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HYPERBOLIC FRACTIONAL DIFFERENTIAL OPERATOR

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Abstract
In this paper, we introduce a new definition of fractional derivative by using the limit approach and based on hyperbolic functions for $\alpha \in (0, 1]$ which obeys classical properties including linearity, product rule and many fractional versions of other properties and results,such as Rolle's theorem, and the mean value theorem. Further, if $\alpha = 1$, the definition coincides with the classical definition of first derivative. We give some applications to fractional differential equations.

1 Introduction

Actually, fractional calculus is a part of real analysis that studies all the topics assuming arbitrary real powers α of the differential operator; so, in the present situation researchers are showing more interest to work in the field of fractional calculus as a generalization of the ordinary calculus;which is one of our main targets in this work, besides its various applications in physics ,bioengineering,see [16],and [18] and recently a climate change model has been studied et al.[10].

For the many years, many definitions of fractional derivative have been introduced by various researchers,The most known are the Riemann Liouville definition and the Caputo definition, see [12],and [11],for some applications refer to [13],[15],and [14].To mention some:

(i) Riemann-Liouville definition. For $\alpha \in [n-1, n)$, the α-derivative of f is

$$D^{\alpha}_{t_0}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} \frac{f(x)}{(t-x)^{\alpha-n+1}} dx.$$

(ii) Caputo definition. For $\alpha \in [n-1, n)$, the α-derivative of f is

$$D^{\alpha}_{t_0}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx.$$

However, the following are some of the setbacks of one definition or the other:

(i) The Riemann-Liouville derivative does not satisfy $D^{\alpha}_{a}(1) = 0$ ($D^{\alpha}_{a}(1) = 0$ for the Caputo derivative), if α is not a natural number,

(ii) All fractional derivatives do not satisfy the known formula of the derivative of the product of two functions: $D^{\alpha}_{a}(fg) = f\,(D^{\alpha}_{a}g) + g\,(D^{\alpha}_{a}f)$,

(iii) All fractional derivatives do not satisfy the known formula of the derivative of the quotient of two functions: $D^{\alpha}_{a}\left(\frac{f}{g}\right) = \frac{g(D^{\alpha}_{a}f) - f(D^{\alpha}_{a}g)}{g^2}$,

(iv) All fractional derivatives do not satisfy the chain rule: $D^{\alpha}_{a}(f \circ g) = f^{(\alpha)}(g(t))g^{(\alpha)}(t)$,

(v) All fractional derivatives do not satisfy: $D^{\alpha}D^{\beta}(f) = D^{\alpha+\beta}(f)$, in general.

Khalil et al.[1] has introduced a new derivative called the conformable fractional derivative of f of order $\alpha \in (0,1]$ and is defined by

$$T_\alpha f(t) = \lim_{h\to 0} \frac{f\left(t + ht^{1-\alpha}\right) - f(t)}{h},$$

which is a natural extension and almost satisfies all the classical properties of the usual first derivative.

T. Abdeljawad has a long history and an important contribution in this field ,for instance et al.[2] ,[3],and [17],and other articles,they have developed and implemented the fundamentals of the analytic theory of the conformable fractional calculus and studied different topics and applications in the general fractional calculus.Recently,they have combined fuzzy calculus,and conformable calculus to introduce the fuzzy conformable calculus et al. [4].

Ajay Dixit, Amit Ujlayan in [5] and [6] have introduced a U-D fractional derivative as a convex combination of the function and its first derivative, where $(D^\alpha f)(t) = (1-\alpha)f(t) + \alpha f'(t)$ for $\alpha \in (0,1]$, they have studied the main results of this operator.

Many types of fractional differential equations are studied and solved with respect to the conformable fractional differential operator,and other fractional operators ,for instance,refer to [7], [8],and[9].

Our new definition of fractional derivative is also an extension of the usual first derivative, we use the limit approach definition and is associated with the hyperbolic function $y = \cosh((1-\alpha)t), \alpha \in (0,1]$; so, we can work on hyperbolic functions which is a rich area of identities and nice properties that facilitates our computations, especially in solving fractional differential equations.

