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Linear Mapping Preserving Non-Zero Angles

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Abstract In this paper, we investigate linear mappings on real Hilbert spaces, which preserve some non-zero angles, and show that they are similar to linear mappings that preserve orthogonal vectors. We show that these mappings have the same properties as those that preserve orthogonal vectors, and are close to isometric mappings.

1 Introduction and Preliminaries

Orthogonality preserving property can be introduced for linear mappings between inner product spaces. We begin with a fairly general setting. Let X, and Y be real or complex inner product spaces with an inner product $\langle \cdot, \cdot \rangle$.

Definition 1.1. An operator $T : X \to Y$ is said orthogonality preserving, if for every orthogonal vectors $x, y \in X$, the vectors Tx, Ty are also orthogonal. That is

$$x \perp y \Rightarrow T(x) \perp T(y).$$

Moreover, an operator T is said strongly orthogonality preserving, if

$$x \perp y \Leftrightarrow T(x) \perp T(y).$$

These mappings are not necessarily linear (see [2, Example 1]). But, if they are linear, we can characterize them more precisely. The next theorem was proved by Chmieliński [2] and which shows that a linear mapping preserving orthogonality must be a similarity.

Theorem 1.2. (*see* [2]) *Let X and Y be two inner product spaces over a same field. For a non-vanishing mapping $T : X \to Y$ the following conditions are equivalent:*

(a) T is linear and there is a $\alpha > 0$ such that for all x in X, $\|Tx\| = \alpha\|x\|$.
(b) There is a $\alpha > 0$ such that for all x, y in X, $\langle Tx, Ty \rangle = \alpha \langle x, y \rangle$.
(c) T is linear and strongly orthogonality preserving.
(d) T is linear, and orthogonality is preserving.

Blanco and Turšnek [1], extended this result to the case of the linear mapping between normed spaces with Birkhoff-James orthogonality. Recently the linear mappings preserving approximately Roberts orthogonality in normed spaces have been studied [7]. Further, Chmieliński studied the stability of angle preserving mappings on the plane [3]. Also, in [4], orthogonality preserving is considered for mappings on Hilbert C^*-modules.

In this paper, we define a kind of angles between two vectors in real Hilbert spaces and use it to replace orthogonality and approximately orthogonality. More precisely, we give some generalization in which the preservation of a given angle replaces the preservation of orthogonality.

2 Main results

First, we present a nice property for vectors in a Hilbert space:

Lemma 2.1. *Let $\mathcal{H}$ be a real Hilbert space and x, y be non-orthogonal vectors in $\mathcal{H}$. Then the followings are equivalent:*

(a) $\|x\| = \|y\|$.
(b) There is a nonzero scalar a such that

$$(x + ay) \perp y, \quad (y + ax) \perp x.$$

Proof. $(a) \Rightarrow (b)$. We assume that $\|x\| = \|y\| = b$ and we consider

$$a = -\frac{\langle x, y\rangle}{b^2}.$$

Since $\langle x, y\rangle \neq 0$ we have $a \neq 0$. Then it is easy to check that

$$(x + ay) \perp y, \quad (y + ax) \perp x.$$

$(b) \Rightarrow (a)$. Assume that there is a non-zero a such that

$$(x + ay) \perp y, \quad (y + ax) \perp x,$$

then we have

$$\langle x + ay, y\rangle = 0, \quad \langle y + ax, x\rangle = 0,$$

or equivalently,

$$\langle x, y\rangle + a\langle y, y\rangle = 0, \quad \langle x, y\rangle + a\langle x, x\rangle = 0.$$

Since $a \neq 0$, from this equations we deduce that

$$\langle x, x\rangle = \langle y, y\rangle.$$

The proof is complete. □

Now we consider linear maps preserves some non-zero angles:

Definition 2.2. Let $\mathcal{H}$ be a real Hilbert space and x, y are non-zero elements in $\mathcal{H}$. By the Cauchy-Schwarz inequality, we have

$$-1 \leq \frac{\langle x, y\rangle}{\|x\|\|y\|} \leq 1.$$

So there is a unique $0 \leq \alpha \leq \pi$ such that

$$\cos\alpha = \frac{\langle x, y\rangle}{\|x\|\|y\|}.$$

In this case, we call α the angle between x and y, and will be denoted by $x\Delta_\alpha y$.

