I)((a - e) + I) = (a - e) + I$. From here, we get $(a - e) - (a - e)x(a - e) \in I$. Since I is a regular ideal, then according to Lemma 3.5, we get $a - e \in Reg(R)$. Since idempotents can be lifted modulo I, we may assume that e is an idempotent of R. Thus, a is an r-clean element of R, which implies P is an r-clean ideal of R. □

Next, we give a sufficient and necessary condition for the infinite direct product of ideals to form an r-clean ideal.

Proposition 3.7. *Let R_i be a ring, and P_i an ideal of R_i, for each $i \in \Lambda$. The ideal $\prod_{i\in\Lambda} P_i$ is an r-clean ideal of $\prod_{i\in\Lambda} R_i$ if and only if P_i is an r-clean ideal of R_i for each $i \in \Lambda$.*

Proof. Let any $i \in \Lambda$ and assume that

$$\begin{aligned} f : \prod_{i\in\Lambda} R_i &\to R_i \\ (r_i)_{i\in\Lambda} &\mapsto r_i, \end{aligned}$$

for each $(r_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} R_i$, is a ring epimorphism. Since $\prod_{i\in\Lambda} P_i$ is an r-clean ideal of $\prod_{i\in\Lambda} R_i$, referring to Proposition 3.2 we have $f(\prod_{i\in\Lambda} P_i) = P_i$ is an r-clean ideal of R_i. Thus, P_i is an r-clean ideal of R_i, for each $i \in \Lambda$. Conversely, let $(x_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} P_i$. Since P_i is an r-clean ideal of R_i for each $i \in \Lambda$, we obtain $x_i = e_i + r_i$ with $e_i \in Id(R_i)$ and $r_i \in Reg(R_i)$, for every $i \in \Lambda$. As a result, we obtain

$$(x_i)_{i\in\Lambda} = (e_i + r_i)_{i\in\Lambda} = (e_i)_{i\in\Lambda} + (r_i)_{i\in\Lambda}.$$

Since $e_i \in Id(R_i)$ for each $i \in \Lambda$, we have $e_i e_i = e_i$. So,

$$(e_i e_i)_{i\in\Lambda} = (e_i)_{i\in\Lambda}(e_i)_{i\in\Lambda} = (e_i)_{i\in\Lambda}.$$

Thus, $(e_i)_{i\in\Lambda} \in Id(\prod_{i\in\Lambda} R_i)$. Moreover, since $r_i \in Reg(R_i)$ for each $i \in \Lambda$, we get $r_i = r_i s_i r_i$ for an $s_i \in R_i$, for every $i \in \Lambda$. Hence, we get

$$(r_i)_{i\in\Lambda} = (r_i s_i r_i)_{i\in\Lambda} = (r_i)_{i\in\Lambda}(s_i)_{i\in\Lambda}(r_i)_{i\in\Lambda},$$

for an $(r_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} R_i$. Thus, $(r_i)_{i\in\Lambda} \in Reg(\prod_{i\in\Lambda} R_i)$. Therefore, we get $(x_i)_{i\in\Lambda}$ is an r-clean element of $\prod_{i\in\Lambda} R_i$. This shows that $\prod_{i\in\Lambda} P_i$ is an r-clean ideal of $\prod_{i\in\Lambda} R_i$. □

Recall that an ideal I of R is an exchange ideal provided that for any $x \in I$, there exists an idempotent $e \in I$ such that $e - x \in R(x - x^2)$. Moreover, an Abelian ring is a ring in which each idempotent element is central. According to [8], the Abelian ring R is r-clean if and only if R exchange ring. Using this property, we give the necessary and sufficient conditions for an ideal to form an r-clean ideal.

