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Flag curvature and homogeneous geodesics of homogeneous Finsler space with (α, β)-metric

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Abstract The computation of flag curvature of Finsler metrics is very difficult, therefore it is important to find an explicit formula for the flag curvature. In this paper, we have studied the existence of homogeneous geodesics for a naturally reductive homogeneous Finsler space M with metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$. We have discussed the necessary and sufficient condition for (M, F) to be naturally reductive. Further, by using Puttmann's formula we give the formula for flag curvature of naturally reductive homogeneous Finsler space with (α, β)-metric.

1 Introduction and Definitions

Flag curvature is considered a generalization of sectional curvature from Riemannian manifolds in Finsler geometry. Flag curvature was first introduced by Berwald (1926). It has an important role in characterizing Finsler spaces. The notion of naturally reductive Riemannian metrics was first introduced by Kobayashi and Nomizu [7]. It is well-known that the geodesics of a naturally reductive homogeneous space are the orbits of one-parameter subgroups of isometries [1]. In the field of mechanics homogeneous geodesics has important applications.

In 1979, the naturally reductive metrics and Einstien metrics on compact Lie groups were studied by D. Atri and Ziller [3]. H. R. Salimi Moghaddam gives the formula for flag curvature of invariant metrics of form $F = \frac{(\alpha+\beta)^2}{\alpha}$ such that α is induced by an invariant Riemannian metric g on the homogeneous space and the Chern connection of F coincides with the Levi-Civita connection of g [14]. In recent years, many authors have given the formula for flag curvature of a naturally reductive homogeneous Finsler space with (α, β)-metrics [3, 11, 10, 12], and also discussed the naturally reductiveness of homogeneous Finsler space. We studied the existence of homogeneous geodesics for the homogeneous Finsler space with metric, $F = \alpha + \sqrt{\alpha^2 + \beta^2}$. In this paper, first, we revive some basic concepts related to the homogeneous Finsler space. Next, we deduce the formula for flag curvature of Finsler space (M, F) with metric F and studied the condition for naturally reductiveness of homogeneous space (M, F). Further, we discussed the existence of homogeneous geodesics for the space (M, F). In the last part, we have obtained the formula for flag curvature of naturally reductive homogeneous Finsler space (M, F).

Definition 1.1. A connected Finsler space (M, F) is called a homogeneous Finsler space if $I(M, F)$, the group of isometries of (M, F), acts transitively on M.

The connected homogeneous Finsler space M can be written in the form of $M = G/H$, where G and H are the Lie group of isometries of M and isotropy subgroup of G a point in M respectively. Here homogeneous Finsler space is reductive decomposition if there exists an $Ad(H)$- invariant decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, here $\mathfrak{h}$ and $\mathfrak{g}$ represents the Lie algebras of H and G, respectively and $\mathfrak{m}$ be the subspace of $\mathfrak{g}$ [9].

Definition 1.2. Let G be a Lie group and M a smooth manifold. If G has a smooth action on M, then G is called a Lie transformation group of M.

Definition 1.3. [7] A homogeneous space G/H of a connected Lie group G is called reductive if the following conditions are satisfied:

- In the Lie algebra $\mathfrak{g}$ of G, there exists a subspace $\mathfrak{m}$ such that $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ (direct sum of vector subspaces).
- ad(h)$\mathfrak{m} \subset \mathfrak{m}$, for all $h \in H$, where $\mathfrak{h}$ is the subalgebra of $\mathfrak{g}$ corresponding to the identity component H_0 of H and $ad(h)$ denotes the adjoint representation of H in $\mathfrak{g}$.

Definition 1.4. A Riemannian homogeneous space $(G/H, g)$ is said to be naturally reductive if there exists a connected Lie group G of isometries acting transitively on G/H is a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ of $\mathfrak{g}$ satisfies the following condition:

$$\langle [u, w]_{\mathfrak{m}}, v\rangle + \langle u, [w, v]_{\mathfrak{m}}\rangle = 0, \forall\ u, v, w \in \mathfrak{m}. \tag{1.1}$$

Here, the subscript $\mathfrak{m}$ indicates the projection of an element of $\mathfrak{g}$ into $\mathfrak{m}$. And $\langle , \rangle$ denotes the inner product on $\mathfrak{m}$ induced by the metric g [7].

Definition 1.5. [10] Let (M, F) be a homogeneous Finsler space. If every geodesic in M is an orbit of a one-parameter group of isometries, i.e., there exists a transitive group G of isometries such that every geodesic in M is of the form exp$(tu)p$ with $u \in \mathfrak{g},\ p \in M$. Then is said to be geodesic orbit space.

One parameter subgroups are the mappings $t \to \exp tu$, where u is an element of Lie algebra[6].