This paper is organized as follows: In section 2,we first present our new Hyperbolic fractional derivative, its main properties including linearity, product rule, quotient rule, power rule, chain rule,also we have discussed some important theorems based on our hyperbolic fractional derivative such as Rolle's theorem, and the mean value theorem .In section 3, we establish anti-derivative corresponding to the proposed derivative (Hyperbolic fractional integral),and so,as an application, we solve some well known fractional deferential equations based on our new fractional differential operator.

2 Hyperbolic Fractional Derivative

Definition 2.1. Given a function $f : [0,\infty) \to \mathbb{R}$, and $\alpha \in (0,1]$, the hyperbolic fractional derivative of order α is defined by

$$(D^\alpha f)(t) = \lim_{h\to 0} \frac{f(t + h\cosh(1-\alpha)t) - f(t)}{h}, \text{ for all } t > 0, \alpha \in (0,1].$$

We will, sometimes, write $f^{(\alpha)}$ for $(D^\alpha f)(t)$, to denote the hyperbolic fractional derivatives of f of order α. In addition, if the Hyperbolic fractional derivative of f of order α exists, then we simply say f is α-differentiable.

If f is hyperbolic α-differentiable in the interval $(0,a)$ for $a > 0$ and $\alpha \in (0,1]$ such that $\lim_{t\to 0^+}(D^\alpha f)(t)$ exists, then $(D^\alpha f)(0) = \lim_{t\to 0^+}(D^\alpha f)(t)$.

In the case of the conformable fractional derivative proposed in [1],we have two important remarks that are considered as the main motivation for our definition:

Remark 2.2. A function could be α - differentiable at a point but not differentiable, for example, take $f(t) = 2\sqrt{t}, T_{\frac{1}{2}}(f)(0) = \lim_{t\to 0^+} T_{\frac{1}{2}}(f)(t) = 1$ where $T_{\frac{1}{2}}(f)(t) = 1$ for $t > 0$. But $T_1(f)(0)$ does not exist. While for our definition, hyperbolic α-differentiable implies differentiable which is an advantage of our derivative.

Remark 2.3. If f and $T_\alpha f(t)$ are differentiable, we have

$$\frac{d}{dt} T_\alpha f(t) = (1-\alpha)t^{-\alpha} \frac{d}{dt} f(t) + t^{1-\alpha} \frac{d^2}{dt^2} f(t).$$

Therefore,this expression tends to infinity when t is very small, but this brings regularities in several mathematical problems especially when one seeks to bounded $T_\alpha f(t)$.

Now, we present the main properties and results concern to our new fractional derivative.

Theorem 2.4. *If a function $f:[0,\infty)\to\mathbb{R}$ is hyperbolic α-differentiable at $t_0>0$, then f is continuous at t_0.*

Proof. Since $f(t_0+h\cosh(1-\alpha)t_0)-f(t_0)=\frac{f(t_0+h\cosh(1-\alpha)t_0)-f(t_0)}{h}\cdot h$, then

$$\lim_{h\to 0} f(t_0+h\cosh(1-\alpha)t_0)-f(t_0)=\lim_{h\to 0}\frac{f(t_0+h\cosh(1-\alpha)t_0)-f(t_0)}{h}\cdot\lim_{h\to 0}h.$$

Let $\varepsilon=h\cosh((1-\alpha)t_0)$, then

$$\lim_{h\to 0} f(t_0+h\cosh(1-\alpha)t_0)-f(t_0),$$

$$=\cosh((1-\alpha)t_0)\lim_{\varepsilon\to 0}\frac{f(t_0+\varepsilon)-f(t_0)}{\varepsilon}\cdot\lim_{h\to 0}h,$$

$$=\cosh((1-\alpha)t_0)\,f'(t_0)\cdot 0=0.$$

Which implies $\lim_{\varepsilon\to 0} f(t_0+\varepsilon)=f(t_0)$,hence f is continuous at t_0. □

Theorem 2.5. *Let f,g be hyperbolic α-differentiable at a point $t>0$, then for $0<\alpha\le 1$*

