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SKEW SPECTRUM OF SOME GRAPHS RELATED TO THE PETERSEN GRAPH

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Abstract In this paper we obtain the skew spectrum of the directed Petersen graph, the generalized directed Petersen graph $\widetilde{P}(n,k)$ and the directed I-graph $\widetilde{I}(n,j,k)$ with respect to some orientation on their edges. Some bounds on the skew eigenvalues based on the lexicographic ordering of complex numbers are also given.

1 Introduction

A directed graph(digraph) $\widetilde{D}$ consists of a finite non empty set $V(\widetilde{D})$ of vertices and a finite set $A(\widetilde{D})$ of arcs which are ordered pairs of distinct vertices.

The skew adjacency matrix $S_A(\widetilde{D}) = [s_{ij}]$ of a digraph $\widetilde{D}$ is the $n \times n$ matrix, where

$$s_{ij} = \begin{cases} 1 & \text{, if there is an arc from } v_i \text{ to } v_j \\ -1 & \text{, if there is an arc from } v_j \text{ to } v_i \\ 0 & \text{, otherwise} \end{cases}$$

All the eigenvalues of $S_A(\widetilde{D})$ are purely imaginary or zero because the matrix is skew symmetric. The skew energy of $\widetilde{D}$ is sum of the absolute values of eigenvalues of $S_A(\widetilde{D})$.

The skew adjacency matrix of an oriented graph G^σ was first introduced in 1947 by Tutte and William T [1]. In [2], Adiga, Balakrishnan, and So gave the definition of skew energy for oriented graphs and established some bounds for the skew energy of oriented graph G^σ in terms of its order, size, as well as the maximum degree of its underlying graph. Anuradha and Balakrishnan studied the skew spectra of oriented bipartite graphs in [3]. The study of 3,4-regular oriented graphs with optimum skew energy was done in [4] and [5] respectively. In [6] and [7] authors describe the skew spectra and skew energy of various products of graphs.

The Petersen graph serves as a useful example and counterexample for many problems in graph theory. The generalized Petersen graph is a graph with vertex set $\{u_i, v_i | 1 \leq i \leq n\}$ and edge set $\{u_iu_{i+1}, u_iv_i, v_iv_{i+k} | 1 \leq i \leq n\}$, where the subscripts are expressed as integers modulo $n(n \geqslant 5)$ and k is the 'skip'. Note that $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$.This family of graph was introduced in 1950 by H.S.M. Coxeter [8]. In 2011 Ralucca Gera and Pantelimon Stanica completely describe the spectrum of the generalized Petersen graph [9]. In Foster Census [10], the class of I-graphs was introduced as a generalization of generalized Petersen graphs. In 2018 S.S. Allana, T.M. Vinagre, de Oliveira and Cybele completely describe the spectrum of the I-graph [11].

In this paper we use a particular orientation on the edges of Petersen graph and generalize the graph with same orientation to generalized directed Petersen graph $\widetilde{P}(n,k)$ and again to the directed I-graph $\widetilde{I}(n,j,k)$. Then we completely describe the skew spectrum of these graphs and also find some bounds for the skew eigenvalues in some particular cases. The discussion in this paper is based upon the following definitions and lemmas.

Definition 1.1. [13] A circulant matrix is a square matrix in which each row vector is rotated one element to the right relative to the preceding row vector. If $a_1, a_2, a_3, \cdots, a_n$ are the elements of the circulant matrix of order n then the eigenvalues are given by $\lambda = \{a_1 + a_2\omega + a_3\omega^2 + \cdots + a_n\omega^{n-1} \mid \omega^n = 1\}$.

Lemma 1.2. *[2] The skew spectrum of directed cycle is* $2i \sin \frac{2\pi r}{n}$ *where* $0 \le r \le n-1$.

