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Some remarks on the Solovay–Kitaev approximations in a C*–algebra setting

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Abstract The Solovay–Kitaev theorem is of fundamental importance in fault–tolerant quantum computing in the circuit–gate framework which is based on the Heisenberg–Born interpretation of quantum mechanics. Considering the C*–algebra approach to quantum mechanics, we explore some aspects of the question whether a generalized version of this theorem can be established in a C*–algebra setting.

1 Introduction

According to the circuit–gate framework of quantum computing, a quantum program can be regarded as the application of several quantum gates in the form of unitary matrices on qubits in the form of state vectors, together with measurements at certain instances. In a fault–tolerant setting, the universality and the supremacy of quantum computing thus depends upon the cost to implement an arbitrary unitary operation to a given accuracy. This problem has been answered elegantly, and independently, by Solovay and Kitaev [7, 3], resulting is what is known today as the *Solovay–Kitaev theorem*.

According to the Solovay–Kitaev theorem, it is possible to approximate any 2×2 unitary matrix with unit determinant to an arbitrary accuracy ϵ by a product of $O(\log^4 \frac{1}{\epsilon})$ physically realizable 2×2 unitaries with unit determinant. More precisely, the theorem for $SU(2)$ can be stated as follows.

Theorem 1.1(Solovay–Kitaev) *Let $\mathcal{G}$ be a finite subset of $SU(2)$ such that $\mathcal{G}$ contains its own inverse and $\langle \mathcal{G} \rangle$ is dense in $SU(2)$. Then for any $\epsilon > 0$, $\mathcal{G}_l$, the set of all strings that can be made from $\mathcal{G}$ without using more than l elements, provides an ϵ–net for $SU(2)$ where $l = O(\log^4(\frac{1}{\epsilon}))$.*

This leads to an interesting theoretical question: what other mathematical structures than special unitary matrices would support this type of approximation and provide a platform for quantum–like computation?

Several previous researchers have investigated the possibility of having different algebraic structures that support quantum–like computing. Aerts and Czachor [1] explored the possibility of using geometric algebras for efficient computing , demonstrating a simulation of the Deutsch–Jozsa algorithm. Later, Cafaro and Mancini [2] formalized the geometric algebra approach of quantum gates. This work is of particular significance as it generalizes the notion of quantum gates. Fernandez and Schneeberger [5] explored possibility of adopting quarternions instead of unitary matrices, relating it to Bernstein–Vazirani theorem. Recently, Mahasinghe et al [8] investigated the possibility of using rotations, by proving a version of the Solovay–Kitaev theorem for the rotation group $SO(3)$.

It is well–known that the circuit–gate framework is based on the Heisenberg and Born interpretation of quantum mechanics [4, 9]. On the other hand, one should not forget in this regard the C*–algebra formalism of quantum mechanics. Therefore, considering the geometric algebra approach [2] and the C*–algebra formalism, it is a natural question to ask whether quantum–like computations are supported in a C*–algebra setting. If the question whether Solovay–Kitaev type approximations are possible in such a setting is explored, it would be helpful for a better understanding of the scope of quantum–like computations. In this context, we attempt to investigate the question whether a generalized version can be established in a C*–algebra setting.

2 Preliminaries

Definition 2.1. A complex algebra A is called a C*–algebra, if there exists a norm, $\|\cdot\|: A \longrightarrow \mathbb{R}$ and an involution, $* : A \longrightarrow \mathbb{C}$ satisfying:

1. A is complete with respect to $\|\cdot\|$
2. $\| ab \| \leq \| a \| \| b \|, \forall a, b \in A$, and
3. $\| a^*a \| = \| a \|^2, \forall a \in A$.

A C*–algebra A is said to be unital if A has an multiplicative identity. In the usual scenario this identity is denoted by 1_A.

Definition 2.2. An element u of a unital C*–algebra A is said to be unitary if $u^*u = 1_A = uu^*$.