(i) $D^\alpha(af+bg)=a(D^\alpha f)+b(D^\alpha g)$ *for all* $a,b\in\mathbb{R}$,

(ii) $D^\alpha(t^p)=p\cosh((1-\alpha)t)t^{p-1}$ *for all* $p\in\mathbb{R}$,

(iii) $D^\alpha(\lambda)=0$ *for all constant function* $f(t)=\lambda$,

(iv) $D^\alpha(fg)=f(D^\alpha g)+g(D^\alpha f)$,

(v) $D^\alpha\left(\frac{f}{g}\right)=\frac{g(D^\alpha f)-f(D^\alpha g)}{g^2}$,

(vi) $D^\alpha(f\circ g)(t)=f'(g(t))D^\alpha(g)(t)$,

(vii) In addition, if f is differentiable, then $(D^\alpha f)(t)=\cosh((1-\alpha)t)f'(t)$.

Proof. In fact, we need only to prove (vii) and (iv), since the other rules are direct consequences.

(vii):

$$(D^\alpha f)(t)=\lim_{h\to 0}\frac{f(t+h\cosh(1-\alpha)t)-f(t)}{h}.$$

Let $\varepsilon=h\cosh((1-\alpha)t)$. Therefore

$$(D^\alpha f)(t)=\cosh((1-\alpha)t)\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon)-f(t)}{\varepsilon},$$
$$=\cosh((1-\alpha)t)f'(t).$$

(iv):

$$(D^\alpha fg)(t)=\lim_{h\to 0}\frac{f(t+h\cosh(1-\alpha)t)g(t+h\cosh(1-\alpha)t)-f(t)g(t)}{h},$$

$$=\lim_{h\to 0}\frac{f(t+h\cosh(1-\alpha)t)g(t+h\cosh(1-\alpha)t)-f(t)g(t+h\cosh(1-\alpha)t)+f(t)g(t+h\cosh(1-\alpha)t)-f(t)g(t)}{h},$$

$$=\lim_{h\to 0}\frac{f(t+h\cosh(1-\alpha)t)-f(t)}{h}g(t+h\cosh(1-\alpha)t)+f(t)\lim_{h\to 0}\frac{g(t+h\cosh(1-\alpha)t)-g(t)}{h},$$

$$=(D^\alpha f)(t)\lim_{h\to 0}g(t+h\cosh(1-\alpha)t)+f(t)(D^\alpha g)(t).$$

Since g is continuous at t, then $\lim_{h\to 0}g(t+h\cosh(1-\alpha)t)=g(t)$,hence

$$D^\alpha(fg)=f(D^\alpha g)+g(D^\alpha f).$$

□

Example 2.6. It is worth noting the following hyperbolic fractional derivatives of certain functions for $\alpha \in (0,1]$.

(i) $D^{\alpha}(sinh(1-\alpha)t) = (1-\alpha)\cosh^2((1-\alpha)t)$,

(ii) $D^{\alpha}(tanh(1-\alpha)t) = (1-\alpha)\,\mathrm{sech}((1-\alpha)t)$,

(iii) $D^{\alpha}(2\cosh(1-\alpha)t) = (1-\alpha)\sinh(2(1-\alpha)t)$,

(iv) $D^{\alpha}(2\tan^{-1}\left(e^{(1-\alpha)t}\right)) = 1-\alpha$.

We generalize the definition of hyperbolic fractional derivative for $\alpha \in (n, n+1], n \in \mathbb{N}$.

Definition 2.7. Let $\alpha \in (n, n+1]$, for some $n \in \mathbb{N}$, and f is n-differentiable at $t > 0$, then the hyperbolic $\alpha-$ fractional derivative is defined by:

$$(D^{\alpha} f)(t) = \lim_{h\to 0} \frac{f^{(n)}(t + h\cosh(1-\alpha)t) - f^{(n)}(t)}{h}$$

if the limit exists.