Definition 1.3. [12] Let A be a set equipped with a total order $<$, and let $A^n = A \times \cdots \times A$ be the n-fold Cartesian product of A. Then the lexicographic order $<$ on A^n is defined as follows: If $a = (a_1, \cdots, a_n) \in A^n$ and $b = (b_1, \cdots, b_n) \in A^n$, then $a < b$ if $a_1 < b_1$ or

$$\begin{gathered} a_1 = b_1 \\ \vdots \\ a_k = b_k \\ a_{k+1} < b_{k+1} \end{gathered}$$

for some $k = 1, \cdots, n-1$.

2 An orientation on Petersen graph

In this section we use an orientation on the Petersen graph to create the directed Petersen graph as shown in Figure 1 in such a way that $\widetilde{P}(5,2)$ has vertex set $\{v_i, u_i; i = 1, 2, \cdots, 5\}$ and arc set $\{v_iv_{i+1}, v_iu_i, u_iu_{i+2}; i = 1, 2, \cdots, 5\}$ where the subscripts are expressed as integers modulo 5.

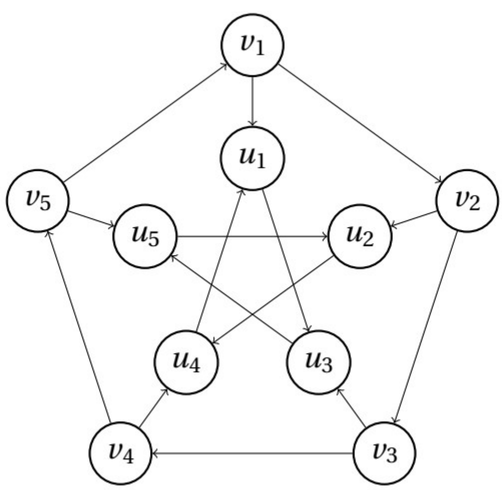

Figure 1. The directed Petersen graph $\widetilde{P}(5,2)$

3 Generalized directed Petersen graph

Generalized directed Petersen graph $\widetilde{P}(n,k)$ has vertex set $\{v_i, u_i; 0 \le i \le n\}$ and arc set $\{v_iv_{i+1}, v_iu_i, u_iu_{i+k}; 0 \le i \le n\}$ where the subscripts are expressed as integers modulo $n (n \ge 5)$, and k is the 'skip'. Note that $k \le \lfloor \frac{n-1}{2} \rfloor$, because of the obvious isomorphism $\widetilde{P}(n,k) \le \widetilde{P}(n, n-k)$. Let $\widetilde{A}(n,k)$ and $\widetilde{B}(n,k)$ are the subgraphs of $\widetilde{P}(n,k)$ consisting of the vertices $\{v_i; 0 \le i \le n\}$ and $\{u_i; 0 \le i \le n\}$ respectively. We will call $\widetilde{A}(n,k)$ and $\widetilde{B}(n,k)$ as the outer and inner subgraph of $\widetilde{P}(n,k)$ respectively . We display in Figure 2 the generalized directed Petersen graph $\widetilde{P}(10,4)$.

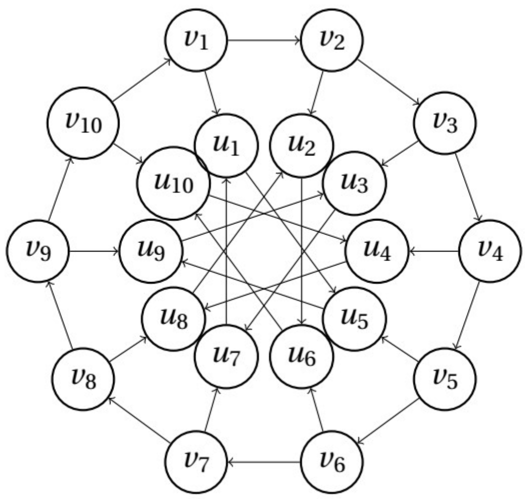

Figure 2. The generalized directed Petersen Graph $\widetilde{P}(10,4)$

Lemma 3.1. *The skew adjacency matrix of the generalized directed Petersen graph $\widetilde{P}(n,k)$ of order $2n$ has the block form*

$$S_A(\widetilde{P}(n,k)) = \begin{bmatrix} C_k^n & I_n \\ -I_n & C_n \end{bmatrix}$$