Proposition 3.8. *Let R be an Abelian ring and I an ideal of R. Then, I is an r-clean ideal if and only if I is an exchange ideal.*

Proposition 3.9. *Let I be an ideal of R. Then, I is an r-clean ideal if and only if each $x \in I$ can be written as $x = r - e$ where $r \in Reg(R)$ and $e \in Id(R)$.*

Proof. Let $x \in I$. We have $-x \in I$ and satisfy $-x = e + r$, where $e \in Id(R)$ and $r \in Reg(R)$. So, we get $x = (-r) - e$ with $e \in Id(R)$ and $-r \in Reg(R)$. Conversely, let $x \in I$. Then, $-x \in I$. According to the hypothesis, $-x = r - e$ with $r \in Reg(R)$ and $e \in Id(R)$ is obtained. Consequently, $x = (-r) + e$ with $-r \in Reg(R)$ and $e \in Id(R)$. This implies x is an r-clean element of R, so I is an r-clean ideal of R. □

Let I be an ideal of ring R and e a central idempotent element of R. Then, we can form a ring eRe. In the following, we give the necessary condition for an ideal I of R to be an r-clean ideal.

Proposition 3.10. *Let I be an r-clean ideal of R, and e a central idempotent of R. Then, eIe is an r-clean ideal of eRe.*

Proof. Let the function $f : R \to eRe$ with $f(r) = ere$ for each $r \in R$. Then, f is a ring epimorphism. Since I is an r-clean ideal of R, referring to Proposition 3.2 we obtain eIe is also an r-clean ideal of eRe. □

Next, we provide the necessary conditions for the ideal $(1_R - e)I(1_R - e)$ to be an r-clean ideal of $(1_R - e)R(1_R - e)$.

Proposition 3.11. *Let I be an ideal of ring R, e a central idempotent element of R, eIe an r-clean ideal of eRe, and $(1_R - e)I(1_R - e)$ an r-clean ideal of $(1_R - e)R(1_R - e)$. Then, I is an r-clean ideal of R.*

Proof. Let $\bar{e} = 1_R - e$. By using Pierce Decomposition, we get

$$R = eRe \oplus eR\bar{e} \oplus \bar{e}Re \oplus \bar{e}R\bar{e}$$

and

$$I = eIe \oplus eI\bar{e} \oplus \bar{e}Ie \oplus \bar{e}I\bar{e}.$$

Since e is a central idempotent element, We obtain

$$R = eRe \oplus \bar{e}R\bar{e} \cong \begin{bmatrix} eRe & 0_R \\ 0_R & \bar{e}R\bar{e} \end{bmatrix}$$

and

$$I = eIe \oplus \bar{e}I\bar{e} \cong \begin{bmatrix} eIe & 0_R \\ 0_R & \bar{e}I\bar{e} \end{bmatrix}.$$

Let $A \in I$ with $A = \begin{bmatrix} a & 0_R \\ 0_R & b \end{bmatrix}$, where $a \in eIe$ and $b \in \bar{e}I\bar{e}$. Since eIe is an r-clean ideal of eRe and $\bar{e}I\bar{e}$ is an r-clean ideal of $\bar{e}R\bar{e}$, we have $a = r_1 + e_1$ and $b = r_2 + e_2$ with $r_1 \in Reg(eRe)$, $r_2 \in Reg(\bar{e}R\bar{e})$, $e_1 \in Id(eRe)$, and $e_2 \in Id(\bar{e}R\bar{e})$. As a result, we get

$$A = \begin{bmatrix} a & 0_R \\ 0_R & b \end{bmatrix} = \begin{bmatrix} r_1 + e_1 & 0_R \\ 0_R & r_2 + e_2 \end{bmatrix} = \begin{bmatrix} r_1 & 0_R \\ 0_R & r_2 \end{bmatrix} + \begin{bmatrix} e_1 & 0_R \\ 0_R & e_2 \end{bmatrix}.$$