Remark 2.8. As a direct consequence of this definition, we can show that

$$(D^{\alpha} f)(t) = \cosh(((n+1)-\alpha)t) f^{(n+1)},$$

where $\alpha \in (n, n+1]$ and f is $(n+1)$- differentiable at $t > 0$.

The previous definitions of fractional derivative Riemann–Liouville and Caputo do not enable us to study the analysis of $\alpha-$ differentiable functions. However, our definition makes it possible to prove basic analysis theorems such as Rolle's theorem and the mean value theorem.

Theorem 2.9. *Rolle's theorem for hyperbolic fractional differentiable functions.*
Let $a > 0$ and $f : [0,\infty) \to \mathbb{R}$ be a given function that satisfies

(i) f is continuous on $[a,b]$,

(ii) f is α-differentiable for some $\alpha \in (0,1]$,

(iii) $f(a) = f(b)$.

Then there exists $c \in (a,b)$, such that $f^{(\alpha)}(c) = 0$.

Proof. Suppose f is continuous on $[a,b]$, and $f(a) = f(b)$, then there is a local extreme point $c \in (a,b)$. Without loss of generality, assume c is a point of local minimum. So,

$$\begin{aligned}(D^{\alpha} f)(c) &= \lim_{h\to 0^+} \frac{f(c + h\cosh(1-\alpha)c) - f(c)}{h} \\ &= \lim_{h\to 0^-} \frac{f(c + h\cosh(1-\alpha)c) - f(c)}{h}.\end{aligned}$$

But, the first limit is nonnegative, and the second limit is nonpositive. Hence $(D^{\alpha} f)(c) = 0$. □

Theorem 2.10. *Mean value theorem for hyperbolic fractional differentiable functions.*
Let $a > 0$ and $f : [0,\infty) \to \mathbb{R}$, be a given function that satisfies

(i) f is continuous on $[a,b]$,

(ii) f is α-differentiable for some $\alpha \in (0,1]$.

Then there exists $c \in (a,b)$, such that

$$(D^{\alpha} f)(c) = \frac{f(b) - f(a)}{\frac{2}{1-\alpha}\left[\tan^{-1}\left(e^{(1-\alpha)b}\right) - \tan^{-1}\left(e^{(1-\alpha)a}\right)\right]}.$$

Proof. Consider the function: $g(x) = f(x) - f(a) -$

$$\frac{f(b)-f(a)}{\frac{2}{1-\alpha}\left[\tan^{-1}\left(e^{(1-\alpha)b}\right)-\tan^{-1}\left(e^{(1-\alpha)a}\right)\right]}\left[\frac{2}{1-\alpha}\tan^{-1}\left(e^{(1-\alpha)x}\right)-\frac{2}{1-\alpha}\tan^{-1}\left(e^{(1-\alpha)a}\right)\right].$$

Then,

$$g(a) = g(b) = 0,$$

hence by Rolle's theorem, there exists $c \in (a, b)$, such that

$$g^{(\alpha)}(c) = 0.$$

Using the fact

$$D^{\alpha}\left(\frac{2\tan^{-1}\left(e^{(1-\alpha)x}\right)}{1-\alpha}\right) = 1,$$

we get

$$\left(D^{\alpha} f\right)(c) = \frac{f(b)-f(a)}{\frac{2}{1-\alpha}\left[\tan^{-1}\left(e^{(1-\alpha)b}\right)-\tan^{-1}\left(e^{(1-\alpha)a}\right)\right]}.$$

□

In order to work on the calculus of the hyperbolic fractional derivative, we need to define a corresponding antiderivative.

3 Hyperbolic Fractional Integral and applications

We introduce the hyperbolic α-fractional integral as follows:

Definition 3.1. Let $\alpha \in (0, 1]$ and $a \geq 0$, let f be a function defined on $(a, t]$, then the hyperbolic $\alpha-$ fractional integral of f is defined by:

$$I_{\alpha}^{a}(f)(t) = \int_{a}^{t} \operatorname{sech}((1-\alpha)s) f(s) ds$$

Example 3.2. Evaluate the following hyperbolic fractional integrals.