The following geometrical property and the above mentioned condition (1.1) both are equivalent:

For any vector $u \in \mathfrak{m} \setminus \{0\}$, the curve $\gamma(t) = \tau(\exp tu)(p)$ is a geodesic with respect to the Riemannian connection. Here, exp and $\tau(h)$ denote the Lie exponential map of G and the left transformation of G/H induced by $h \in G$, respectively. Thus, for a naturally reductive homogeneous space, every geodesic on $(G/H, g)$ is an orbit of a one-parameter subgroup of the group of isometries [11]. Let $(G/H, g)$ be a homogeneous Riemannian manifold with a fixed origin p, and

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$$

a reductive decomposition. A homogeneous geodesic through the origin $p \in G/H$ is a geodesic $\gamma(t)$ which is an orbit of a one-parameter subgroup of G, that is

$$\gamma(t) = \exp(tw)(p),\ \ t \in R,$$

where, w is a nonzero vector of $\mathfrak{g}$.

A non-zero vector $w \in \mathfrak{g}$ is called a geodesic vector if the curve exp$(tw)(p)$ is a geodesic. Kowalski and Venhecke [8] proved the following characterization of geodesic vectors.

Lemma 1.6. *A non-zero vector $w \in \mathfrak{g}$ is a geodesic if and only if*

$$\langle [w, v]_{\mathfrak{m}}, w_{\mathfrak{m}}\rangle = 0, \forall\, v \in \mathfrak{m}.$$

2 Flag curvature of homogeneous Finsler spaces

Let $(G/H, \alpha)$ be a homogeneous Riemannian manifold. Then the Lie algebra has a decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, where $\mathfrak{h}$ and $\mathfrak{g}$ are the Lie algebras of H and G respectively. We consider $\mathfrak{m}$ along the $T_0(G/H)$ is a tangent space of the $H = o(origin)$ (means isomorphism between the $\mathfrak{m}$ and $T_0(G/H)$). Here a left-invariant metric denoted by $\bar{\alpha}$ on G was generalized by a G-invariant Riemannian metric on G/H. The concept of a naturally reductive homogeneous Riemannian space is a generalization of the concept of bi-invariant Riemannian metric denoted by $\bar{\alpha_0}$ on G. The values of $\bar{\alpha_0}$ and $\bar{\alpha}$ are the inner products on G and denote them as $\langle\langle , \rangle\rangle$ and $\langle , \rangle$ respectively. The inner product $\langle , \rangle$ induces an endomorphism ψ of $\mathfrak{g}$ such that $\langle v, w\rangle = \langle\langle \psi(v), w\rangle\rangle, \forall\, v, w \in \mathfrak{g}$. In fact, if $H = e$ then $\mathfrak{m} = \mathfrak{g}$, the condition (1.1) is just the condition

$$\langle [u, v], w\rangle + \langle v, [u, w]\rangle = 0, \tag{2.1}$$

for a bi-invariant Riemannian metric on G.

Theorem 2.1. *Let every geodesic of $(G/H, F)$ is an orbit of a one-parameter subgroup of G, every geodesic of $(G/H, \alpha)$ is an orbit of a one-parameter subgroup of $\tilde{G}$, β is a G-invariant Killing form of $(G/H, \alpha)$. Assume that $F = \alpha + \sqrt{\alpha^2 + \beta^2}$ be a special (α, β)-metric and $\{P, v\}$ is a flag in $T_H(G/H)$ such that $\{x, v\}$ be the orthonormal basis of P with respect to $\langle , \rangle$. Then the flag curvature of the flag $\{P, v\}$ is given by*

$$K(P,v) = \left[\frac{\langle x, R(x,v)v\rangle S_1 + \langle u,x\rangle\langle u, R(x,v)v\rangle S_2 + \langle v, R(x,v)v\rangle S_3}{8 + 4\langle u,v\rangle^2 + 2\langle u,x\rangle^2 + \frac{S_4}{\sqrt{1+\langle u,v\rangle^2}} + \frac{2\langle u,x\rangle^2}{1+\langle u,v\rangle^2} + \frac{S_5}{(1+\langle u,v\rangle^2)^{\frac{3}{2}}} - \frac{\langle u,v\rangle^2\langle u,x\rangle^2}{(1+\langle u,v\rangle^2)^3}} \right] \tag{2.2}$$

where,

$$S_1 = 2 + \frac{2+\langle u,v\rangle^2}{\sqrt{1+\langle u,v\rangle^2}}, \quad S_2 = \frac{(1+\langle u,v\rangle^2)^{\frac{3}{2}} + 1}{(1+\langle u,v\rangle^2)^{\frac{3}{2}}}, \quad S_3 = \frac{\langle u,v\rangle^3\langle u,x\rangle}{(1+\langle u,v\rangle^2)^{\frac{3}{2}}},$$
$$S_4 = 8 + 8\langle u,v\rangle^2 + 2\langle u,x\rangle^2(1+\langle u,v\rangle^2) + \langle u,v\rangle^4, \quad S_5 = 2\langle u,x\rangle^2 - \langle u,v\rangle^2\langle u,x\rangle^2.$$