Definition 2.3. Let A and B be two C*–algebras. Then a *–homomorphism is a linear, multiplicative mapping, $\varphi : A \to B$ which satisfies $\varphi(a^*) = \varphi(a)^*, \forall a \in A$. A unital or a unit preserving *–homomorphism is a *-homomorphism defined between two unital C*–algebras A and B with $\varphi(1_A) = 1_B$.

Definition 2.4. Trace on a C* algebra A is a bounded linear functional $\mathcal{T} : A \to \mathbb{C}$ satisfying : $\mathcal{T}(a^*) = \overline{\mathcal{T}(a)}$, $\mathcal{T}(a^*a) \geq 0$ and $\mathcal{T}(ab) = \mathcal{T}(ba)$ for all $a, b \in A$. If A is unital and $\mathcal{T}$ is a trace with $\mathcal{T}(1) = 1$, $\mathcal{T}$ is said to be normalised.

Definition 2.5. [6] Suppose A is a unital C*–algebra and $\mathcal{T}$ is a normalized trace on A. Then the Fuglede–Kadison determinant $det : U(A) \to \mathbb{R}_+$ is defined by,

$$det(a) = exp[T(log(a^*a)^{\frac{1}{2}})], \forall a \in U(A) \tag{2.1}$$

An inductive sequence in C^*-algebras is a sequence $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \ldots$ where A_n is a C^*-algebra and $\varphi_n : A_n \to A_{n+1}$ is a $*$-homomorphism for each $n \in \mathbb{N}$. If A_n and φ_n is unital for every $n \in \mathbb{N}$ the above sequence is said to be a unital indcutive sequence.

Definition 2.6. A C^*-algebra A is said to be the inductive limit of the inductive sequence $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \ldots$ if the following are true:

1. There exists a sequence $\{\mu_n\}_{n=1}^{\infty}$ of $*$-homomorphisms such that $\mu_n : A_n \to A$ and $\mu_n = \mu_{n+1} \circ \phi_n$ for every $n \in \mathbb{N}$.
2. If B is a C^*-algebra and $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of $*$-homomorphisms such that $\lambda_n : A_n \to B$ and $\lambda_n = \lambda_{n+1} \circ \phi_n$ for every $n \in \mathbb{N}$, then there exists a unique $*$-homomorphism $\lambda : A \to B$ such that $\lambda_n = \lambda \circ \mu_n$ for every $n \in \mathbb{N}$.

Proposition 2.7. Every inductive sequence of C^*-algebras $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \ldots$ has an inductive limit A with $*$-homomorphisms $\mu_n : A_n \to A$ as in Definition 2.6 (1). Moreover;

(i) $A = \overline{\bigcup_{n=1}^{\infty} \mu_n(A_n)}$.

(ii) $\mu_n(a) = \mu_{n+1}(\varphi_n(a))$ for each $n \in \mathbb{N}$ and for all $n \in \mathbb{N}$ and $a \in A_n$.

(iii) $\| \mu_n(a) \| = \lim_{m\to\infty} \| \varphi_{m,n}(a) \|$ for all $n \in \mathbb{N}$ and $a \in A$.

(iv) If $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \ldots$ is unital then A is unital and μ_n is unital for every $n \in \mathbb{N}$.

More on inductive limits can be found in [11].

Definition 2.8. A C^* algebra A is said to be an UHF algebra if A is isomorphic to an inductive limit of a unital inductive sequence of the form,

$$M_{k_1}(\mathbb{C}) \xrightarrow{\varphi_1} M_{k_2}(\mathbb{C}) \xrightarrow{\varphi_2} M_{k_3}(\mathbb{C}) \xrightarrow{\varphi_3} \ldots \tag{2.2}$$

where, $\{k_n\}_{n=1}^{\infty}$ is a sequence of natural numbers and $\{\varphi_n\}_{n=1}^{\infty}$ is a sequence of unital * - homomorphisms. From Proposition 2.7 it follows that $A = \overline{\bigcup_{n=1}^{\infty} \mu_n(M_{k_n}(\mathbb{C}))}$ where $\{\mu_n\}$ are as in Definition 2.6. Note that since each φ_n is unital, k_n devides k_{n+1} for each $n \in \mathbb{N}$.