where I_n is the identity matrix of order n. C_n, C_k^n are circulant matrices, with
$C_n = circ(0,1,0,0,\cdots,0,-1)$ *and* $C_k^n = circ(\overbrace{0,\cdots,0}^{k\ times},1,0,\cdots,0,-1,\overbrace{0,\cdots,0}^{k-1\ times})$ *being the skew adjacency matrix for $\widetilde{A}(n,k)$ and $\widetilde{B}(n,k)$ respectively. Thus, C_n is the skew adjacency matrix of directed cycle of order n and C_k^n is the union of m directed cycles $C_{n/m}$ on n/m vertices, where $m = gcd(n,k)$.*

Proof. In generalized directed Petersen graph the outer subgraph $\widetilde{A}(n,k)$ is the directed cycle of order n and the inner subgraph $\widetilde{B}(n,k)$ consists of m connected components each isomorphic to directed cycle of order n/m. Then the corresponding skew adjacency matrix of $\widetilde{A}(n,k)$ and $\widetilde{B}(n,k)$ are the circulant matrix C_n and $C_{n/m}$ respectively. If we label the vertices of $\widetilde{A}(n,k)$ with consecutive numbers $1,2,\cdots,n$ and the vertices of $\widetilde{B}(n,k)$ with $i, i+k, i+2k,\cdots$ (where $i+dk$ is understood as $1+(i-1+dk)(mod n)$) then the skew adjacency matrix of $\widetilde{P}(n,k)$ has the claimed form.

□

Lemma 3.2. *The eigenvalues corresponding to the circulant matrix C_k^n are $2i\sin\frac{2\pi rk}{n}, 0 \le r \le n-1$.*

Proof. By definition 1.1,

eigenvalues of circulant matrix $C_k^n = circ(\overbrace{0,\cdots,0}^{k\ times},1,\overbrace{0,\cdots,0}^{n-2k-1\ times},-1,\overbrace{0,\cdots,0}^{k-1\ times})$ is

$$\begin{aligned}
\lambda &= (\omega^k - \omega^{n-k}) \\
&= \cos\tfrac{2\pi rk}{n} + i\sin\tfrac{2\pi rk}{n} - \cos\tfrac{2\pi r(n-k)}{n} - i\sin\tfrac{2\pi r(n-k)}{n} \\
&= \cos\tfrac{2\pi rk}{n} + i\sin\tfrac{2\pi rk}{n} - \cos(2\pi r - \tfrac{2\pi rk}{n}) - i\sin(2\pi r - \tfrac{2\pi rk}{n}) \\
&= \cos\tfrac{2\pi rk}{n} + i\sin\tfrac{2\pi rk}{n} - \cos\tfrac{2\pi rk}{n} - (-i\sin\tfrac{2\pi rk}{n}) \\
&= 2i\sin\tfrac{2\pi rk}{n}
\end{aligned}$$

where $r = 0, 1, \cdots, n-1$. □

Theorem 3.3. *The eigenvalues of the generalized directed Petersen graph $\widetilde{P}(n,k)$ are given by*

$$\delta = \left(i\sin\tfrac{2\pi r}{n} + i\sin\tfrac{2\pi rk}{n}\right) \pm i\sqrt{\left(\sin\tfrac{2\pi r}{n} - \sin\tfrac{2\pi rk}{n}\right)^2 + 1} \quad ,0 \le r \le n-1$$

Proof. We first consider the case of $m = gcd(n,k) = 1$
Since $m = 1$, C_k^n is the skew adjacency matrix of a directed cycle isomorphic to C_n and so it is similar to C_n. There exists a permutation matrix P, such that $P^{-1}C_k^nP = C_n \Leftrightarrow C_n$ and C_k^n will have the same eigenvalues and eigenvectors. Then $\alpha_r = 2i\sin\frac{2\pi r}{n}$ and $\beta_r = 2i\sin\frac{2\pi rk}{n}$ are eigenvalues corresponding to the eigenvector say $\mathbf{v}_r = (1, \zeta_n^r, \cdots, \zeta_n^{(n-1)r})^t$. We are looking for an eigenvector for $S(\widetilde{P}(n,k))$ of the form $\mathbf{w}_r = (a_r\mathbf{v}_r, \mathbf{v}_r)^t$ where a_r will be determined later. With this value for $\mathbf{w}_r$ we need to find an eigenvalue δ such that