Proof. We can write $F(v) = \sqrt{\langle v,v\rangle} + \sqrt{\langle v,v\rangle + \langle u,v\rangle^2}, \forall u \in \mathfrak{m}$. By the formula,

$$g_v(x,y) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t} F^2(v + sx + ty)\Big|_{t=s=0},$$

after some calculations for the above metric F, we get

$$g_v(x,y) = 2\langle x,y\rangle + \langle u,x\rangle\langle u,y\rangle + \frac{L_1}{\sqrt{\langle v,v\rangle^2 + \langle u,v\rangle^2\langle v,v\rangle}} - \frac{L_2}{(\langle v,v\rangle^2 + \langle u,v\rangle^2\langle v,v\rangle)^{\frac{3}{2}}}, \tag{2.3}$$

where,

$$\begin{aligned} L_1 =& 4\langle v,x\rangle\langle v,y\rangle + 2\langle v,v\rangle\langle x,y\rangle + 2\langle u,v\rangle\langle u,y\rangle\langle v,x\rangle + \langle u,x\rangle\langle u,y\rangle\langle v,v\rangle + 2\langle u,v\rangle\langle u,x\rangle\langle v,y\rangle \\ &+ \langle u,v\rangle^2\langle x,y\rangle, \\ L_2 =& 4\langle v,v\rangle^2\langle v,x\rangle\langle v,y\rangle + 2\langle u,v\rangle\langle u,x\rangle\langle v,y\rangle\langle v,v\rangle^2 + 2\langle u,v\rangle^2\langle v,v\rangle\langle v,x\rangle\langle v,y\rangle \\ &+ 2\langle u,v\rangle\langle v,v\rangle^2\langle u,y\rangle\langle v,x\rangle + \langle u,v\rangle^2\langle v,v\rangle^2\langle u,x\rangle\langle u,y\rangle + \langle u,v\rangle^3\langle v,v\rangle\langle u,y\rangle\langle v,x\rangle \\ &+ 2\langle u,v\rangle^2\langle v,v\rangle\langle v,x\rangle\langle v,y\rangle + \langle u,v\rangle^3\langle u,x\rangle\langle v,v\rangle\langle v,y\rangle + \langle u,v\rangle^4\langle v,x\rangle\langle v,y\rangle. \end{aligned}$$

Since, an orthonormal basis $\{x, v\}$ with respect to $\langle , \rangle$, equation (2.3) can be reduced as follows:

$$g_v(x,y) = 2\langle x,y\rangle + \langle u,x\rangle\langle u,y\rangle + \frac{L_3}{\sqrt{1+\langle u,v\rangle^2}} - \frac{L_4}{(1+\langle u,v\rangle^2)^{\frac{3}{2}}}, \tag{2.4}$$

where,

$$\begin{aligned} L_3 =& 2\langle x,y\rangle + \langle u,x\rangle\langle u,y\rangle + 2\langle u,v\rangle\langle u,x\rangle\langle v,y\rangle + \langle u,v\rangle^2\langle x,y\rangle, \\ L_4 =& 2\langle u,v\rangle\langle u,x\rangle\langle v,y\rangle + \langle u,v\rangle^2\langle u,x\langle u,y\rangle + \langle u,v\rangle^3\langle u,x\rangle\langle v,y\rangle. \end{aligned}$$

from the equation (2.3) we get the following:

$$\begin{aligned} g_v(v,v) =& 2 + \langle u,v\rangle^2 + 2\sqrt{1+\langle u,v\rangle^2}, \\ g_v(x,x) =& 2 + \langle u,x\rangle^2 + \frac{2 + \langle u,x\rangle^2 + 3\langle u,v\rangle^2 + \langle u,v\rangle^4}{(1+\langle u,v\rangle^2)^{\frac{3}{2}}}, \\ g_v(v,x) =& \langle u,v\rangle\langle u,x\rangle + \frac{\langle u,v\rangle\langle u,x\rangle}{(1+\langle u,v\rangle^2)^{\frac{3}{2}}}. \end{aligned}$$