(i) $I_{\frac{1}{2}}^{1}\left(\cosh\left(\frac{1}{2}t\right)\right) = \int_{1}^{t} 1 ds = t - 1.$

(ii) $I_{\frac{1}{2}}^{0}(1) = \int_{0}^{t} \operatorname{sech}\left(\frac{1}{2}s\right) ds = 4\tan^{-1}\left(e^{\frac{1}{2}t}\right) - \pi.$

Remark 3.3. Since $\operatorname{sech}((1-\alpha)s)$ is continuous and bounded,then if $f(s)$ is continuous and bounded on $(a, t]$,then $I_{\alpha}^{a}(f)(t) = \int_{a}^{t} \operatorname{sech}((1-\alpha)s) f(s) ds$ is convergent;which is an extra advantage to the hyperbolic fractional integral.

The following result shows the inverse property of the hyperbolic fractional operator.

Theorem 3.4. *If $f : [0, \infty) \to \mathbb{R}$ is any continuous function in the domain of I_{α} and $0 < \alpha \leq 1$. then, for $t > a$, we have $D_{a}^{\alpha}(I_{\alpha}^{a} f(t)) = f(t)$.*

Proof. Since f is continuous on, then $I_{\alpha}^{a} f(t)$ is clearly differentiable.
Hence, $D_{a}^{\alpha}(I_{\alpha}^{a} f(t)) = \cosh((1-\alpha)t)\frac{d}{dt}\left(I_{\alpha}^{a} f(t)\right),$

$$\begin{aligned}
&= \cosh((1-\alpha)t)\frac{d}{dt}\int_{a}^{t} \operatorname{sech}((1-\alpha)s) f(s) ds,\\
&= \cosh((1-\alpha)t)\operatorname{sech}((1-\alpha)t) f(t),\\
&= f(t).
\end{aligned}$$

□

As an application, we solve certain well known fractional differential equations with respect to our hyperbolic differential operator with $\alpha \in (0,1]$.

Definition 3.5. The general form of the linear hyperbolic fractional differential equation of order α is given by:

$$y^{(\alpha)} + p(x)y = f(x), \tag{3.1}$$

where $p(x)$ and $f(x)$ are α-differentiable functions.

Clearly equation (3.1)is equivalent to

$$\cosh((1-\alpha)x)y' + p(x)y = f(x). \tag{3.2}$$

If we Divide equation (3.2) by $\cosh((1-\alpha)x)$, we get

$$y' + p(x)\,\text{sech}((1-\alpha)x)y = f(x)\,\text{sech}((1-\alpha)x). \tag{3.3}$$

Now, equation (3.3) is a first order linear ordinary differential equation that has the general solution

$$y = \frac{1}{\mu(x)}\left(I^{\alpha}(f(x)\mu(x)\right),$$

where $\mu(x)$ is the integrating factor given by:

$$\mu(x) = e^{I^{\alpha}(p(x))}.$$

Example 3.6. Solve the hyperbolic fractional differential equation
$y^{(\alpha)} + (1-\alpha)\sinh((1-\alpha)x)y = 1.$

Solution. *This equation is transformed to the linear equation*

$$y' + (1-\alpha)\tanh((1-\alpha)x)y = \text{sech}((1-\alpha)x).$$

We compute the integrating factor

$$\mu(x) = e^{\int(1-\alpha)\tanh((1-\alpha)x)dx} = e^{\ln(\cosh((1-\alpha)x)} = \cosh((1-\alpha)x).$$

Hence, the general solution is given by

$$y = \text{sech}((1-\alpha)x)\int 1dx = \text{sech}((1-\alpha)x)(x+c),$$
$$= x\,\text{sech}((1-\alpha)x) + c\,\text{sech}((1-\alpha)x).$$

Definition 3.7. The general form of the Bernoulli hyperbolic fractional differential equation of order α is given by:

$$y^{(\alpha)} + p(x)y = f(x)y^n, n \neq 0, 1. \tag{3.4}$$

where $p(x)$ and $f(x)$ are α-differentiable functions.