$$\begin{pmatrix} C_k^n & I_n \\ -I_n & C_n \end{pmatrix} \begin{pmatrix} a_r\mathbf{v}_r \\ \mathbf{v}_r \end{pmatrix} = \delta \begin{pmatrix} a_r\mathbf{v}_r \\ \mathbf{v}_r \end{pmatrix}$$

Thus we get the system

$$\begin{cases} a_rC_k^n\mathbf{v}_r + \mathbf{v}_r = \delta a_r\mathbf{v}_r \\ -a_r\mathbf{v}_r + C_n\mathbf{v}_r = \delta\mathbf{v}_r \end{cases} \Leftrightarrow \begin{cases} a_r\beta_r\mathbf{v}_r + \mathbf{v}_r = \delta a_r\mathbf{v}_r \\ -a_r\mathbf{v}_r + \alpha_r\mathbf{v}_r = \delta\mathbf{v}_r \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} a_r(\beta_r - \delta)\mathbf{v}_r = -\mathbf{v}_r \\ (\alpha_r - \delta)\mathbf{v}_r = a_r\mathbf{v}_r \end{cases}$$

and so $(\beta_r - \delta)(\alpha_r - \delta) = -1$. That is the eigenvalue δ satisfy the quadratic equation $\delta^2 - (\alpha_r + \beta_r)\delta + \alpha_r\beta_r + 1 = 0$.

Similarly for the case $m \ge 1$ the eigenvectors $\mathbf{w}_r$ must have the form $\mathbf{w}_r = (a_1\mathbf{v}_r', a_2\mathbf{v}_r', \cdots, a_m\mathbf{v}_r', \mathbf{v}_r)^t$, with $\mathbf{v}_r$ as before and $\mathbf{v}_r' = (1, \zeta_n^r, \cdots, \zeta_n^{(n'-1)r})^t$ where $n' = \frac{n}{m}$ for some appropriate multiplier a_i. A similar system to the one for $m = 1$ case will be obtained and the eigenvalue satisfies the same polynomial will be found. Therefore to find the eigenvalue of generalized directed Petersen graph $\widetilde{P}(n,k)$, we want to solve the quadratic equation $\delta^2 - (\alpha_r + \beta_r)\delta + \alpha_r\beta_r + 1 = 0$.

$$\delta = \frac{(\alpha_r + \beta_r) \pm \sqrt{(\alpha_r + \beta_r)^2 - 4(\alpha_r\beta_r + 1)}}{2}$$

$$\delta = \frac{(\alpha_r + \beta_r) \pm \sqrt{(\alpha_r - \beta_r)^2 - 4}}{2}.$$

Substitute the value for α_r and β_r and simplify we get,

$$\delta = \left(i\sin\frac{2\pi r}{n} + i\sin\frac{2\pi rk}{n}\right) \pm i\sqrt{\left(\sin\frac{2\pi r}{n} - \sin\frac{2\pi rk}{n}\right)^2 + 1} \quad ,0 \le r \le n-1$$

□

We know that $\mathbb{C}$ is an unordered field and complex numbers cannot be 'compared' in the sense that one is less than or greater than another. So bounds of these skew eigenvalues make no sense in the complex field. Here we compare the skew eigenvalues by using lexicographic ordering and try to find some bounds, since the lexicographic order yields a total order on the field of complex numbers.