Therefore,

$$\begin{aligned} g_v(v,v)g_v(x,x) - g_v^2(v,x) =& 8 + 4\langle u,v\rangle^2 + 2\langle u,x\rangle^2 + \frac{8 + 8\langle u,v\rangle^2 + 2\langle u,x\rangle^2(1+\langle u,v\rangle^2)}{\sqrt{1+\langle u,v\rangle^2}} \\ &+ \frac{2\langle u,x\rangle^2}{1+\langle u,v\rangle^2} + \frac{2\langle u,x\rangle^2}{(1+\langle u,v\rangle^2)^{\frac{3}{2}}} - \frac{\langle u,v\rangle^2\langle u,x\rangle^2}{(1+\langle u,v\rangle^2)^{\frac{3}{2}}} - \frac{\langle u,v\rangle^2\langle u,x\rangle^2}{(1+\langle u,v\rangle^2)^3} \\ &+ \frac{\langle u,v\rangle^4}{\sqrt{1+\langle u,v\rangle^2}}, \end{aligned} \tag{2.5}$$

and also,

$$\begin{aligned} g_v(x,R(x,v)v) =& \langle x,R(x,v)v\rangle \left[2 + \frac{2+\langle u,v\rangle^2}{\sqrt{1+\langle u,v\rangle^2}}\right] + \langle u,x\rangle\langle u,R(x,v)v\rangle \\ &\times \left[\frac{(1+\langle u,v\rangle^2)^{\frac{3}{2}}+1}{(1+\langle u,v\rangle^2)^{\frac{3}{2}}}\right] + \frac{\langle u,v\rangle^3\langle u,x\rangle\langle v,R(x,v)v\rangle}{(1+\langle u,v\rangle^2)^{\frac{3}{2}}}. \end{aligned} \tag{2.6}$$

Here, we are using the Puttmann's formula [13], and obtain the following:

$$\begin{aligned} \langle x,R(x,v)v\rangle =& \frac{1}{2}\left(\langle\langle[\psi x,v]+[x,\psi v],[v,x]\rangle\rangle\right) + \frac{3}{4}\langle[v,x],[v,x]_{\mathfrak{m}}\rangle + \langle\langle[x,\psi x],\psi^{-1}([v,\psi v])\rangle\rangle \\ &- \frac{1}{4}\langle\langle[x,\psi v]+[v,\psi x],\psi^{-1}([v,\psi x]+[x,\psi v])\rangle\rangle. \end{aligned} \tag{2.7}$$

Hence the flag curvature is given by

$$K(P,v) = \frac{g_v(x,R(x,v)v)}{g_v(v,v)g_v(x,x) - g_v^2(v,x)}.$$

By substituting the equations (2.5) to (2.7) in $K(P,v)$, we obtain the required equation (2.2). □

3 Naturally Reductive spaces

Let G be a connected Lie group. Then there exists a bi-invariant Finsler metric on G if and only if there exists a Minkowski norm F on $\mathfrak{g}$ such that the below condition (3.1) is considered as a natural generalization of the condition (2.1), i.e.,

$$g_v([u,x]_{\mathfrak{m}},y) + g_v(x,[u,y]_{\mathfrak{m}}) + 2C_v([u,v]_{\mathfrak{m}},x,y) = 0,\ v\neq 0,\ u,x,y\in\mathfrak{m}. \tag{3.1}$$

The naturally reductive homogeneous Finsler space was first proposed by D. Latifi [10].

Definition 3.1. A homogeneous manifold $M = G/H$ with an invariant Finsler metric F is called naturally reductive if there exists an $Ad(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{m}\oplus\mathfrak{h}$ such that

$$g_v([w,x]_{\mathfrak{m}},y) + g_v(x,[w,y]_{\mathfrak{m}}) + 2C_v([w,v]_{\mathfrak{m}},x,y) = 0, v\neq 0, w,x,y\in\mathfrak{m}. \tag{3.2}$$

Definition 3.2. [4] A homogeneous space $(M = G/H, F)$ with an invariant Finsler metric is called naturally reductive if there exists an invariant Riemannian metric $\tilde{\alpha}$ is such that $(M,\tilde{\alpha})$ is naturally reductive and the Chern connection of F coincides with the Levi-Civita connection of $\tilde{\alpha}$.

In this definition, the authors have assumed that the metric must be of Berwald type.
Remark: If $(G/H,F)$ is naturally reductive in accordance with the definition (3.1), then it implies that $(G/H,F)$ must be naturally reductive in the sense of definition (3.2)[5].

Theorem 3.3. *Let a homogeneous Finsler space $(G/H,F)$ with (α,β)-metric $F = \alpha+\sqrt{\alpha^2+\beta^2}$ be defined by an invariant Riemannian metric $\tilde{\alpha}$ and an invariant vector field u such that the Chern connection of F coincides the Levi-Civita connection of $\tilde{\alpha}$. Then $(G/H,F)$ is naturally reductive if and only if the underlying Riemannian space $(G/H,\tilde{\alpha})$ is naturally reductive.*

