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On besselian Schauder frames, shrinking Schauder frames and boundedly complete Schauder frames in Banach spaces

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Abstract: In this paper we introduce, for a Banach space, a new notions of besselian paires and of besselian Schauder frames and we extend, to Schauder frames, the notions of shrinking and boundedly complete Schauder basis. Our aim is to study the relations which exist between these notions and the links these notions have with the Banach space structure. Among the multiple results that we obtain, we prove that a paire $\mathcal{F}$ is besselian if and only if its dual paire $\mathcal{F}^*$ is a besselian paire of its topological dual E^*, we generalize to besselian Schauder frames, the well-known James's theorem which characterizes reflexive Banach spaces by means of shrinking and boundedly complete Schauder basis, and finally we prove that a Banach space E which has a besselian Schauder frame is reflexive if and only if the space E and E^* are both weakly sequentially complete.

1 Introduction

In 1946, Gabor [8] performed a new method for the signal reconstruction from elementary signals. In 1952, Duffin and Schaeffer [6] developped, in the field of nonharmonic series, a similar tool and introduced frame theory for Hilbert spaces. For more than thirty years, the results of Duffin and Schaeffer has not received from the mathematical community, the interest they deserve, until the publication of the work of Higgins and Young [21] where the authors studied frames in abstract Hilbert spaces. In 1986, the work of Daubechies, Grossmann and Meyer [5] gave to frame theory the momentum it lacked and allowed it to be widely studied. This contributed, among other things, to the wider developpement of wavelet theory. The concept of atomic decompositions was introduced, in 1988, by Feichtinger and Gröchenig [7], in order to extend the definition of frames from the seting of Hilbert spaces to that of general separable Banach spaces. In 1991, Gröchenig [9], presented a generalisation of the notions of atomic decomposition and of synthesis operator and introduced the definition of Banach frames. In 2001, Aldroubi, Sun and Tang [1] introduced the concepts of p-frames. In 2003, Christensen and Stoeva [4] extended the definition of p-frames, by replacing the sequence space L_p by a more general scalar sequence space X_d. By getting rid of the sequence spaces X_d in the definition of atomic decompositions, Cassaza, Han, Larson in 1999 [3] and in 2000 Han and Larson [12], generalized the notion of atomic decompositions by introducing the new notion of Schauder frames. One of the peculiarities of Schauder frames is that they constitute a natural extension of the concept of Schauder basis.

The growing interest in Schauder frames has led us to contribute to the generalization to Schauder frames of the properties and results specific to Schauder bases. Indeed In this paper we introduce, for a Banach space, a new notions of besselian paires and of besselian Schauder frames and we extend, to Schauder frames, the notions of shrinking and boundedly complete Schauder basis. Our aim is to study the relations which exist between these notions and the links these notions have with the Banach space structure. Among the multiple results that we obtain, we prove that a paire $\mathcal{F}$ is besselian if and only if its dual paire $\mathcal{F}^*$ is a besselian paire of E^*, we generalize to besselian Schauder frames, the well-known James [10],[11, page 62] theorem which characterizes reflexive Banach spaces by means of shrinking and boundedly complete Schauder basis, and finally we prove that a Banach space E which has a besselian Schauder frame is reflexive if and only if the space E and its topological dual E^* are both weakly sequentially complete.

For all the material on Banach spaces, one can refer to [17], [11], [13], [14], [22]. In the sequel $(E, \|\cdot\|_E)$ is a given separable Banach space, $((a_n, b_n^*))_{n\in\mathbb{N}^*}$ a paire of E and $p \in]1, +\infty[$ is a given constant and we set $p^* = \dfrac{p}{p-1}$. Finally we will index all the series by $\mathbb{N}^*$.

2 Main definitions, principal notations and useful remarks

Let $(X, \|.\|_X)$ be a Banach space on $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and X^*, $X^{**}$and X^{***} respectively its first, second and third topological duals.

(i) We denote by $\mathbb{B}_X$ the closed unit ball of X :

$$\mathbb{B}_X := \{x \in X : \|x\|_X \leq 1\}$$

(ii) We denote by $l^1(\mathbb{K})$ the $\mathbb{K}$-vector space of sequences $\lambda := (\lambda_n)_{n\in\mathbb{N}^*}$ such that $\lambda_n \in \mathbb{K}$ for each $n \in \mathbb{N}^*$ and $\sum_{n=1}^{+\infty} |\lambda_n| < +\infty$. It is a classical result that $l^1(\mathbb{K})$ is a Banach space for the norm :

$$\begin{array}{rccc} \|.\|_{l^1(\mathbb{K})} : & l^1(\mathbb{K}) & \rightarrow & \mathbb{R}^+ \\ & (\lambda_n)_{n\in\mathbb{N}^*} & \mapsto & \sum\limits_{n=1}^{+\infty} |\lambda_n| \end{array}$$

(iii) We denote, for each $n \in \mathbb{N}^*$, by e_n the element of $l^1(\mathbb{K})$ defined by the relation $e_n := (\delta_{k,n})_{k\in\mathbb{N}^*}$, where $\delta_{k,n} = 1$ if $k = n$ and $\delta_{k,n} = 0$ if $k \neq n$.

(iv) We denote, for each $n \in \mathbb{N}^*$, by u_n^* the element of $\left(l^1(\mathbb{K})\right)^*$ defined, for each $\lambda := (\lambda_k)_{k\in\mathbb{N}^*} \in l^1(\mathbb{K})$ by the relation $u_n^*(\lambda) := \lambda_n$.