Before proving Theorem 2.5, we present a useful lemma:

Lemma 2.3. *Let $\mathcal{H}$ be a real Hilbert space and x, y be vectors in $\mathcal{H}$ such that the angle between them is not equal to α, where $\alpha \neq 0, \frac{\pi}{2}, \pi$. Then the followings are equivalent:*

(a) $\|x\| = \|y\|$.
(b) There is a nonzero scalar a such that

$$(x + ay)\Delta_\alpha(-y), \quad (y + ax)\Delta_\alpha(-x).$$

Proof. $(a) \Rightarrow (b)$. Assume that $\|x\| = \|y\| = b$. Put $a := \frac{-2\langle x,y\rangle}{b^2}$. We have

$$\left[\left(a^2+1\right)b^2 + 2a\langle x,y\rangle\right]^{\frac{1}{2}} b\cos\alpha + \langle x,y\rangle + ab^2 = 0$$

and so

$$\cos\alpha = \frac{\langle x+ay, -y\rangle}{\|x+ay\|\|y\|}.$$

Therefore,

$$(x+ay)\Delta_\alpha(-y).$$

Employing an argument similar to that used in the above, gives $(y+ax)\Delta_\alpha(-x)$.
$(b) \Rightarrow (a)$. Assume that there is a nonzero scalar a such that

$$(x+ay)\Delta_\alpha(-y), \quad (y+ax)\Delta_\alpha(-x),$$

then we have

$$-\langle x,y\rangle - a\langle y,y\rangle = \|x+ay\|\|y\|\cos\alpha, \quad -\langle x,y\rangle - a\langle x,x\rangle = \|y+ax\|\|x\|\cos\alpha.$$

By squaring in the two relations above we get

$$(\cos^2\alpha - 1)\langle x,x\rangle^2 a^2 + 2(\cos^2\alpha - 1)\langle x,y\rangle\langle x,x\rangle a + \cos^2\alpha\langle x,x\rangle\langle y,y\rangle - \langle x,y\rangle^2 = 0,$$
$$(\cos^2\alpha - 1)\langle y,y\rangle^2 a^2 + 2(\cos^2\alpha - 1)\langle x,y\rangle\langle y,y\rangle a + \cos^2\alpha\langle y,y\rangle\langle x,x\rangle - \langle x,y\rangle^2 = 0.$$

Finally, from these two equations, we have

$$(\cos^2\alpha - 1)\left[\langle x,x\rangle^2 - \langle y,y\rangle^2\right] a^2 + 2(\cos^2\alpha - 1)\langle x,y\rangle\left[\langle x,x\rangle - \langle y,y\rangle\right] a = 0,$$

and taking into account that $\alpha \neq 0, \pi$, we obtain $\cos^2\alpha \neq 1$. Therefore

$$\left[\langle x,x\rangle^2 - \langle y,y\rangle^2\right] a^2 + 2\langle x,y\rangle\left[\langle x,x\rangle - \langle y,y\rangle\right] a = 0,$$

one can express $\langle x,x\rangle = \langle y,y\rangle$. The proof is complete.□

As a generalization of [2, Theorem 1], we shall show the following theorem.

Theorem 2.4. *Let $\mathcal{H}$ be a real Hilbert space and T be a linear operator on $\mathcal{H}$. Then the followings are equivalent:*

(a) There is a nonzero scalar a such that $\|Tx\| = a\|x\|$ for all x in $\mathcal{H}$.
(b) For all non-zero x, y in $\mathcal{H}$, we have

$$\frac{\|Tx\|}{\|x\|} = \frac{\|Ty\|}{\|y\|}.$$

(c) $\|x\| = \|y\|$ *implies that* $\|Tx\| = \|Ty\|$.
(d) $x \perp y$ implies that $Tx \perp Ty$. That is T preserves orthogonality on the space $\mathcal{H}$.