Proof. Let $v \neq 0, w \in \mathfrak{m}$. From the formula (2.3), we get

$$g_v(v,[v,w]_{\mathfrak{m}}) = \langle v,[v,w]_{\mathfrak{m}}\rangle \left\{2+\frac{L_5}{\sqrt{\langle v,v\rangle^2+\langle u,v\rangle^2\langle v,v\rangle}}-\frac{L_6}{(\langle v,v\rangle^2+\langle u,v\rangle^2\langle v,v\rangle)^{\frac{3}{2}}}\right\}$$
$$+\langle u,[v,w]_{\mathfrak{m}}\rangle \left\{\langle u,v\rangle+\frac{L_7}{\sqrt{\langle v,v\rangle^2+\langle u,v\rangle^2\langle v,v\rangle}}\right.$$
$$\left.-\frac{L_8}{(\langle v,v\rangle^2+\langle u,v\rangle^2\langle v,v\rangle)^{\frac{3}{2}}}\right\} \tag{3.3}$$

where,

$$L_5 = 6\langle v,v\rangle+3\langle u,v\rangle^2, \quad L_6 = 4\langle v,v\rangle^3+6\langle u,v\rangle^2\langle v,v\rangle^2+2\langle u,v\rangle^4\langle v,v\rangle,$$
$$L_7 = 3\langle u,v\rangle\langle v,v\rangle, \quad L_8 = 2\langle u,v\rangle\langle v,v\rangle^3+2\langle u,v\rangle^3\langle v,v\rangle^2.$$

Since F is of Berwald type, $(G/H,F)$ and $(G/H,\tilde{\alpha})$ have the same connection and are coincides. Thus the equation (3.3) implies that

$$\langle u,[v,w]_{\mathfrak{m}}\rangle = 0, \forall w \in \mathfrak{m}. \tag{3.4}$$

Now, let $(G/H,F)$ be naturally reductive, as per the reference [10], and we can write the equation (3.2) as follows:

$$g_v([v,x]_{\mathfrak{m}},y)+g_v(x,[v,y]_{\mathfrak{m}})+2C_v([v,v]_{\mathfrak{m}},x,y)=0, v\neq 0.$$

which implies

$$g_v([v,x]_{\mathfrak{m}},y)+g_v(x,[v,y]_{\mathfrak{m}})=0, \tag{3.5}$$

from (3.5), we have

$$g_v([v,w]_{\mathfrak{m}},v)=0. \tag{3.6}$$

From (3.3), (3.4) and (3.6), we obtain

$$\langle [v,w]_{\mathfrak{m}},v\rangle = 0. \tag{3.7}$$

From the equation (2.3), also using the equations (3.4) and (3.7) we have

$$g_v([v,x]_{\mathfrak{m}},y) = \langle [v,x]_{\mathfrak{m}},y\rangle \left\{2+\frac{2\langle v,v\rangle+\langle u,v\rangle^2}{\sqrt{\langle v,v\rangle^2+\langle u,v\rangle^2\langle v,v\rangle}}\right\},$$

similarly,

$$g_v(x,[v,y]_{\mathfrak{m}}) = \langle x,[v,y]_{\mathfrak{m}}\rangle \left\{2+\frac{2\langle v,v\rangle+\langle u,v\rangle^2}{\sqrt{\langle v,v\rangle^2+\langle u,v\rangle^2\langle v,v\rangle}}\right\}.$$

Adding the terms $g_v([v,x]_{\mathfrak{m}},y)$ and $g_v(x,[v,y]_{\mathfrak{m}})$, we obtain

$$g_v(x,[v,y]_{\mathfrak{m}})+g_v([v,x]_{\mathfrak{m}},y) = \{\langle x,[v,y]_{\mathfrak{m}}\rangle+\langle [v,x]_{\mathfrak{m}},y\rangle\} \left\{2+\frac{2\langle v,v\rangle+\langle u,v\rangle^2}{\sqrt{\langle v,v\rangle^2+\langle u,v\rangle^2\langle v,v\rangle}}\right\}. \tag{3.8}$$

From (3.5) and (3.8), $\left\{2+\frac{2\langle v,v\rangle+\langle u,v\rangle^2}{\sqrt{\langle v,v\rangle^2+\langle u,v\rangle^2\langle v,v\rangle}}\right\} \neq 0$. Thus

$$\langle x,[v,y]_{\mathfrak{m}}\rangle+\langle [v,x]_{\mathfrak{m}},y\rangle = 0.$$

Hence, the Riemannian metric $(G/H, \tilde{\alpha})$ is naturally reductive.
On the other hand, let Riemannian metric $(G/H, \tilde{\alpha})$ is naturally reductive. Thus, by using equation (2.3), we can write