(v) We denote by $l^\infty(\mathbb{K})$ the $\mathbb{K}$-vector space of sequences $\lambda := (\lambda_n)_{n\in\mathbb{N}^*}$ such that $\lambda_n \in \mathbb{K}$ for each $n \in \mathbb{N}^*$ and $\sup_{n\in\mathbb{N}^*}(|\lambda_n|) < +\infty$. It is a classical result that $l^\infty(\mathbb{K})$ is a Banach space for the norm :

$$\begin{array}{rccc} \|.\|_{l^\infty(\mathbb{K})} : & l^\infty(\mathbb{K}) & \rightarrow & \mathbb{R}^+ \\ & (\mu_n)_{n\in\mathbb{N}^*} & \mapsto & \sup\limits_{n\in\mathbb{N}^*}(|\mu_n|) \end{array}$$

(vi) We denote by Ψ the mapping :

$$\begin{array}{rccc} \Psi : & l^\infty(\mathbb{K}) & \rightarrow & \left(l^1(\mathbb{K})\right)^* \\ & \mu := (\mu_n)_{n\in\mathbb{N}^*} & \mapsto & \Psi(\mu) \end{array}$$

where :

$$\Psi(\mu)\left((\lambda_n)_{n\in\mathbb{N}^*}\right) := \sum_{n=1}^{+\infty} \mu_n \lambda_n$$

It is well-known that Ψ is an isometric isomorphism from $l^\infty(\mathbb{K})$ onto $l^1(\mathbb{K})$ [17, page 85, example 1.10.3] .

(vii) A mapping $\omega : X \rightarrow \mathbb{R}^+$ is said to be countably subadditive [17, page 42, definition 1.6.2] if $\omega\left(\sum\limits_{n=1}^{+\infty} z_n\right) \leq \sum\limits_{n=1}^{+\infty} \omega(z_n)$ for each convergent series $\sum z_n$ in X.

(viii) Given a Banach space $(Y, \|.\|_Y)$ and a real number $\alpha > 0$. A linear mapping $\varphi : X \rightarrow Y$ is said to be an α-isometry [16] if the following condition holds for each $x \in X$:

$$(1-\alpha)\|x\|_X \leq \|\varphi(x)\|_Y \leq (1+\alpha)\|x\|_X$$

(ix) We denot by J_X the canonical linear mapping :

$$\begin{array}{rccc} J_X : & X & \to & X^{**} \\ & x & \mapsto & J_X(x) \end{array}$$

defined for each $x \in X$ and $x^* \in X^*$ by the formula $J_X(x)(x^*) = x^*(x)$. It is well-known [17, page 98, proposition 1.11.3] that the linear mapping J_X is an isometry from X into X^{**}.

(x) We denote by P_X the mapping :

$$\begin{array}{rccc} P_X : & X^{***} & \to & X^{*} \\ & u^{***} & \mapsto & u^{***} \circ J_X \end{array}$$

It is well-known [2] that the mapping P_X is a continuous linear mapping from X^{***} onto X^*.

(xi) Let $(x_n)_{n\in\mathbb{N}^*}$ be a sequence of elements of X. The series $\sum x_n$ of X is said to be weakly unconditionally convergent [22, pages 58-59] if the series $\sum |x^*(x_n)|$ is convergent for each $x^* \in X^*$.

(xii) The Banach space X is said to be weakly sequentially complete [17, page 218, definition 2.5.23] [11, pages 37-38] if for each sequence $(x_n)_{n\in\mathbb{N}^*}$ of X such that $\lim\limits_{n\to+\infty} x^*(x_n)$ exists for every $x^* \in X^*$, there exists $x \in X$ such that $\lim\limits_{n\to+\infty} x^*(x_n) = x^*(x)$ for every $x^* \in X^*$.

(xiii) The Banach space X is a Schur space [17, page 220, definition 2.5.25] [11, page 37, definition 2.3.4] [20] if it satisfies the following condition : Whenever $(x_n)_{n\in\mathbb{N}^*}$ a sequence of X and $x \in X$ such that $\lim\limits_{n\to+\infty} x^*(x_n) = x^*(x)$ for every $x^* \in X^*$, then $\lim\limits_{n\to+\infty} x_n = x$.

(xiv) A sequence $\mathfrak{X} := ((x_n, y_n^*))_{n\in\mathbb{N}^*} \subset X \times X^*$ is called a paire of X.

(xv) The sequence $\mathfrak{X}^* := ((y_n^*, J_X(x_n)))_{n\in\mathbb{N}^*} \subset X^* \times X^{**}$ is called the dual paire of the paire $\mathfrak{X}$.

(xvi) The paire $\mathfrak{X}$ is called a Schauder frame (resp. unconditional Schauder frame) of X if for all $x \in X$, the series $\sum y_n^*(x)\, x_n$ is convergent (resp. unconditionally convergent) in X to x.

(xvii) The paire $\mathfrak{X}$ is said to be a besselian paire of X if there exists a constant $A > 0$ such that :

$$\sum_{n=1}^{+\infty} |y_n^*(x)|\, |x^*(x_n)| \leq A \|x\|_X \|x^*\|_{X^*}$$

for each $x \in X$ and $x^* \in X^*$.

(xviii) The paire $\mathfrak{X}$ is said to be a besselian Schauder frame of X if it is both a besselian paire and a Schauder frame of X .

(xix) A Schauder frame $\mathfrak{X}$ of X is said to be shrinking if the series $\sum x^*(x_n)\, y_n^*$ is convergent for every $x^* \in X^*$.