As $r_1 \in Reg(eRe)$ and $r_2 \in Reg(\bar{e}R\bar{e})$, there exists $y_1 \in eRe$ and $y_2 \in \bar{e}R\bar{e}$ such that $r_1 = r_1y_1r_1$ and $r_2 = r_2y_2r_2$. So, we obtain

$$\begin{bmatrix} r_1 & 0_R \\ 0_R & r_2 \end{bmatrix}\begin{bmatrix} y_1 & 0_R \\ 0_R & y_2 \end{bmatrix}\begin{bmatrix} r_1 & 0_R \\ 0_R & r_2 \end{bmatrix} = \begin{bmatrix} r_1y_1r_1 & 0_R \\ 0_R & r_2y_2r_2 \end{bmatrix} = \begin{bmatrix} r_1 & 0_R \\ 0_R & r_2 \end{bmatrix}.$$

Thus, $\begin{bmatrix} r_1 & 0_R \\ 0_R & r_2 \end{bmatrix} \in Reg(R)$. Clearly, $\begin{bmatrix} e_1 & 0_R \\ 0_R & e_2 \end{bmatrix} \in Id(R)$. Hence, A is an r-clean ideal of I, so I is an r-clean ideal of R. □

In the following, we generalize the Proposition 3.11.

Proposition 3.12. *Let I be an ideal of ring R, and $e_1, e_2, \cdots, e_n$ orthogonal central idempotents of R with $e_1 + e_2 + \cdots + e_n = 1_R$. Then, e_iIe_i is an r-clean ideal of e_iRe_i for each $i = 1, 2, \cdots n$ if and only if I is an r-clean ideal of R.*

Proof. One direction allows from Proposisi 3.11 by induction. Conversely, let I be an r-clean ideal of R and $e_1, e_2, \cdots, e_n$ orthogonal central idempotents of R with $e_1 + e_2 + \cdots + e_n = 1_R$. Consider that

$$I = \bigoplus_{i=1}^{n} e_iIe_i \cong \begin{bmatrix} e_1Ie_1 & 0_R & \cdots & 0_R \\ 0_R & e_2Ie_2 & \cdots & 0_R \\ \vdots & \vdots & \ddots & \vdots \\ 0_R & 0_R & \cdots & e_nIe_n \end{bmatrix}.$$

Let any $i \in \{1, 2, \cdots, n\}$ and we form a function $f : I \to e_iIe_i$. We have f a ring epimorphism. Since I is an r-clean ideal of R, we have e_iIe_i also an r-clean ideal of e_iRe_i. Thus, e_iIe_i is an r-clean ideal of e_iRe_i for each $i = 1, 2, \cdots, n$. □

Let $T = \begin{bmatrix} R & 0 \\ M & S \end{bmatrix}$ be the lower triangular matrix ring and $K = \begin{bmatrix} I & 0 \\ M & J \end{bmatrix}$ ideal of T. Next, we provide necessary conditions for the ideal $K = \begin{bmatrix} I & 0 \\ M & J \end{bmatrix}$ to be r-clean.

Proposition 3.13. *Let R and S be rings, M an (R, S)-bimodule, I an ideal of R, J an ideal of S, and $T = \begin{bmatrix} R & 0 \\ M & S \end{bmatrix}$ the lower triangular matrix ring. If $K = \begin{bmatrix} I & 0 \\ M & J \end{bmatrix}$ is an r-clean ideal of T, then I is an r-clean ideal of R and J is an r-clean ideal of S.*

Proof. Let $K = \begin{bmatrix} I & 0 \\ M & J \end{bmatrix}$ is an r-clean ideal of T. Let $\begin{bmatrix} a & 0 \\ m & b \end{bmatrix} \in K$. Since K is an r-clean ideal, then

$$\begin{bmatrix} a & 0 \\ m & b \end{bmatrix} = \begin{bmatrix} f_1 & 0 \\ f_2 & f_3 \end{bmatrix} + \begin{bmatrix} r_1 & 0 \\ r_2 & r_3 \end{bmatrix},$$