$$\begin{aligned} g_v([w,x]_{\mathfrak{m}}, y) =& \langle [w,x]_{\mathfrak{m}}, y\rangle \left\{ 2 + \frac{2\langle v,v\rangle + \langle u,v\rangle^2}{\sqrt{\langle v,v\rangle^2 + \langle u,v\rangle^2 \langle v,v\rangle}} \right\} \\ &+ 2\langle [w,x]_{\mathfrak{m}}, v\rangle \left\{ \frac{2\langle v,y\rangle + \langle u,v\rangle\langle u,y\rangle}{\sqrt{\langle v,v\rangle^2 + \langle u,v\rangle^2 \langle v,v\rangle}} \right\} \\ &- \frac{\langle [w,x]_{\mathfrak{m}}, v\rangle}{(\langle v,v\rangle^2 + \langle u,v\rangle^2\langle v,v\rangle)^{\frac{3}{2}}} [4\langle v,v\rangle^2\langle v,y\rangle + 4\langle u,v\rangle^2\langle v,v\rangle\langle v,y\rangle \\ &+ 2\langle u,v\rangle\langle v,v\rangle^2\langle u,y\rangle + \langle u,v\rangle^3\langle v,v\rangle\langle u,y\rangle + \langle u,v\rangle^4\langle v,y\rangle], \end{aligned} \tag{3.9}$$

and similarly,

$$\begin{aligned} g_v(x, [w,y]_{\mathfrak{m}}) =& \langle [w,y]_{\mathfrak{m}}, x\rangle \left\{ 2 + \frac{2\langle v,v\rangle + \langle u,v\rangle^2}{\sqrt{\langle v,v\rangle^2 + \langle u,v\rangle^2 \langle v,v\rangle}} \right\} \\ &+ 2\langle [w,y]_{\mathfrak{m}}, v\rangle \left\{ \frac{2\langle v,x\rangle + \langle u,v\rangle\langle u,x\rangle}{\sqrt{\langle v,v\rangle^2 + \langle u,v\rangle^2 \langle v,v\rangle}} \right\} \\ &- \frac{\langle [w,y]_{\mathfrak{m}}, v\rangle}{(\langle v,v\rangle^2 + \langle u,v\rangle^2\langle v,v\rangle)^{\frac{3}{2}}} [4\langle v,v\rangle^2\langle v,x\rangle + 4\langle u,v\rangle^2\langle v,v\rangle\langle v,x\rangle \\ &+ 2\langle u,v\rangle\langle v,v\rangle^2\langle u,x\rangle + \langle u,v\rangle^3\langle v,v\rangle\langle u,x\rangle + \langle u,v\rangle^4\langle v,x\rangle] \end{aligned} \tag{3.10}$$

By the fundamental Cartan tensor,

$$C_v(w,x,y) = \frac{1}{2}\frac{d}{dt}[g_{v+ty}(w,x)]\big|_{t=0}.$$

Thus,

$$\begin{aligned} 2C_v([w,v]_{\mathfrak{m}}, x, y) =& 2\langle [w,v]_{\mathfrak{m}}, x\rangle \left\{ \frac{2\langle v,y\rangle + \langle u,v\rangle\langle u,y\rangle}{\sqrt{\langle v,v\rangle^2 + \langle u,v\rangle^2 \langle v,v\rangle}} \right\} \\ &+ 2\langle [w,v]_{\mathfrak{m}}, y\rangle \left\{ \frac{2\langle v,x\rangle + \langle u,v\rangle\langle u,x\rangle}{\sqrt{\langle v,v\rangle^2 + \langle u,v\rangle^2 \langle v,v\rangle}} \right\} \\ &- \frac{\langle [w,v]_{\mathfrak{m}}, x\rangle}{(\langle v,v\rangle^2 + \langle u,v\rangle^2\langle v,v\rangle)^{\frac{3}{2}}} [4\langle v,v\rangle^2\langle v,y\rangle + 4\langle u,v\rangle^2\langle v,v\rangle\langle v,y\rangle \\ &+ 2\langle u,v\rangle\langle v,v\rangle^2\langle u,y\rangle + \langle u,v\rangle^3\langle v,v\rangle\langle u,y\rangle + \langle u,v\rangle^4\langle v,y\rangle] \\ &- \frac{\langle [w,v]_{\mathfrak{m}}, y\rangle}{(\langle u,v\rangle^2 + \langle u,v\rangle^2\langle v,v\rangle)^{\frac{3}{2}}} [4\langle v,v\rangle^2\langle v,x\rangle + 4\langle u,v\rangle^2\langle v,v\rangle\langle v,x\rangle \\ &+ 2\langle u,v\rangle\langle v,v\rangle^2\langle u,x\rangle + \langle u,v\rangle^3\langle v,v\rangle\langle u,x\rangle + \langle u,v\rangle^4\langle v,x\rangle] \end{aligned} \tag{3.11}$$

Since, $(G/H, \tilde{\alpha})$ is naturally reductive. Adding equation (3.9), (3.10) and (3.11),

$$g_v([w,x]_{\mathfrak{m}}, y) + g_v([w,y]_{\mathfrak{m}}, x) + 2C_v([w,v]_{\mathfrak{m}}, x, y) = 0.$$

Hence, the homogeneous Finsler space $(G/H, F)$ is naturally reductive.
Hence the proof. □

4 Existence of Homogeneous geodesics

D. Latifi, Toomanian [11] and some other authors discussed the results of homogeneous geodesics in homogeneous Finsler manifold. In Finsler space, the basic formula characterizing the geodesic vector was given in [10]. Let $(G/H, g)$ be a homogeneous Riemannian manifold with a fixed origin p, and $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ a reductive decomposition.