(xx) A Schauder frame $\mathfrak{X}$ of X is said to be boundedly complete if the series $\sum x^{**}(y_n^*)\, x_n$ is convergent in X for every $x^{**} \in X^{**}$.

(xxi) We denote by Φ_p the following mapping :

$$\begin{array}{rccc} \Phi_p : & L_{p^*}([0,1]) & \to & (L_p([0,1]))^* \\ & f & \mapsto & \Phi_p(f) \end{array}$$

defined for each $f \in L_{p^*}([0,1])$ and $g \in L_p([0,1])$ by the formula

$$\Phi_p(f)(g) = \int_0^1 fg\,dx$$

It is well known [17, page 85, example 1.10.2] that the mapping Φ_p is an isometric isomorphism from $L_{p^*}([0,1])$ onto $(L_p([0,1]))^*$.

(xxii) We consider the Haar system $(h_n)_{n\in\mathbb{N}^*}$ of $L_p([0,1])$ defined as follows [17, pages:359-361]. Let h_1 be 1 on $[0,1[$ and 0 at 1. When $n \geq 2$, we define h_n by letting m be the positive integer such that $2^{m-1} < n \leq 2^m$, then let

$$h_n(t) = \begin{cases} 1 \text{ if } t \in \left[\frac{2n-2}{2^m} - 1, \frac{2n-1}{2^m} - 1\right[\\ -1 \text{ if } t \in \left[\frac{2n-1}{2^m} - 1, \frac{2n}{2^m} - 1\right[\\ 0 \text{ else} \end{cases}$$

We denote by $\mathfrak{h}(p)$ the paire $((h_n, \Phi_p(h_n)))_{n\in\mathbb{N}^*}$ of $L_p([0,1])$ and by $\mathfrak{h}(p)^*$ the dual paire of the paire $\mathfrak{h}(p)$, that is $\mathfrak{h}(p)^* := \left(\left(\Phi_p(h_n), J_{L_p([0,1])}(h_n),\right)\right)_{n\in\mathbb{N}^*}$.

Remark 2.1. For each $\lambda = (\lambda_n)_{n\in\mathbb{N}^*} \in l^1(\mathbb{K})$, it is clear that the series $\sum u_n^*(\lambda) e_n$ is convergent and that we have :

$$\lambda = \sum_{n=1}^{+\infty} u_n^*(\lambda) e_n$$

On the other hand let us given $\lambda = (\lambda_n)_{n\in\mathbb{N}^*} \in l^1(\mathbb{K})$ and $\xi^* \in \left(l^1(\mathbb{K})\right)^*$. We set $\mu := \Psi^{-1}(\xi^*) \in l^\infty(\mathbb{K})$. Hence $\xi^* = \Psi(\mu)$ and we have :

$$\begin{aligned} \sum_{n=1}^{+\infty} |u_n^*(\lambda)| |\xi^*(e_n)| &= \sum_{n=1}^{+\infty} |\lambda_n| |\mu_n| \\ &\leq \left(\sum_{n=1}^{+\infty} |\lambda_n|\right) \sup_{n\in\mathbb{N}^*} |\mu_n| \\ &\leq \|\lambda\|_{l^1(\mathbb{K})} \|\mu\|_{l^\infty(\mathbb{K})} \\ &\leq \|\lambda\|_{l^1(\mathbb{K})} \|\xi^*\|_{(l^1(\mathbb{K}))^*} \end{aligned}$$

It follows that $((e_n, u_n^*))_{n\in\mathbb{N}^*}$ is a besselian Schauder frame of $l^1(\mathbb{K})$.

Remark 2.2. Assume that the paire $\mathfrak{X}$ and its dual $\mathfrak{X}^*$ are a Schauder frames of X and X^* respectively. Then $\mathfrak{X}$ will be a shrinking Schauder frame of X.

Proof. Since $\mathfrak{X}^*$ is a Schauder frame of X^* it follows that the series $\sum J_X(x_n)(x^*) y_n^* = \sum x^*(x_n) y_n^*$ is convergent for each $x^* \in X^*$. Consequently $\mathfrak{X}$ is a shrinking Schauder frame of X. □

Remark 2.3. For a besselian paire $\mathfrak{X}$ of X, the quantity

$$\mathcal{L}_{\mathfrak{X}} := \sup_{(u,u^*)\in\mathbb{B}_X\times\mathbb{B}_{X^*}} \left(\sum_{n=1}^{+\infty} |y_n^*(u)| |u^*(x_n)|\right)$$

is finite and for each $x \in X$ and $x^* \in X^*$, the following inequality holds

$$\sum_{n=1}^{+\infty} |y_n^*(x)| |x^*(x_n)| \leq \mathcal{L}_{\mathfrak{X}} \|x\|_X \|x^*\|_{X^*}$$

In the sequel $\mathcal{F} := ((a_n, b_n^*))_{n\in\mathbb{N}^*}$ is a fixed paire of a Banach space E.

3 Fundamental results

Theorem 3.1. *The paire $\mathcal{F}$ of E is a besselian paire of E if and only if its dual paire $\mathcal{F}^*$ is a besselian paire of E^*. In this case we have $\mathcal{L}_{\mathcal{F}} = \mathcal{L}_{\mathcal{F}^*}$.*

Proof. Assume that $\mathcal{F}$ is a besselian paire of E. Let $x^* \in E^*$, $x^{**} \in E^{**}$. We set for each $n \in \mathbb{N}^*$: $U = span(x^{**})$ and $V_n = span(b_1^*, ..., b_n^*)$. It is clear that U and V_n are finite dimensional subspaces of E^{**} and E^* respectively.