Proof. The implications $(a) \Rightarrow (b)$, $(b) \Rightarrow (a)$, $(b) \Rightarrow (c)$, $(c) \Rightarrow (b)$ are trivial. $(c) \Rightarrow (d)$. Assume that x, y are orthogonal. In this case we have

$$\|x+y\| = \|x-y\|.$$

Consequently, it follows from (c) that

$$\|Tx+Ty\| = \|Tx-Ty\|,$$

it turn implies that $\langle Tx, Ty\rangle = 0$ and obviously the result holds.
$(d) \Rightarrow (c)$. Assume that $\|x\| = \|y\|$. We consider two cases:

(1) If x, y are orthogonal, then $x+y$, $x-y$ are also orthogonal. Hence by (d), we see that $Tx+Ty$, $Tx-Ty$ are also orthogonal. Therefore

$$\langle Tx+Ty, Tx-Ty\rangle = 0,$$

we can conclude

$$\langle Tx, Tx\rangle = \langle Ty, Ty\rangle$$

and the proof is complete.
(2) If x, y are non-orthogonal, then by Lemma 2.1, there is a non-zero a such that

$$(x+ay) \perp y, \quad (y+ax) \perp x.$$

Since T preserves orthogonality, this means

$$(Tx+aTy) \perp Ty, \quad (Ty+aTx) \perp Tx.$$

Again by Lemma 2.1, we can write $\|Tx\| = \|Ty\|$.

This completes the proof of Theorem 2.4. □

In the remainder of this paper, we shall consider the continuation of our general theorem. The following theorem is a complementary result to Theorem 2.4.

Theorem 2.5. *Let $\mathcal{H}$ be a real Hilbert space and T be a linear operator on $\mathcal{H}$. Then the followings are equivalent:*

(a) There is a nonzero scalar a such that $\|Tx\| = a\|x\|$ for all x in $\mathcal{H}$.
(b) For all non-zero x, y in $\mathcal{H}$, we have

$$\frac{\|Tx\|}{\|x\|} = \frac{\|Ty\|}{\|y\|}.$$

(c) $\|x\| = \|y\|$ implies that $\|Tx\| = \|Ty\|$.
(d) For any $\alpha \in (0,\pi) - \{\frac{\pi}{2}\}$ and $a \in \mathbb{R} - \{0\}$, if $(x+ay)\Delta_\alpha(-y)$ and $(y+ax)\Delta_\alpha(-x)$ then there exist a nonzero scalar a_1 such that $(Tx+a_1Ty)\Delta_\alpha(-Ty)$ and $(Ty+a_1Tx)\Delta_\alpha(-Tx)$, $x, y \in \mathcal{H}$.

Proof. The implications $(a) \Rightarrow (b)$, $(b) \Rightarrow (a)$, $(b) \Rightarrow (c)$ and $(c) \Rightarrow (b)$ are trivial. The details are omitted.

To show that $(c) \Rightarrow (d)$, suppose that $a \neq 0$, $\alpha \in (0,\pi) - \{\frac{\pi}{2}\}$, $(x+ay)\Delta_\alpha(-y)$ and $(y+ax)\Delta_\alpha(-x)$ for $x, y \in \mathcal{H}$. By Lemma 2.3, $\|x\| = \|y\|$ and so $\|Tx\| = \|Ty\|$. By using the Lemma 2.3, there exist a nonzero scalar a_1, such that $(Tx+a_1Ty)\Delta_\alpha(-Ty)$ and $(Ty+a_1Tx)\Delta_\alpha(-Tx)$. This proves $(c) \Rightarrow (d)$.
Finally, we show $(d) \Rightarrow (c)$ (and this will prove the theorem). If we assume (d) holds and $\|x\| = \|y\|$, then by Lemma 2.3 there is a nonzero scalar a, such that $(x+ay)\Delta_\alpha(-y)$ and $(y+ax)\Delta_\alpha(-x)$, it follows that $(Tx+a_1Ty)\Delta_\alpha(-Ty)$ and $(Ty+a_1Tx)\Delta_\alpha(-Tx)$. Now the result follows from Lemma 2.3. □

Corollary 2.6. *The condition that α must be non-zero is essential. In fact, for $\alpha = 0, \pi$, the implications may fail. We know from Cauchy-Schwarz that two vectors x, y are parallel if and only if there is a scalar a such that $x = ay$ or $y = ax$. Therefore, every linear mapping on a Hilbert space $\mathcal{H}$ preserves parallel vectors.*

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