with $\begin{bmatrix} f_1 & 0 \\ f_2 & f_3 \end{bmatrix} \in Id(T)$ and $\begin{bmatrix} r_1 & 0 \\ r_2 & r_3 \end{bmatrix} \in Reg(T)$. So, $a = f_1 + r_1$ dan $b = f_3 + r_3$. Since $\begin{bmatrix} r_1 & 0 \\ r_2 & r_3 \end{bmatrix} \in Reg(T)$, there exists $\begin{bmatrix} y_1 & 0 \\ y_2 & y_3 \end{bmatrix} \in T$ such that

$$\begin{bmatrix} r_1 & 0 \\ r_2 & r_3 \end{bmatrix} = \begin{bmatrix} r_1 & 0 \\ r_2 & r_3 \end{bmatrix}\begin{bmatrix} r_1 & 0 \\ r_2 & r_3 \end{bmatrix}\begin{bmatrix} y_1 & 0 \\ y_2 & y_3 \end{bmatrix}$$

$$\begin{bmatrix} r_1 & 0 \\ r_2 & r_3 \end{bmatrix} = \begin{bmatrix} r_1y_1r_1 & 0 \\ r_2y_1r_1 + r_3y_2r_1 + r_3y_3r_2 & r_3y_3r_3 \end{bmatrix}.$$

Hence, $r_1 \in Reg(R)$ and $r_3 \in Reg(S)$. Moreover, since $\begin{bmatrix} f_1 & 0 \\ f_2 & f_3 \end{bmatrix} \in Id(T)$, we obatin

$$\begin{bmatrix} f_1 & 0 \\ f_2 & f_3 \end{bmatrix}\begin{bmatrix} f_1 & 0 \\ f_2 & f_3 \end{bmatrix} = \begin{bmatrix} f_1 & 0 \\ f_2 & f_3 \end{bmatrix}$$
$$\begin{bmatrix} f_1^2 & 0 \\ f_2f_1 + f_3f_2 & f_3^2 \end{bmatrix} = \begin{bmatrix} f_1 & 0 \\ f_2 & f_3 \end{bmatrix}.$$

So we get $f_1 \in Id(R)$ and $f_3 \in Id(S)$. Hence, a is an r-clean element of R and b is an r-clean element of S. Thus, I is an r-clean ideal of R and J is an r-clean ideal of S. □

Next, the following is given the converse of Proposition 3.13.

Proposition 3.14. *Let R and S be rings, M an (R, S)-bimodule, I an ideal of R, J an ideal of S, $T = \begin{bmatrix} R & 0 \\ M & S \end{bmatrix}$ the lower triangular matrix ring, and $K = \begin{bmatrix} I & 0 \\ M & J \end{bmatrix}$ an ideal of T. Assume that one of the following conditions holds:*

(i) I and J are clean.

(ii) one of the ideals I and J is clean and the other one is r-clean.

Then, the ideal K of T is r-clean.

Proof. Referring to [7], it is clear that if I and J are clean, then K is clean. So it is r-clean. On the other hand, let I be r-clean and let J be clean. Then for every $A = \begin{bmatrix} x & 0 \\ m & y \end{bmatrix} \in K$, we have $x = e_1 + r$ dan $y = e_2 + u$ for $e_1, e_2 \in Id(R)$, $r \in Reg(R)$, and $u \in U(R)$. Assume that $r = rpr$ for some $p \in R$. Let $A = E + W$ where $E = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}$ and $W = \begin{bmatrix} r & 0 \\ m & u \end{bmatrix}$. It is clear that $E \in Id(T)$ and we have the equality $\begin{bmatrix} r & 0 \\ m & u \end{bmatrix} = \begin{bmatrix} r & 0 \\ m & u \end{bmatrix}\begin{bmatrix} p & 0 \\ -u^{-1}mp & u^{-1} \end{bmatrix}\begin{bmatrix} r & 0 \\ m & u \end{bmatrix}$. This condition implies that W is a regular element if T. Hence, A is an r-clean element of K. Thus, K is r-clean. □

According to [4], we know that if I is a clean ideal of R, then $M_n(I)$ is a clean ideal of $M_n(R)$. So, using this property, we give the sufficient condition for $M_n(I)$ to be an r-clean ideal of $M_n(R)$.