According to the principle of local reflexivity [16, Theorem.2], we can find for each $\alpha > 0$ an α-isometry $T_n : U \longrightarrow E$ such that :

$$\begin{cases} b_j^*\left(T_n(x^{**})\right) = x^{**}(b_j^*),\ j \in \{1, ...n\} \\ \|T_n(x^{**})\|_E \leq (1+\alpha)\,\|x^{**}\|_{E^{**}} \end{cases}$$

It follows that :

$$\begin{aligned}\sum_{j=1}^{n} \left|x^{**}(b_j^*)\right| \left|J_E(a_j)\,(x^*)\right| &= \sum_{j=1}^{n} \left|b_j^*(T_n(x^{**}))\right| \left|x^*(a_j)\right| \\ &\leq \mathcal{L}_{\mathcal{F}}\, \|T_n(x^{**})\|_E\, \|x^*\|_{E^*} \\ &\leq (1+\alpha)\mathcal{L}_{\mathcal{F}}\, \|x^{**}\|_{E^{**}}\, \|x^*\|_{E^*}\end{aligned}$$

Consequently, we have

$$\sum_{j=1}^{+\infty} \left|x^{**}(b_j^*)\right| \left|J_E(a_j)\,(x^*)\right| \leq (1+\alpha)\mathcal{L}_{\mathcal{F}}\, \|x^{**}\|_{E^{**}}\, \|x^*\|_{E^*}$$

for each $\alpha > 0$, $x^{**} \in E^{**}$ and $x^* \in E^*$. Hence :

$$\sum_{j=1}^{+\infty} \left|x^{**}(b_j^*)\right| \left|J_E(a_j)\,(x^*)\right| \leq \mathcal{L}_{\mathcal{F}}\, \|x^{**}\|_{E^{**}}\, \|x^*\|_{E^*}$$

for each $x^{**} \in E^{**}$ and $x^* \in E^*$. Consequently $\mathcal{F}^*$ is a besselian paire of $E^*$and we have $\mathcal{L}_{\mathcal{F}^*} \leq \mathcal{L}_{\mathcal{F}}$.

Assume now that $\mathcal{F}^*$ is a besselian paire of E^*. Let $(x, x^*) \in E \times E^*$, then we have :

$$\begin{aligned}\sum_{j=1}^{+\infty} \left|b_j^*(x)\right| \left|x^*(a_j)\right| &= \sum_{j=1}^{+\infty} \left|J_E(x)(b_j^*)\right| \left|J_E\,(a_j)\,(x^*)\right| \\ &\leq \mathcal{L}_{\mathcal{F}^*}\, \|J_E(x)\|_{E^{**}}\, \|x^*\|_{E^*} \\ &\leq \mathcal{L}_{\mathcal{F}^*}\, \|x\|_E\, \|y^*\|_{E^*}\end{aligned}$$

Hence $\mathcal{F}$ is a besselian paire of E
and we have $\mathcal{L}_{\mathcal{F}} \leq \mathcal{L}_{\mathcal{F}^*}$.

We conclude that :

(i) $\mathcal{F}$ is a besselian paire of E if and only $\mathcal{F}^*$ is a besselian paire of E^*.

(ii) If $\mathcal{F}$ is a besselian paire of E then we will have $\mathcal{L}_{\mathcal{F}^*} \leq \mathcal{L}_{\mathcal{F}}$ and $\mathcal{L}_{\mathcal{F}} \leq \mathcal{L}_{\mathcal{F}^*}$, hence $\mathcal{L}_{\mathcal{F}} = \mathcal{L}_{\mathcal{F}^*}$.

The proof of the theorem is then complete. □

Proposition 3.2. *(i) We assume that E is weakly sequentially complete and that $\mathcal{F}$ is a besselian paire of E. Then for each $x \in E$, the series $\sum b_n^*\,(x)\,a_n$ is unconditionally convergent.*

(ii) We assume that E^* is weakly sequentially complete and that $\mathcal{F}$ is a besselian paire of E. Then for each $x^* \in E^*$, the series $\sum x^*\,(a_n)\,b_n^*$ is unconditionally convergent.

Proof. (i) Since $\mathcal{F}$ is a besselian paire of E, it follows that we have for each $x \in E$ and $x^* \in E^*$:

$$\begin{aligned}\sum_{n=1}^{+\infty} \left|x^*\left(b_n^*\,(x)\,a_n\right)\right| &= \sum_{n=1}^{+\infty} \left|b_n^*\,(x)\right| \left|x^*\,(a_n)\right| \\ &\leq \mathcal{L}_{\mathcal{F}}\, \|x\|_E\, \|x^*\|_{E^*} \\ &< +\infty\end{aligned}$$

Hence the series $\sum b_n^*\,(x)\,a_n$ is weakly unconditionally convergent. Hence, since E is weakly sequentially complete, it follows, thanks to Orlicz's theorem [18]; [22, theorem of page 66], , that the series $\sum b_n^*\,(x)\,a_n$ is unconditionally convergent.