4 Bounds on the eigenvalues of $\widetilde{P}(n,2)$

In this section we find the bounds on the eigenvalues of $\widetilde{P}(n,2)$.
Spectra of $\widetilde{P}(n,2)$ is

$$\delta = \left(i\sin\frac{2\pi r}{n} + i\sin\frac{4\pi r}{n}\right) \pm i\sqrt{\left(\sin\frac{2\pi r}{n} - \sin\frac{4\pi r}{n}\right)^2 + 1} \quad ,0 \le r \le n-1$$

Substitute $\sin 2\theta = 2\sin\theta cos\theta$

$$\delta = \left(i\sin\frac{2\pi r}{n} + i2\sin\frac{2\pi r}{n}\cos\frac{2\pi r}{n}\right) \pm i\sqrt{\left(\sin\frac{2\pi r}{n} - 2\sin\frac{2\pi r}{n}\cos\frac{2\pi r}{n}\right)^2 + 1}$$

To find the bounds on the eigenvalues, we look for the extreme points of the function

$$F_{\pm}(x) = if_{\pm}(x) = i(x + 2x\sqrt{1-x^2} \pm \sqrt{(x - 2x\sqrt{1-x^2})^2 + 1})$$

in the interval $-1 \le x \le 1$. There is no way to expect exact or even tight values, since our sequence $\frac{2\pi r}{n}, 0 \le r \le n-1$ is finite, so $\sin\frac{2\pi r}{n}$ is not dense in the interval $-1 \le x \le 1$; we can only expect lower and upper bounds. Because any differentiable function in a compact domain reaches its extreme point either at the critical points or on the boundary, we first examine the functions critical points. Since the function $F_{\pm}(x)$ is purely an imaginary function we calculate the critical points of $f_{\pm}(x)$.

$$f_{\pm}^{'}(x) = 1 - \frac{2x^2}{\sqrt{1-x^2}} + 2\sqrt{1-x^2} \pm \frac{(1 + \frac{2x^2}{\sqrt{1-x^2}} - 2\sqrt{1-x^2})(x - 2x\sqrt{1-x^2})}{\sqrt{1 + (x - 2x\sqrt{1-x^2})^2}}$$

has solutions at $x_1 \sim -0.823087, x_2 \sim 0.778257$(for f_+) ,$x_3 \sim -0.778257$,and $x_4 \sim 0.823087$(for f_-). The value of the corresponding $F_{\pm}$ at this critical points are

$$F_+(x_1) = -0.751744\, i$$
$$F_+(x_2) = 2.7753\, i$$
$$F_-(x_3) = -2.7753\, i$$
$$F_-(x_4) = 0.751744\, i$$

Further, we look at the values of $F_{\pm}$ at $|x| = 1$. Thus $F_+(1) = (1-\sqrt{2})i, F_+(-1) = (-1-\sqrt{2})i, F_-(1) = (1+\sqrt{2})i$, and $F_-(-1) = (-1+\sqrt{2})i$. Certainly the maximum value is approximately $2.7753i$ at x_2 and minimum value is approximately $2.7753i$ at x_3. We sketch in Figure 3 the two functions $f_{\pm}$ to visualize our analysis from above and it is obvious that every value of f_+ is above every value of f_-, and so the minimum is attained by f_- and the maximum is attained by f_+.

5 Directed I-graph

A directed I-graph $\widetilde{I}(n,j,k)$ for $1 \le j,k < n$ and $j,k \ne \frac{n}{2}$ is a generalization of a generalized directed Petersen graph and has vertex set $V(\widetilde{I}(n,j,k)) = \{v_i, u_i; 0 \le i \le n\}$ and arc set $E(\widetilde{I}(n,j,k)) = \{v_i v_{(i+k)}\ , v_i u_i\ , u_i u_{(i+j)}; 0 \le i \le n\}$, where the subscripts are expressed as integers modulo n with k and j are the 'skips'. We may assume that $j \le k$ since $\widetilde{I}(n,j,k) = \widetilde{I}(n,k,j)$ .We consider $j,k < \frac{n}{2}$ because $\widetilde{I}(n,j,k), \widetilde{I}(n,n-j,k)$ and $\widetilde{I}(n,j,n-k)$ are isomorphic. Let $\widetilde{A}(n,j,k)$ and $\widetilde{B}(n,j,k)$ are the subgraphs of $\widetilde{I}(n,j,k)$ consisting of the vertices $\{v_i; 0 \le i \le n\}$ and $\{u_i; 0 \le i \le n\}$ and arcs $\{v_i v_{(i+k)}; 0 \le i \le n\}$ and $\{u_i u_{(i+j)}; 0 \le i \le n\}$ respectively. We will call $\widetilde{A}(n,j,k)$ and $\widetilde{B}(n,j,k)$ as the outer and inner subgraph of $\widetilde{I}(n,j,k)$ respectively . We display in Figure 4 the directed I-graph $\widetilde{I}(10,3,4)$.