Definition 2.9. Let H be a Hilbert space and $B(H)$ the space of all bounded operators of H. Consider the linear mapping, $\alpha : H \to B(H)$, where for all x, y belong to H, $\alpha(x)\alpha(y) + \alpha(y)\alpha(x) = 0$ and $\alpha(x)\alpha(y)^* + \alpha(y)^*\alpha(x) = \langle y|x\rangle I$, where I is the identity operator on H. These are called *canonical anti-commutation relations*. The CAR algebra is the C*–algebra generated from $\alpha(H)$. i.e., $\mathfrak{A} = C^*(\alpha(H))$.

Proposition 2.10. Let H be a separable Hilbert space and $A = C^*(\alpha(H))$ where $\alpha : H \to B(H)$ as in Definition 2.9. Then A is UHF algebra and is independent (up to isomorphisms) of the choice of H, α. In particular, A is isomoprhic the limit of the inductive sequence $M_{2^1}(\mathbb{C}) \xrightarrow{\varphi_1} M_{2^2}(\mathbb{C}) \xrightarrow{\varphi_2} M_{2^3}(\mathbb{C}) \xrightarrow{\varphi_3} \dots$, where $\varphi_n : M_{2^n}(\mathbb{C}) \to M_{2^{n+1}}(\mathbb{C})$ is defined by $\varphi_n(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, for each $n \in \mathbb{N}$ and $a \in M_{2^n}(\mathbb{C})$.

Corollary 2.11. Let A and $M_{2^1}(\mathbb{C}) \xrightarrow{\varphi_1} M_{2^2}(\mathbb{C}) \xrightarrow{\varphi_2} M_{2^3}(\mathbb{C}) \xrightarrow{\varphi_3} \dots$ be as in Proposition 2.10. Then there exist $*$-homomorphisms $\mu_n : M_{2^n}(\mathbb{C}) \to A$ such that $\mu_n = \mu_{n+1} \circ \varphi_n$ for every $n \in \mathbb{N}$ and

$$A = \overline{\bigcup_{n=1}^{\infty} \mu_n(M_{2^n}(\mathbb{C}))} \tag{2.3}$$

3 C*–algebra approach

Throughout this section we will write A to denote the CAR algebra. In exploring the possibility of establishing Soloway - Kitaev type approximation results for A one requires to extend the notion of special unitary matrices to A.

Let $\tau_n : M_{2^n}(\mathbb{C}) \to \mathbb{C}$ be the usual normalised trace. i.e $\tau_n(a) = \frac{1}{2^n}\sum_{i=1}^{n} a_{ii}, \forall a = (a_{ij})_{2^n \times 2^n} \in M_{2^n}(\mathbb{C})$. Note that for every $n \in \mathbb{N}, \tau_n = \tau_{n+1} \circ \varphi_n$ where φ_n is as in Corollary 2.11.

Proposition 3.1: There exists a unique normalised trace $\tau : A \to \mathbb{C}$.

Proof. As observed above $\tau_n = \tau_{n+1} \circ \varphi_n$. Hence, $\tau : \bigcup_{n=1}^{\infty} \mu_n(M_{2^n}(\mathbb{C}))$ defined by $\tau(\mu_n(a)) = \tau_n(a), \forall a \in M_{2^n}(\mathbb{C})$ is well defined and is bounded linear. Since each τ_n is a trace it follows that τ satisfy the trace properties given in Definition 2.4. Now from density of $\bigcup_{n=1}^{\infty} \mu_n(M_{2^n}(\mathbb{C}))$ in A (Corollary 2.11), τ extends to trace on A. Since $\tau_n(1_{2^n}) = 1$ where 1_{2^n} denotes the identity in $M_{2^n}(\mathbb{C})$ and μ_n is unital for each n, we have $\tau(1) = 1$. To observe the uniqueness of τ note that for any normalised trace τ' on A, $\tau' \circ \mu_n$ is a normalised trace on $M_{2^n}(\mathbb{C})$. But τ_n is the only normalised trace on $M_{2^n}(\mathbb{C})$. Hence $\tau_n = \tau' \circ \mu_n$ and $\tau = \tau'$ on $(M_{k_n}(\mathbb{C})$. Therefore, by density of $\bigcup_{n=1}^{\infty} \mu_n(M_{2^n}(\mathbb{C}))$, we get $\tau = \tau'$. □

With τ as above we have Fuglede–Kadison determinant (Definition 2.5) defined in A. We will use det to denote this.