(ii) Since $\mathcal{F}$ is a besselian paire of E, it follows, thanks to the theorem 3.1, that the paire $\mathcal{F}^*$ is a besselian paire of E^*. Hence we have for each $x^* \in E^*$ and $x^{**} \in E^{**}$:

$$\begin{aligned}\sum_{n=1}^{+\infty} \left|x^{**}\left(x^*\left(a_n\right) b_n^*\right)\right| &= \sum_{n=1}^{+\infty} \left|x^*\left(a_n\right)\right| \left|x^{**}\left(b_n^*\right)\right| \\ &\leq \mathcal{L}_{\mathcal{F}^*} \left\|x^*\right\|_{E^*} \left\|x^{**}\right\|_{E^{**}} \\ &< +\infty\end{aligned}$$

Hence for each $x^* \in E^*$, the series $\sum x^*\left(a_n\right) b_n^*$ is weakly unconditionally convergent. Hence since E^* is weakly sequentially complete it follows thanks to Orlicz's theorem [18]; [22, theorem of page 66] that the series $\sum x^*\left(a_n\right) b_n^*$ is, for each $x^* \in E^*$, unconditionally convergent.

The proof of the proposition is then complete. □

Proposition 3.3. *(i) We assume that E is weakly sequentially complete and that the paire $\mathcal{F}^*$ is a besselian Schauder frame of E^*. Then the paire $\mathcal{F}$ is a besselian Schauder frame of E.*

(ii) We assume that the dual space E^* is weakly sequentially complete and that the paire $\mathcal{F}$ is a besselian Schauder frame of E. Then the paire $\mathcal{F}^*$ is a besselian Schauder frame of E^*.

Proof. (i) Since E is weakly sequentially complete and that $\mathcal{F}^*$ is a besselian Schauder frame of E^* it follows, from the theorem 3.1 and the proposition 3.2, that $\mathcal{F}$ is a besselian paire of E and that the series $\sum b_n^*(x) a_n$ is unconditionally convergent for each $x \in E$. Let us then consider the mapping:

$$\begin{aligned} S: \quad E &\rightarrow \quad E \\ x &\mapsto \sum_{n=1}^{+\infty} b_n^*\left(x\right) a_n \end{aligned}$$

Then we have for each $x^* \in E^*$ and $x \in E$:

$$\begin{aligned} x^*\left(S(x)\right) &= \sum_{n=1}^{+\infty} x^*\left(a_n\right) b_n^*(x) \\ &= J_E\left(x\right)\left(\sum_{n=1}^{+\infty} x^*\left(a_n\right) b_n^*\right) \\ &= J_E\left(x\right)\left(x^*\right) \\ &= x^*\left(x\right) \end{aligned}$$

Consequently, $S(x) = x$, $x \in E$. So $\mathcal{F}$ is a besselian Schauder frame of E.

(ii) Since E^* is weakly sequentially complete it follows, from the theorem (3.1) and the proposition 3.2, that $\mathcal{F}^*$ is a besselian paire of E^* and that the series $\sum J_E(a_n)(x^*) b_n^*$ is unconditionally convergent for each $x^* \in E^*$. Let us then consider the mapping :

$$\begin{aligned} T: \quad E^* &\rightarrow \quad E^* \\ x^* &\mapsto \sum_{n=1}^{+\infty} J_E\left(a_n\right)\left(x^*\right) b_n^* \end{aligned}$$

Then we have for each $x^* \in E^*$ and $x \in E$:

$$\begin{aligned} T(x^*)(x) &= \sum_{n=1}^{+\infty} x^*\left(a_n\right) b_n^*(x) \\ &= J_E\left(x\right)\left(\sum_{n=1}^{+\infty} J_E(a_n)\left(x^*\right) b_n^*\right) \\ &= J_E\left(x\right)\left(x^*\right) \\ &= x^*\left(x\right) \end{aligned}$$

Consequently we have, for each $x^* \in E^*$, $T(x^*) = x^*$. So $\mathcal{F}^*$ is a besselian Schauder frame of E^*.

The proof of the proposition is then complete. □

Proposition 3.4. *(i) Assume that E^* is weakly sequentially complete and that $\mathcal{F}$ is a besselian Schauder frame of E. Then $\mathcal{F}$ is shrinking.*

(ii) Assume that E is weakly sequentially complete and that $\mathcal{F}$ is a besselian Schauder frame of E. Then $\mathcal{F}$ is boundedly complete.

Proof. (i) Since E^* is weakly sequentially complete and $\mathcal{F}$ is a besselian Schauder frame of E, it follows from the proposition 3.2 that the series $\sum x^*(a_n)\, b_n^*$ is, for each $x^* \in E^*$, unconditionally convergent of E. Hence $\mathcal{F}$ is a shrinking Schauder frame in E^*.

(ii) Since $\mathcal{F}$ is a besselian paire of E it follows, thanks to theorem 3.1, that $\mathcal{F}^*$ is a besselian paire of E^*. It follows that we have for each $x^* \in E^*$ and $x^{**} \in E^{**}$:

$$\begin{aligned}\sum_{n=1}^{+\infty} \left|x^*\left(x^{**}\left(b_n^*\right) a_n\right)\right| &= \sum_{n=1}^{+\infty} \left|x^*\left(a_n\right)\right| \left|x^{**}\left(b_n^*\right)\right| \\ &\leq \mathcal{L}_{\mathcal{F}^*} \left\|x^*\right\|_{E^*} \left\|x^{**}\right\|_{E^{**}} \\ &< +\infty\end{aligned}$$

It follows that the series $\sum x^{**}\left(b_n^*\right) a_n$ is, for each $x^{**} \in E^{**}$, weakly unconditionally convergent. Hence since E is weakly sequentially complete it follows, thanks to Orlicz's theorem [18], [22, theorem of page 66] that the series $\sum x^{**}\left(b_n^*\right) a_n$ is unconditionally convergent for each $x^{**} \in E^{**}$. Consequently $\mathcal{F}$ is a boundedly complete Schauder frame of E.