Theorem 5.1. *The eigenvalues of directed I-graph $\widetilde{I}(n,j,k)$ are given by*

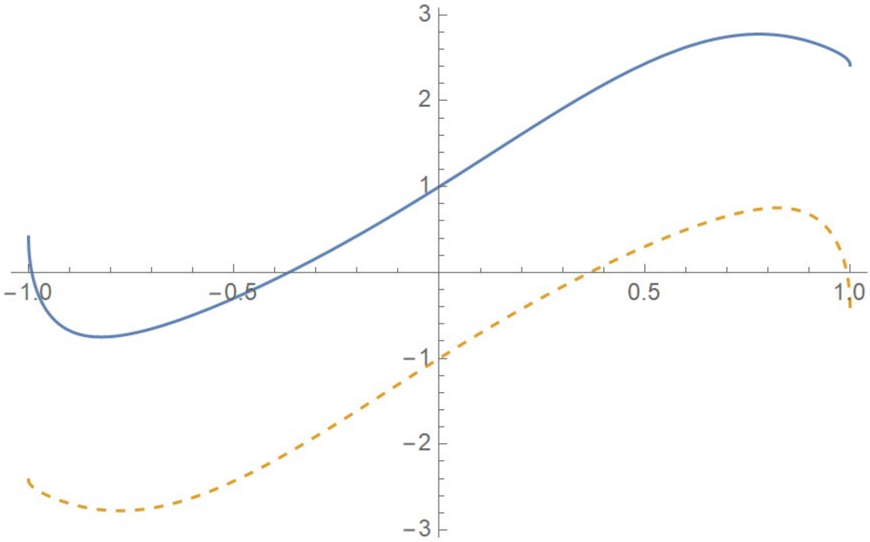

Figure 3. The top function is f_+ and the bottom function is f_-

$$\zeta = \left(i\sin\frac{2\pi rj}{n} + i\sin\frac{2\pi rk}{n}\right) \pm i\sqrt{\left(\sin\frac{2\pi rj}{n} - \sin\frac{2\pi rk}{n}\right)^2 + 1}\,,\quad 0 \le r \le n-1$$

Proof. The skew adjacency matrix of the directed I-graph $\widetilde{I}(n,j,k)$ of order $2n$ has the block form

$$S_A(\widetilde{I}(n,j,k)) = \begin{bmatrix} C_k^n & I_n \\ -I_n & C_j^n \end{bmatrix}$$

where I_n is the identity matrix of order n. C_j^n, C_k^n are circulant matrices, with

$C_j^n = circ(\overbrace{0,\cdots,0}^{j\ times}, 1, \overbrace{0,\cdots,0}^{n-2j-1\ times}, -1, \overbrace{0,\cdots,0}^{j-1\ times})$ and

$C_k^n = circ(\overbrace{0,\cdots,0}^{k\ times}, 1, \overbrace{0,\cdots,0}^{n-2k-1\ times}, -1, \overbrace{0,\cdots,0}^{k-1\ times})$ being the skew adjacency matrix for $\widetilde{A}(n,j,k)$ and $\widetilde{B}(n,j,k)$ respectively. The eigenvalues corresponding to the circulant matrix C_j^n and C_k^n are $2i\sin\frac{2\pi rj}{n}$ and $2i\sin\frac{2\pi rk}{n}$, $0 \le r \le n-1$ respectively from lemma 3.2.