Proposition 3.15. *Let I be a clean ideal of R. Then, $M_n(I)$ is an r-clean ideal of $M_n(R)$.*

Proof. Let I be a clean ideal of R. Referring to [4], we have $M_n(I)$ is a clean ideal of $M_n(R)$. Thus, $M_n(I)$ is an r-clean ideal of $M_n(R)$. □

We have stated earlier that the r-clean ideal is a generalization of the clean ideal. Thus, every clean ideal is an r-clean ideal, but the converse is not necessarily true. In the following, we give several sufficient conditions for an r-clean ideal to be clean.

Proposition 3.16. *Let I be a non-zero ideal of ring R. Then, if I is an r-clean ideal of R and 0_R and 1_R are the only idempotents in R, then I is a clean ideal of R.*

Proof. Let I be an r-clean ideal of R. Let $x \in I$. We have $x = e + r$ for an $e \in Id(R)$ and $r \in Reg(R)$. If $r = 0_R$, then $x = e = (2e - 1_R)(1_R - e)$. Clearly, $1_R - e \in Id(R)$. Considering $(2e - 1_R)(2e - 1_R) = 1_R$, so we get $2e - 1_R \in U(R)$. Thus, x is a clean element of R. But if $r \neq 0_R$, then there exists $y \in R$ such that $r = ryr$. As a result, we get $ry \in Id(R)$. Since 0_R and 1_R are the only idempotents in R, we have $ry = 0_R$ or $ry = 1_R$. If $ry = 0_R$, then $r = ryr = 0_R$ is a contradiction. Therefore, $ry = 1_R$. Similarly, $yr = 1_R$. Thus, $r \in U(R)$. Hence, x is a clean element of R, so I is a clean ideal of R. □

Furthermore, let R be an Abelian ring. According to [8], if $a \in R$ is a clean element of R and $e \in Id(R)$, then

(i) ae is a clean element of R.

(ii) If $-a$ is a clean element of R, then $a + e$ is also a clean element of R.

Using the properties above, we give sufficient and necessary conditions for an ideal to be an r-clean ideal.

Proposition 3.17. *Let R be an Abelian ring. The ideal I of R is an r-clean ideal if and only if I is a clean ideal of R.*

Proof. Let I be an r-clean ideal of R. Let $x \in I$. Then, $x = e + r$ with $e \in Id(R)$ and $r \in Reg(R)$. Since $r \in Reg(R)$, there exists $y \in R$ such that $r = ryr$. Clearly, $ry, yr \in Id(R)$. Assume $e' = ry$, so

$$\begin{aligned}(re' + (1_R - e'))(ye' + (1_R - e')) &= re'ye' + re'(1_R - e') + (1_R - e')ye' \\ &\quad + (1_R - e')(1_R - e') \\ &= rye' + 0_R + 0_R + (1_R - e') \\ &= e' + 1_R - e' \\ &= 1_R.\end{aligned}$$

Next, since R is an Abelian ring, we get

$$\begin{aligned}(ye' + (1_R - e'))(re' + (1_R - e')) &= ye're' + ye'(1_R - e') + (1_R - e')re' \\ &\quad + (1_R - e')(1_R - e') \\ &= yre' + 1_R - e' \\ &= e'yr + 1_R - e' \\ &= ry(yr) + 1_R - e' \\ &= r(yr)y + 1_R - e' \\ &= (ry)^2 + 1_R - e' \\ &= e'^2 + 1_R - e' \\ &= e' + 1_R - e' \\ &= 1_R.\end{aligned}$$