We use the following geodesic lemma:

Lemma 4.1. *[10] A vector $w \in \mathfrak{g}$ is a geodesic vector if and only if*

$$g_{w_m}([w, v]_{\mathfrak{m}}, w_{\mathfrak{m}}) = 0, \forall v \in \mathfrak{m}.$$

The following theorem shows the existence of homogeneous geodesics.

Theorem 4.2. *Let a homogeneous Finsler space (M, F) with $F = \alpha + \sqrt{\alpha^2 + \beta^2}$ defined by the Riemannian metric, where $\alpha = a_{ij} dx^i \otimes dx^j$ and the vector field u corresponding to 1-form β. Then the homogeneous Finsler space (M, F) with the origin $p = \{H\}$ and with an $Ad(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ is naturally reductive with respect to this decomposition if and only if for any vector $u \in \mathfrak{m} \setminus \{0\}$, the curve $\gamma(t)$ is geodesic of homogeneous Finsler manifold, here $\gamma(t)$ is $\exp tu(p)$.*

Proof. Suppose that a homogeneous Finsler space (M, F) is naturally reductive, then the Riemannian metric α is naturally reductive and the connection of Finsler metric F and Riemannian metric α are coinciding. This means Finsler metric is of Berwald metric. Let the decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ be naturally reductive, thus

$$g_v([u, x]_{\mathfrak{m}}, y) + g_v([u, y]_{\mathfrak{m}}, x) + 2C_v([u, v]_{\mathfrak{m}}, x, y) = 0,$$

where $v \neq 0, u, x, y \in \mathfrak{m}$. Then for $w \in \mathfrak{m}$,

$$g_w([w, v]_{\mathfrak{m}}, w) = a(w, [w, v]_{\mathfrak{m}}) = 0, \forall\, v \in \mathfrak{m}.$$

Thus, according to D. Latifi [10], each geodesic of $(G/H, F)$ obtained from the fixed origin $p = \{H\}$ is nothing but $\exp(tw)p, w \in \mathfrak{m}$.

On the other hand, assume that for any vector $u \in \mathfrak{m} \setminus \{0\}$, the curve $\gamma(t)$ is a geodesic of homogeneous Finsler space $(G/H, F)$, where $\gamma(t) = \exp tu(p)$. Here first thing is to prove the F be a Berwald metric. The smooth mapping $\pi : G \to G/H$ is a canonical projection that instigates an isomorphism between the tangent space $T_x M$ and the subspace $\mathfrak{m}$. In fact, the tangent space can be identified with $\mathfrak{m}$ by the correspondence

$$u \in \mathfrak{m} \to \frac{d}{dt} \exp(tu) z\big|_{t=0}.$$

And then due to geodesics of (M, F), the exponential mapping $Exp|_z$ of (M, F) at z is $Exp(u) = \pi(\exp(u)),\ u \in \mathfrak{m}$, which is a smooth mapping everywhere. Therefore F must be a Berwald metric [2].

As F is of Berwald type, $(G/H, F)$ and $(G/H, \alpha)$ have the same connection as well as have the same geodesics also. Therefore, for any $w \in \mathfrak{m}$, and the curve $\exp(tu)p$ is a homogeneous geodesic of $(G/H, \alpha)$, from the lemma 1.6, we concluded that, the vector $w \in \mathfrak{m}$ is a geodesic vector, hence $\langle [w, v]_{\mathfrak{m}}, w \rangle = 0, \forall\, v \in \mathfrak{m}$.

Consider, any $v, w, D \in \mathfrak{m}$, set $D' = D + v,\ v' = v + w$.

Then, we have

$$0 = \langle [D', v']_{\mathfrak{m}}, D' \rangle = \langle v, [D, w]_{\mathfrak{m}} \rangle + \langle [v, w]_{\mathfrak{m}}, D \rangle.$$

This implies that $(G/H, \alpha)$ is naturally reductive. So $(G/H, F)$ is naturally reductive. □

5 Flag curvature of naturally reductive homogeneous space

As per the study of Deng and Hou, in [4] authors find the formula for flag curvature of naturally reductive homogeneous (α, β)-metric spaces. In the sense of Deng and Hou, with special (α, β)-metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$, for this metric we derive the formula for flag curvature of naturally reductive homogeneous Finsler space.