The proof of the proposition is then complete. □

Proposition 3.5. *Assume that $\mathcal{F}$ is a Schauder frame of E. Then $\mathcal{F}$ is shrinking if and only if $\mathcal{F}^*$ is a Schauder frame of E^*.*

Proof. Assume that $\mathcal{F} = \left((a_n, b_n^*)\right)_{n\in\mathbb{N}^*}$ is shrinking. For each $x^* \in E^*$ and $m \in \mathbb{N}^*$ we have

$$\begin{aligned}\left\|x^* - \sum_{n=1}^{m} J_E(a_n)\left(x^*\right) b_n^*\right\|_{E^*} &= \sup_{x\in\mathbb{B}_E} \left|x^*(x) - \sum_{n=1}^{m} x^*(a_n) b_n^*(x)\right| \\ &= \sup_{x\in\mathbb{B}_E} \left|x^*\left(\sum_{n=1}^{+\infty} b_n^*(x) a_n\right) - x^*\left(\sum_{n=1}^{m} b_n^*(x) a_n\right)\right| \\ &= \sup_{x\in\mathbb{B}_E} \left|x^*\left(\sum_{n=m+1}^{+\infty} b_n^*(x) a_n\right)\right| \\ &= \sup_{x\in\mathbb{B}_E} \left|\left(\sum_{n=m+1}^{+\infty} x^*(a_n) b_n^*\right)(x)\right| \\ &= \left\|\sum_{n=m+1}^{+\infty} x^*(a_n) b_n^*\right\|_{E^*}\end{aligned}$$

It follows that the series $\sum J_E(a_n)\left(x^*\right) b_n^*$ is convergent to x^*. Consequently, $\mathcal{F}^*$ is a Schauder frame of E^*.

Assume now that $\mathcal{F}^*$ is a Schauder frame of E^*. Then for each $x^* \in E^*$, the series $\sum J_E(a_n)\left(x^*\right) b_n^* = \sum x^*(a_n) b_n^*$ is convergent to x^*. Hence $\mathcal{F}$ is a shrinking Schauder frame of E.

The proof of the proposition is then achieved. □

Theorem 3.6. *$\mathcal{F}$ is a besselian paire of E if and only if the following condition holds for each $x^* \in E^*$ and $x^{**} \in E^{**}$:*

$$\sum_{n=1}^{+\infty} |x^{**}(b_n^*)| |x^*(a_n)| < +\infty \tag{3.1}$$

Proof. Since $\mathcal{F}$ is a besselian paire of E, it follows thanks to the theorem 3.1, that the paire $\mathcal{F}^*$ is a besselian paire of E^*. Hence the follwing inequality holds for each $x^* \in E^*$ and $x^{**} \in E^{**}$:

$$\begin{aligned}\sum_{n=1}^{+\infty} |x^{**}(b_n^*)| |x^*(a_n)| &= \sum_{n=1}^{+\infty} |x^{**}(b_n^*)| |J_E(a_n)(x^*)| \\ &\leq \mathcal{L}_{\mathcal{F}^*} \|x^{**}\|_{E^{**}} \|x^*\|_{E^*}\end{aligned}$$

Consequently the condition (3.1) holds for each $x^* \in E^*$ and $x^{**} \in E^{**}$.

Assume now that the condition (3.1) holds for each $x^* \in E^*$ and $x^{**} \in E^{**}$. It follows that the series $\sum J_E(x)(b_n^*) x^*(a_n) = \sum x^*(b_n^*(x) a_n)$ and $\sum x^{**}(x^*(a_n) b_n^*)$ are unconditionally convergent for all $x \in E$, $x^* \in E^*$ and $x^{**} \in E^{**}$. Consequently, there exists [22, proposition 4, page 59] for each $x \in E$ and $x^* \in E^*$ a constants $C_x, D_{x^*} > 0$ depending respectively on x and x^* such that :

$$\sum_{n=1}^{+\infty} |b_n^*(x)| |x^*(a_n)| \leq C_x \|x^*\|_{E^*}, \; x^* \in E^* \tag{3.2}$$

$$\sum_{n=1}^{+\infty} |x^{**}(b_n^*)| |x^*(a_n)| \leq D_{x^*} \|x^{**}\|_{E^{**}} \tag{3.3}$$

It follows from the inequality (3.3) that :

$$\sum_{n=1}^{+\infty} |J_E(x)(b_n^*)| |y^*(a_n)| \leq D_{y^*} \|J_E(x)\|_{E^{**}}, \; x \in E, \; y^* \in E^*$$

that is :

$$\sum_{n=1}^{+\infty} |b_n^*(x)| |y^*(a_n)| \leq D_{y^*} \|x\|_E, \; x \in E, \; y^* \in E^* \tag{3.4}$$

The inequalities (3.2) and (3.4) entail that the following mappings :

$$\begin{array}{rccc} f: & E & \to & \mathbb{R}^+ \\ & x & \mapsto & \sup_{u^* \in \mathbb{B}_{E^*}} \left(\sum_{n=1}^{+\infty} |b_n^*(x)| |u^*(a_n)| \right) \end{array}$$

$$\begin{array}{rccc} g: & E^* & \to & \mathbb{R}^+ \\ & y^* & \mapsto & \sup_{u \in \mathbb{B}_E} \left(\sum_{n=1}^{+\infty} |b_n^*(u)| |y^*(a_n)| \right) \end{array}$$

are well-defined. We prove by direct computations that f and g are seminorms on E and E^* respectively and that we have for every $x \in E$ and $x^* \in E^*$:

$$\begin{cases} \sum_{n=1}^{+\infty} |b_n^*(x)| |x^*(a_n)| \leq f(x) \|x^*\|_{E^*} \\ \sum_{n=1}^{+\infty} |b_n^*(x)| |x^*(a_n)| \leq g(x^*) \|x\|_E \end{cases}$$