To solve equation (3.4), we use the substitution $z = y^{1-n}$ that reduce it to linear hyperbolic fractional differential equation

$$z^{(\alpha)} + (1-n)p(x)z = (1-n)f(x), \tag{3.5}$$

that has a general solution

$$y = [\frac{1}{\mu(x)}(I^{\alpha}((1-n)f(x)\mu(x))]^{\frac{1}{1-n}},$$

where the integrating factor

$$\mu(x) = e^{I^{\alpha}((1-n)p(x))}.$$

Example 3.8. Solve the hyperbolic fractional differential equation

$$y^{(\alpha)} + (1-\alpha)\operatorname{sech}((1-\alpha)x)y = (1-\alpha)\frac{\operatorname{sech}((1-\alpha)x)}{y}.$$

Solution. *The substitution $z = y^2$ reduces the equation into linear hyperbolic fractional differential equation*

$$z^{(\alpha)} + 2(1-\alpha)\operatorname{sech}((1-\alpha)x)z = 2(1-\alpha)\operatorname{sech}((1-\alpha)x).$$

We compute the integrating factor

$$\mu(x) = e^{I^{\alpha}(2(1-\alpha)\operatorname{sech}((1-\alpha)x)} = e^{\int 2(1-\alpha)\operatorname{sech}^2((1-\alpha)x)dx} = e^{2\tanh((1-\alpha)x)},$$

hence, the general solution is given by

$$\begin{aligned} y &= \left(e^{-2\tanh((1-\alpha)x)}\left(I^{\alpha}\left(2(1-\alpha)e^{2\tanh((1-\alpha)x)}\operatorname{sech}((1-\alpha)x)\right)\right)^{\frac{1}{2}}, \\ &= \left(e^{-2\tanh((1-\alpha)x)}\left(\int\left(2(1-\alpha)e^{2\tanh((1-\alpha)x)}\operatorname{sech}^2((1-\alpha)x)\right)\right)^{\frac{1}{2}}, \\ &= \left(e^{-2\tanh((1-\alpha)x)}\left(e^{2\tanh((1-\alpha)x)} + c\right)\right)^{\frac{1}{2}} = \sqrt{1 + ce^{-2\tanh((1-\alpha)x)}}. \end{aligned}$$

Definition 3.9. The general form of the Riccati hyperbolic fractional differential equation of order α is given by:

$$y^{(\alpha)} = h(x) + k(x)y + u(x)y^2, \tag{3.6}$$

where $h(x), k(x)$, and $u(x)$ are α-differentiable functions.

To solve equation (3.6).If a specific solution y_1 is known, then the general solution, which comes in the form of $y = y_1 + z$, where z is the general solution to the following Bernoulli hyperbolic fractional differential equation

$$z^{(\alpha)} + (-k(x) - 2u(x)y_1)\, z = u(x)z^2. \tag{3.7}$$

Example 3.10. Find the general solution of the hyperbolic fractional differential equation

$$y^{(\alpha)} = \frac{\left(-2x^4 + x^2y + y^2\right)\left(e^{(1-\alpha)x} + e^{(\alpha-1)x}\right)}{2x^3},$$

given that $y_1 = -x^2$ is a solution.

Solution. *We can simplify the equation to get*

$$y^{(\alpha)} = \frac{\left(-2x^4 + x^2y + y^2\right)}{x^3}\cosh((1-\alpha)x) = \cosh((1-\alpha)x)\left(-2x + x^{-1}y + x^{-3}y^2\right),$$

which is Riccati equation. To solve it, we solve first the corresponding Bernoulli equation (3.7)*.After doing all the simplifications, we get*

$$z^{(\alpha)} + \left(x^{-1}\cosh((1-\alpha)x)\right)z = x^{-3}\cosh((1-\alpha)x)z^2,$$