Theorem 5.1. *A naturally reductive homogeneous Finsler space G/H with (α,β)-metric $F = \alpha + \sqrt{\alpha^2+\beta^2}$, defined by an inavariant Riemannian metric $\tilde{\alpha}$ and an invariant vector field $\tilde{u}$ on G/H, suppose that $\tilde{u}_H = u$. And (P, v) be a flag in $T_H(G/H)$ such that $\{x, v\}$ is an orthonormal basis of P with respect to $\langle,\rangle$. Then the flag curvature of the flag (P, v) is given in (2.2)*

$$K(P,v) = \left[\frac{\begin{array}{r}\langle x, \frac{1}{4}[v,[x,v]_{\mathfrak{m}}]_{\mathfrak{m}} + [v,[x,v]_{\mathfrak{h}}]\rangle S_1 + \langle u, \frac{1}{4}[v,[x,v]_{\mathfrak{m}}]_{\mathfrak{m}} + [v,[x,v]_{\mathfrak{h}}]\rangle S_2 \\ + \langle v, \frac{1}{4}[v,[x,v]_{\mathfrak{m}}]_{\mathfrak{m}} + [v,[x,v]_{\mathfrak{h}}]\rangle S_3\end{array}}{8 + 4\langle u,v\rangle^2 + 2\langle u,x\rangle^2 + \frac{S_4}{\sqrt{1+\langle u,v\rangle^2}} + \frac{2\langle u,x\rangle^2}{1+\langle u,v\rangle^2} + \frac{S_5}{(1+\langle u,v\rangle^2)^{\frac{3}{2}}} - \frac{\langle u,v\rangle^2\langle u,x\rangle^2}{(1+\langle u,v\rangle^2)^3}}\right], \tag{5.1}$$

where, $S_1,\ S_2,\ S_3,\ S_4$ and S_5 are as defined in the equation (2.2).

Proof. Since (M, F) is naturally reductive, from [7] using proposition 3.4, we have

$$R(x,v)v = \frac{1}{4}[v,[x,v]_{\mathfrak{m}}]_{\mathfrak{m}} + [v,[x,v]_{\mathfrak{h}}], \forall x, v \in \mathfrak{m}.$$

Substitute the above equation in (2.2) and after simplification, we get equation (5.1), □

If $H = \{e\}$, then we have the following corollary:

Corollary 5.2. *Let $F = \alpha + \sqrt{\alpha^2+\beta^2}$ be defined by a bi-invariant Riemannian metric $\tilde{\alpha}$ on a Lie group G and a left-invariant vector field u on G such that the Chern connection of F and Riemannian connection of $\tilde{\alpha}$ are coincides. Suppose that (P, v) be a flag in $T_e(G)$ such that $\{x, v\}$ is an orthonormal basis of P with respect to $\langle,\rangle$. Then the flag curvature of the flag (P, v) is given by*

$$K(P,v) = \left[\frac{\left\langle x, \frac{1}{4}[v,[x,v]]\right\rangle S_1 + \left\langle u, \frac{1}{4}[v,[x,v]]\right\rangle S_2 + \left\langle v, \frac{1}{4}[v,[x,v]]\right\rangle S_3}{8 + 4\langle u,v\rangle^2 + 2\langle u,x\rangle^2 + \frac{S_4}{\sqrt{1+\langle u,v\rangle^2}} + \frac{2\langle u,x\rangle^2}{1+\langle u,v\rangle^2} + \frac{S_5}{(1+\langle u,v\rangle^2)^{\frac{3}{2}}} - \frac{\langle u,v\rangle^2\langle u,x\rangle^2}{(1+\langle u,v\rangle^2)^3}}\right] \tag{5.2}$$

where $S_1,\ S_2,\ S_3,\ S_4$ and S_5 are mentioned in the equation (2.2).

Proof. Since $\langle,\rangle$ is bi-invariant, we have

$$R(x,v)v = \frac{1}{4}[v,[x,v]].$$

Substituting the above term in (2.2) and after simplifying this we get the equation (5.2). □

Conclusion

The concept of homogeneity is one of the fundamental notions in Finsler geometry, it means that, for any smooth Finsler manifold (M, F), its group of isometries $I(M, F)$ acts transitively on a smooth manifold M. The notion of naturally reductive Riemannian metric was first introduced by Kobayashi and Nomizu (1969). The definition of naturally reductive homogeneous Finsler space is a generalization of the definition of naturally reductive homogeneous Riemannian spaces. To study the geometric properties of naturally reductive homogeneous Finsler space, we use the Lie theory methods. First, we find the flag curvature formula for homogeneous Finsler space with (α,β)-metric i.e., $F = \alpha + \sqrt{\alpha^2+\beta^2}$. Then we discuss the condition for a Berwald type space (M, F) to be naturally reductive. Further, we extend our study to the existence of homogeneous geodesics and the flag curvature of naturally reductive homogeneous Finsler space. This paper provides a convinient methodology for researchers to construct flag curvature formula for naturally reductive Riemannian or Finsler spaces of any kind.

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