Let us prove now that f and g are both countably subadditive. Indeed, let $\sum v_k$ be a convergent series in the Banach space E. Then we have for each $x^* \in E^*$:

$$\begin{aligned}\sum_{n=1}^{+\infty}\left|b_n^*\left(\sum_{k=1}^{+\infty}v_k\right)\right||x^*(a_n)| &\leq \sum_{n=1}^{+\infty}\left(\sum_{k=1}^{+\infty}|b_n^*(v_k)|\,|x^*(a_n)|\right)\\ &\leq \sum_{k=1}^{+\infty}\left(\sum_{n=1}^{+\infty}|b_n^*(v_k)|\,|x^*(a_n)|\right)\\ &\leq \left(\sum_{k=1}^{+\infty}f(v_k)\right)\|x^*\|_{E^*}\end{aligned}$$

It follows that :

$$f\left(\sum_{k=1}^{+\infty}v_k\right) \leq \sum_{k=1}^{+\infty}f(v_k)$$

Hence f is countably subadditive. We prove similarly that g is countably subadditive. Thanks to Zabreĭko's lemma [23], [17, lemma 1.6.3., page 42], that f (resp. g) is continuous on E (resp. E^*).

We consider now the mapping :

$$\begin{array}{rccc} U: & E\times E^* & \to & l^1(\mathbb{K})\\ & (x,x^*) & \mapsto & (b_n^*(x)x^*(a_n))_{n\in\mathbb{N}^*}\end{array}$$

It is clear that U is well-defined since the numerical series $\sum |b_n^*(x)|\,|y^*(a_n)|$ is convergent for each $x \in E$ and $x^* \in E^*$. Furthermore U is bilinear. Let us show that U is continuous. Indeed, let $x \in E$ and $x^* \in E^*$ and $(x_k, x_k^*)_{k\in\mathbb{N}^*}$ be a sequence in $E \times E^*$ which is convergent to (x, x^*). We have for every $k \in \mathbb{N}^*$:

$$\begin{aligned}\|U((x,x^*)) - U((x_k,x_k^*))\|_{l^1(\mathbb{K})} &= \sum_{n=1}^{+\infty}|b_n^*(x)x^*(a_n) - b_n^*(x_k)x_k^*(a_n)|\\ &= \sum_{n=1}^{+\infty}|b_n^*(x-x_k)x^*(a_n) - b_n^*(x_k)(x_k^*-x^*)(a_n)|\\ &\leq \sum_{n=1}^{\infty}|b_n^*(x-x_k)|\,|x^*(a_n)| + \sum_{k=0}^{\infty}|b_n^*(x_k)|\,|(x_k^*-x^*)(a_n)|\\ &\leq f(x-x_k)\,\|x^*\|_{E^*} + g(x_k^*-x^*)\,\|x_k\|_E\end{aligned}$$

But f is continuous on E and g is continuous on E^*. It follows that :

$$\lim_{k\to+\infty} f(x-x_k) = \lim_{k\to+\infty} g(x_k - x^*) = 0$$

Consequently :

$$\lim_{k\to+\infty}\|U((x,x^*)) - U((x_k,x_k^*))\|_{l^1(\mathbb{K})} = 0$$

Hence the bilinear mapping $U : E \times E^* \longrightarrow l^1(\mathbb{K})$ is continuous. It follows that there exists a constant $C > 0$ such that :

$$\sum_{n=1}^{+\infty}|b_n^*(x)|\,|x^*(a_n)| \leq C\,\|x\|_E\,\|x^*\|_{E^*}$$

for every $x \in E$ and $x^* \in E^*$. It follows that $\mathcal{F}$ is a besselian paire of E.

Hence the proof of the theorem is then complete. □

Remark 3.7. Using the definition of weakly unconditionally convergent series in a Banach, the theorem 3.6 can be rephrased as : *$\mathcal{F}$ is a besselian paire of E if and only if for each $x^* \in E^*$ the series $\sum x^*(a_n)\,b_n^*$ is weakly unconditionally convergent.*

Relying on the theorem 3.6, we obtain a new proof of theorem 3.1 which do not use the principle of local reflexivity.

Proof. (i) Assume that $\mathcal{F}$ is a besselian paire. Then, thanks to the theorem 3.6, the following condition holds for each $x^{**} \in E^{**}$ and $x^{***} \in E^{***}$:

$$\begin{aligned}\sum_{n=1}^{+\infty} |x^{**}(b_n^*)| \, |x^{***}(J_E(a_n))| &= \sum_{n=1}^{+\infty} |P_E(x^{***})(a_n)| \, |x^{**}(b_n^*)| \\ &< +\infty\end{aligned}$$

It follows, thanks to the theorem 3.6, that $\mathcal{F}^*$ is a besselian paire of E^*.

(ii) Assume now that $\mathcal{F}^*$ is a besselian paire of E^*. Let $x^* \in E^*$ and $x^{**} \in E^{**}$. Then there exists $y^{***} \in E^{***}$ such that $x^* = P_E(y^{***})$. It follows, by virtue of the theorem 3.6, that :

$$\begin{aligned}\sum_{n=1}^{+\infty} |y^{**}(b_n^*)| \, |x^*(a_n)| &= \sum_{n=1}^{+\infty} |y^{**}(b_n^*)| \, |P_E(y^{***})(a_n)| \\ &= \sum_{n=1}^{+\infty} |y^{**}(b_n^*)| \, |y^{***}(J_E(a_n))| \\ &< +\infty\end{aligned}$$

Consequently $\mathcal{F}$ is a besselian paire.