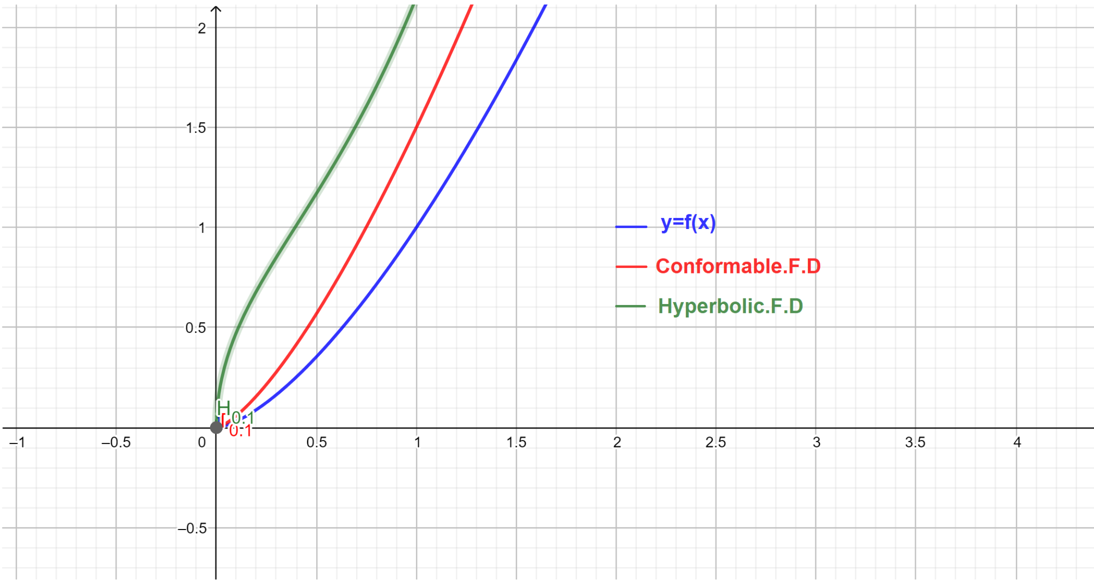
that has a general solution

$$z = \left(x(I^{\alpha}(-x^{-4}\cosh((1-\alpha)x)\right)^{-1} = \left(x(\frac{x^{-3}}{3} + c\right)^{-1} = \frac{3x^2}{1+3cx^3},$$

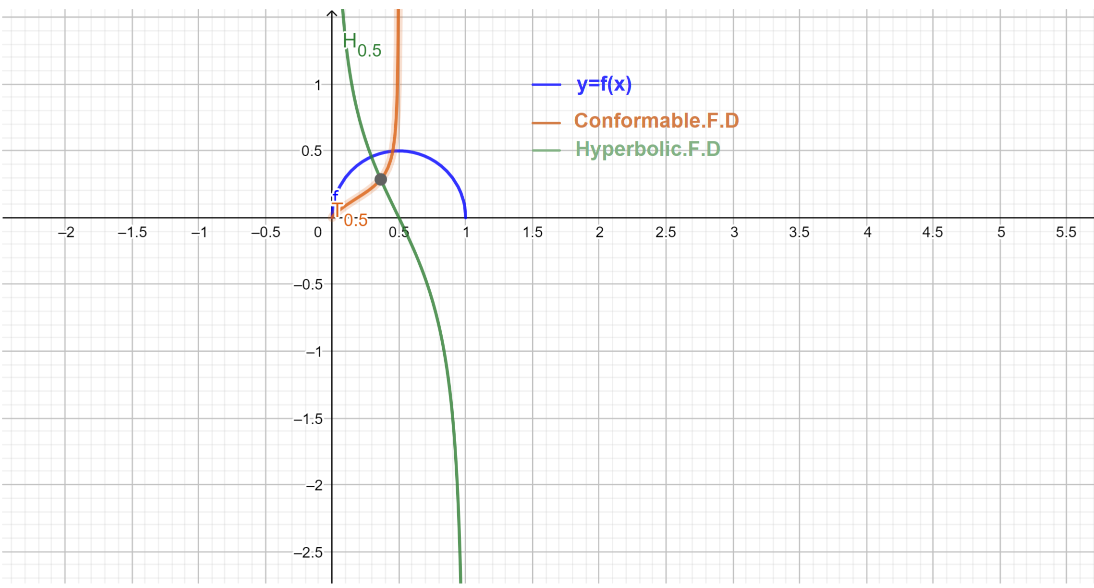
so the general solution for Riccati equation is

$$y = y_1 + z = -x^2 + \frac{3x^2}{1+3cx^3} = \frac{2x^2 - 3cx^5}{1+3cx^3}.$$

Finally,we present some graphical comparison between the conformable fractional derivative vs the hyperbolic fractional derivative for two functions with different values of α.