By similar arguments for the spectrum of generalized directed Petersen graph as in Theorem 3.3, we get that the eigenvalues of directed I-graph satisfies the quadratic equation $\zeta^2 - (\gamma_r + \beta_r)\zeta + \gamma_r\beta_r + 1 = 0$. Substituting the value for $\gamma_r = 2i\sin\frac{2\pi rj}{n}$ and $\beta_r = 2i\sin\frac{2\pi rk}{n}$ and simplify we get,

$$\zeta = \left(i\sin\frac{2\pi rj}{n} + i\sin\frac{2\pi rk}{n}\right) \pm i\sqrt{\left(\sin\frac{2\pi rj}{n} - \sin\frac{2\pi rk}{n}\right)^2 + 1}\quad, 0 \le r \le n-1$$

constitute the spectrum of directed I-graph $\widetilde{I}(n,j,k)$. □

6 Bounds on the eigenvalues of $\widetilde{I}(n,j,2j)$

In this section we find the bounds on the eigenvalues of directed I-graph particularly for the case where the skips of outer subgraph is double the skips of inner subgraph, that is the case of $k = 2j$.
Now the spectrum of $\widetilde{I}(n,j,2j)$ is

$$\zeta = \left(i\sin\frac{2\pi rj}{n} + i\sin\frac{4\pi rj}{n}\right) \pm i\sqrt{\left(\sin\frac{2\pi rj}{n} - \sin\frac{4\pi rj}{n}\right)^2 + 1}\quad, 0 \le r \le n-1$$

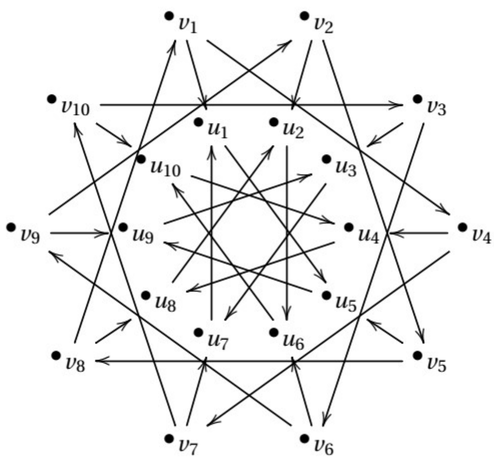

Figure 4. The directed I-graph $\widetilde{I}(10, 3, 4)$

Substitute $\sin 2\theta = 2\sin\theta cos\theta$

$$\zeta = \left(i\sin\frac{2\pi rj}{n} + i2\sin\frac{2\pi rj}{n}\cos\frac{2\pi rj}{n} \right) \pm i\sqrt{\left(\sin\frac{2\pi rj}{n} - 2\sin\frac{2\pi rj}{n}\cos\frac{2\pi rj}{n}\right)^2 + 1}$$

Now we look for the extreme points of the function to find the bounds on the eigenvalues,

$$F_{\pm}(x) = if_{\pm}(x) = i(x + 2x\sqrt{1-x^2} \pm \sqrt{(x - 2x\sqrt{1-x^2})^2 + 1})$$

in the interval $-1 \leq x \leq 1$.

We got the same function as discussed in the section 4 for finding the bounds of $\widetilde{P}(n, 2)$ as $\widetilde{P}(n, 2)$ is a particular case of $\widetilde{I}(n, j, 2j)$ with $j = 1$. So we conclude that approximate bounds of the eigenvalues of $\widetilde{I}(n, j, 2j)$ are -2.773i and 2.773i respectively by the lexicographic ordering of complex field.

7 Further comments

In this paper we consider only a particular orientation with inner and outer subgraphs of generalized Petersen graph. If we choose the orientation in anticlockwise direction it does not affect the eigenvalues that we obtained but the change in direction of edges between inner subgraphs and outer subgraphs will affect the signs of these eigenvalues.

All of our results for $\widetilde{P}(n, 2)$ can be certainly extended to $\widetilde{P}(n, 3)$,$\widetilde{P}(n, 4)$, ... We propose to find the bounds on eigenvalues for arbitrary $\widetilde{P}(n, k)$ and the skew energy of generalized directed Petersen graph $\widetilde{P}(n, k)$ and $\widetilde{I}(n, j, k)$.

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