Thus, $re' + (1_R - e') \in U(R)$. Furthermore, note that $u = re' + (1_R - e')$. So, we have $e'u = e're' + 0_R = e're' = ryr = r$. Now, assume $f = 1_R - e'$. Then, $f \in Id(R)$. Consider the equation $r + f = e'u + f$. So, we get $-r = f + (-(e'u + f))$. Next, note that

$$\begin{aligned}u &= re' + (1_R - e') \\ &= re' + f \\ &= (e'u)e' + f \\ &= e'u + f \in U(R).\end{aligned}$$

As a result, we have $-(e'u + f) \in U(R)$. Hence, $-r$ is a clean element of R. Thus, $r + e = x$ is a clean element of R, so I is a clean ideal of R. Conversely, it is clear and needs no proof. □

Proposition 3.18. *Let R be a ring with no zero divisors and I an ideal of R. Then, I is an r-clean ideal if and only if I is a clean ideal of R.*

Proof. Let I be an r-clean ideal of R. Let $x \in I$. Then, $x = e + r$ with $e \in Id(R)$ and $r \in Reg(R)$. There exists $y \in R$ such that $r = ryr$. If $r = 0_R$, then $x = e = (2e - 1_R) + (1_R - e)$. Clearly, $1_R - e \in Id(R)$ and $(2e - 1_R) \in U(R)$. So, x is a clean element of R. However, if $r \neq 0_R$, since R contains no zero divisor elements, from equation $r = ryr$, we get $r - ryr = 0_R$. So, $yr = 1_R$. Similarly, $ry = 1_R$. Hence, $r \in U(R)$. Thus, x is a clean element of R, so I is a clean ideal of R. Conversely, it is clear and needs no proof. □

Based on [8], a ring is said to be reduced if it has no (non-zero) nilpotent elements. These rings are Abelian. So we have the following property.

Proposition 3.19. *Let R be a reduced ring and I an ideal of R. Then, I is r-clean if and only if I is clean.*

References

[1] W.K. Nicholson, *Lifting Idempotents and Exchange Rings*, Transactions of the American Mathematical Society, 229(1977), 269–278.

[2] W. A. Adkins, Algebra "An Approach via Module Theory", Springer-Verlag New York, Inc., USA, 1992.

[3] N. Ashrafi and E. Nasibi, *On r-Clean Rings*, to appear in Mathematical Reports, 15(65)(2)(2013), 125–132.

[4] H. Chen and M. Chen, *On Clean Ideals*, International Journal of Mathematics and Mathematical Sciences, 62(2003), 3949–3956.

[5] B. Brown and N.H. McCoy, *The Maximal Regular Ideal of a Ring*, Proceedings of the American Mathematical Society, 1 (2)(1950), 165–171.

[6] K. Adarbeh, *Trivial Ring Extension of Suitable-Like Conditions and Some Properties*, An-Najah Univ. J. Res. (N.Sc.), 33 (1)(2019), 115-125.

[7] J. Han and W. K. Nicholson, *Extensions of clean rings*, Comm. Algebra, 29 (6)(2001), 2589–2595.

[8] N. Ashrafi and E. Nasibi, *Rings in Which Elements are the Sum of An Idempotent and A Regular Element*, Bulletin of the Iranian Mathematical Society, 39 (3)(2013), 579–588.

Author information

Dian Ariesta Yuwaningsih, Department of Mathematics, Universitas Gadjah Mada, Yogyakarta 55281 and Department of Mathematics Education, Universitas Ahmad Dahlan, Bantul 55166, Indonesia.
E-mail: `dian.ariesta.yuwaningsih@mail.ugm.ac.id`

Indah Emilia Wijayanti, Department of Mathematics, Universitas Gadjah Mada, Yogyakarta 55281, Indonesia.
E-mail: `ind_wijayanti@ugm.ac.id`

Budi Surodjo, Department of Mathematics, Universitas Gadjah Mada, Yogyakarta 55281, Indonesia.
E-mail: `surodjo_b@ugm.ac.id`

Received: 2022-03-09
Accepted: 2023-04-18