Figure(1)



Figure(2)

Figure(1) represents the graphs of $f(x) = x^{\frac{3}{2}}$,and its conformable fractional derivative vs. its hyperbolic fractional derivative with $\alpha = 0.1$.

Figure(2) represents the graphs of $f(x) = \sqrt{x - x^2}$,and its conformable fractional derivative vs. its hyperbolic fractional derivative with $\alpha = 0.5$.

References

[1] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, "A new definition of fractional derivative," Journal of Computational and Applied Mathematics, vol.264, pp.65-70,2014.

[2] T. Abdeljawad, "On conformable fractional calculus," Journal of Computational and Applied Mathematics, vol.279,pp.57-66, 2015 .

[3] Thabet Abdeljawad, Qasem M. Al-Mdallal, Fahd Jarad, Fractional logistic models in the frame of fractional operators generated by conformable derivatives, Chaos,Solitons and Fractals, Vol.119,pp. 94-101,2019.

[4] Younus, Awais, Muhammad Asif, Usama Atta, Tehmina Bashir, and Thabet Abdeljawad. . “Some Fundamental Results on Fuzzy Conformable Differential Calculus”. Journal of Fractional Calculus and Nonlinear Systems 2 (2),pp.31-61,2021.

[5] Dixit, A. & Ujlayan, A.The theory of UD derivative and its applications. TWMS Journal of Applied and Engineering Mathematics, 11(2),pp.350-358,2021.

[6] Dixit, Ajay, Amit Ujlayan, and Priyanka Ahuja. "On the properties of The UD differential and integral operator." Journal/ MESA 11(2),pp.291-300,(2020).

[7] M. Ilie, J. Biazar, Z. Ayati, General solution of Bernoulli and Riccati fractional differential equations based on conformable fractional derivative, International Journal of Applied Mathematical Research, 6(2),pp.49-51,2017.

[8] R. Khalil, M.A. Al Horani, D. Anderson. Undetermined coefficients for local differential equations.J Math Comput Sci,16, pp. 140-146,2016.

[9] M. Al Horani, M.A. Hammad, R. Khalil. Variations of parameters for local fractional nonhomogeneous linear-differential equations J Math Comput Sci, 16 , pp.147-153 ,2016.

[10] Anwarud Din, Faiz Muhammad Khan, Zia Ullah Khan, Abdullahi Yusuf, Taj Munir, The mathematical study of climate change model under nonlocal fractional derivative, Partial Differential Equations in Applied Mathematics, Volume 5, 2022.

[11] K. S. Miller, An Introduction to Fractional Calculus and Fractional Differential Equations, J. Wiley and Sons, New York, NY, USA, 1993.

[12] A. Kilbas, H. Srivastava, and J. Trujillo, "Theory and applications of fractional differential equations," in Math. Studies, North-Holland, New York, NY, USA, 2006.

[13] K. Oldham and J. Spanier, The Fractional Calculus, Theory and Applications of Differentiation and Integration of Arbitrary Order, Academic Press, Cambridge, MA, USA, 1974.

[14] I. Podlubny, Fractional Differential Equations, Academic Press, Cambridge, MA, USA, 1999.

[15] K. S. Miller, B. Ross, An introduction to fractional calculus and fractional differential equations, John Wiley and Sons, New York, 1993.

[16] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[17] A. Younas, T. Abdeljawad, R. Batool, A. Zehra, M. A. Alqudah, Linear conformable differential system and its controllability, Adv. Differ. Equ., 2020 (2020), 449.

[18] R. L. Magin, Fractional calculus in Bioengineering, House Publishers, Redding, 2006.

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