The new proof of theorem 3.1 is then complete. □

Proposition 3.8. *Assume that $\mathcal{F}^*$ is an unconditional Schauder frame of E^*, then $\mathcal{F}$ is a besselian paire of E.*

Proof. The assumption on $\mathcal{F}^*$ entails that the series $\sum J_E(a_n)(x^*) b_n^* = \sum x^*(a_n) b_n^*$ is weakly unconditionally convergent for each $x^* \in E^*$. Consequently, thanks to the remark 3.7, $\mathcal{F}$ is a besselian paire of E. □

Proposition 3.9. *The paire $\mathfrak{h}(p)$ is a besselian Schauder frame of $L_p([0,1])$.*

Proof. Since $(\Phi_p(h_n))_{n\in\mathbb{N}^*})$ is a Schauder basis of $(L_p([0,1]))^*$ which is isometrically isomorphic to $L_{p^*}([0,1])$ [19], [15], it follows that the paire $\mathfrak{h}(p)^* = \left(\left(\Phi_p(h_n), J_{L_p(0,1)}(h_n)\right)\right)_{n\in\mathbb{N}^*}$ is an unconditionally Schauder frame of $(L_p(0,1))^*$. Hence according to the proposition 3.4 as rephrased in the remark 3.7, the paire $\mathfrak{h}(p)$ is a besselian Schauder frame of $L_p([0,1])$. □

The following result is a generalisation to the well known James's theorem [10] which characterizes reflexive Banach spaces.

Theorem 3.10. *Assume that $\mathcal{F}$ is a besselian Schauder frame of E. Then E is reflexive if and only if $\mathcal{F}$ is shrinking and boundedly complete.*

Proof. Assume that E is reflexive. Then E^* is also reflexive [17, Corollary 1.11.17 page 104]. Consequently, E and E^* are weakly sequentially complete Banach spaces. Since $\mathcal{F}$ is a besselian paire of E, it follows, thanks to the theorem 3.1, that $\mathcal{F}^*$ is a besselian paire of E^*. It follows from proposition 3.4, that $\mathcal{F}$ is shrinking and boundedly complete.

Assume now that $\mathcal{F}$ is shrinking and boundedly complete. Let be given $x^{**} \in E^{**}$. Since $\mathcal{F}$ is boundedly complete then the series $\sum x^{**}(b_n^*) a_n$ convergent to an element x in E, that is:

$$x = \sum_{n=1}^{+\infty} x^{**}(b_n^*) a_n$$

But $\mathcal{F}$ is shrinking. Hence, thanks to the proposition 3.5, $\mathcal{F}^*$ is a Schauder frame of E^*. It follows that :

$$y^* = \sum_{n=1}^{+\infty} y^*(a_n) b_n^*, \; y^* \in E^*$$

Consequently, we have for each $x^* \in E^*$:

$$\begin{aligned} J_E(x)(x^*) &= x^*(x) \\ &= \sum_{n=1}^{+\infty} x^*(a_n)x^{**}\left(b_n^*\right) \\ &= x^{**}\left(\sum_{n=1}^{+\infty} x^*(a_n)b_n^*\right) \\ &= x^{**}(x^*) \end{aligned}$$

It follows that $x^{**} = J_E(x)$. Thus J_E is surjective. Consequently E is reflexive.

The proof of the theorem is then complete. □

Theorem 3.11. *Assume that $\mathcal{F}$ is a besselian Schauder frame of E. Then E is reflexive if and only if the spaces E and E^* are both weakly sequentially complete.*

Proof. Assume that E is reflexive. Then E and E^* are both reflexive. Hence E and E^* are both weakly sequentially complete.
Assume now that E and E^* are both weakly sequentially complete. Since $\mathcal{F}$ is a besselian Schauder frame of E, it follows, according to the proposition 3.4, that $\mathcal{F}$ is a besselian Schauder frame of E which is shrinking and boundedly complete. Consequently the theorem 3.10 entails that the Banach space E is reflexive.
The proof of the theorem is then complete. □

Corollary 3.12. *Assume that E is an infinitely dimensional Schur space which has a besselian Schauder frame. Then the dual space E^* is not weakly sequentially complete.*

Proof. Since E is an infinitely dimensional Schur space, then E is not reflexive [11, corollary 2.3.8 page 37]. But it is also assumed that E has a besselian Schauder frame. Consequently E^* is not weakly sequentially complete. □

Corollary 3.13. *The Banach space $l^\infty(\mathbb{K})$ is not weakly sequentially complete.*

Proof. It is well-known that the space $l^1(\mathbb{K})$ is a Schur space [17, example 2.5.24 pages 218-220], [11, page 37, theorem 2.3.6], [20] which is infinite dimensional and for which the paire $((e_n, u_n^*))_{n\in\mathbb{N}^*}$ is a besselian Schauder frame as proved in the remark 2.1. Hence the dual space $\left(l^1(\mathbb{K})\right)^*$ is not weakly sequentially complete. But since $\left(l^1(\mathbb{K})\right)^*$ is isometrically isomorphic to the Banach space $l^\infty(\mathbb{K})$, it follows that the Banach space $l^\infty(\mathbb{K})$ is not weakly sequentially complete. □

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