Proposition 3.2: *Under the Fuglede—Kadison determinant in A, any unitary element is a special unitary in A.*

Proof. Let $u \in A$ be an unitary. Then $u^*u = 1$. Then $log(u^*u) = log(1) = 0$. Thus $\tau(log(u^*u)^{\frac{1}{2}}) = \tau(0) = 0$. Therefore $\det(u) = \exp 0 = 1$.

□

Proposition 3.3: [10] *For each unitray $u \in A$ and $\epsilon > 0$ there exists a special unitray $u_n \in M_{2^n}(\mathbb{C})$ for some $n \in \mathbb{N}$ such that*

$$\| u - \mu_n(u_n) \| < \epsilon \tag{3.1}$$

Now as mentioned, from the Solovay Kitaev theorem we have the instruction set I_n, where the free group of I_n denoted by $(I_n)_l$ is dense in $SU(M_{2^n}(\mathbb{C}))$. Then for each $u_n \in SU(M_{2^n}(\mathbb{C}))$ there exists a word $(u_n)_l$ such that $\| (u_n)_l - u_n \| < \frac{\epsilon}{2}$. Then since $\mu_n : M_{2^n}(\mathbb{C}) \longrightarrow A$ is a unital *-isomorphism, $\| \mu_n(u_n)_l - \mu_n(u_n) \| < \frac{\epsilon}{2}$. Then combining this with triangle inequality and Proposition 3.2 we get

$$\begin{aligned}
\| \mu_n((u_n)_l) - u \| &\leq \| \mu_n((u_n)_l) - \mu_n(u_n) \| + \| \mu_n(u_n) - u \| \\
&\leq \| (u_n)_l - u_n \| + \| \mu_n(u_n) - u \| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&< \epsilon
\end{aligned}$$

This leads to the following Theorem.

Theorem 3.7: *For each $u \in SU(A)$ and for each $\epsilon > 0$ there exists a $l_{u,\epsilon} > 0$ such that there is a word $w \in \bigcup_{n=1}^{\infty} \mu_n(I_n)$ of length $l_{u,\epsilon}$ such that $\| w - u \| < \epsilon$.*

4 Conclusion

We explored the possibility of performing quantum–like computations in a C*–algebra setting. Specifically, the possibility of establishing a version of the Solovay–Kitaev theorem in a UHF algebra setting was traversed. In this regard a special instance of a UHF algebra, called CAR algebra was studied by concentrating on finding a link between such an algebra and finite matrix algebras, for which the Solovay–Kitaev approximations are fundamentally proven. While investigating a path to achieve the objective it was discovered that for a given element in the CAR algebra there, is a possibility of finding a positive real number l such that a word with length of the said l made up of elements of the CAR algebra, which gives an approximation up to an arbitrary accuracy level. This is primarily different from the word length mentioned under the original theorem in the sense that the said word length should only depend upon the desired accuracy but not on the element that is being approximated.

Further follow up can be done in order to come up with a version which is more inline with the original theorem which is to find either a finite or a countable subset, whose free group will be dense in the CAR algebra. One possible path is to investigate the possibility of considering the matrix algebra consisting of matrices with elements from the CAR algebra and writing and proving the Solovay–Kitaev theorem for the special unitary group of the said matrix algebra. Also being inline with the objective of investigating the possibility of expanding the scope on which the Solovay–Kitaev approximations can be applied, a perusal can be done to reexamine the theorem from a geometric algebra perspective.

5 Declaration

Several results in this paper were presented at the International Conference on Multidisciplinary Approaches in Science, Colombo.

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