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Some methods of constructing Transcendental numbers and Rational approximation

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Abstract It is one thing to show the existence of transcendental numbers, and another thing to construct explicitly the transcendental numbers. Apart from these two things there is much more difficult thing is to investigate the specific number is transcendental. The aim of this article is to take the reader to this three fold nature displayed in the classical literature on transcendental numbers.

1 Introduction

Let $\mathbb{Z}[x]$ denote the ring of polynomials with integer coefficients. We define

$$A = \{a \in \mathbb{C} : p(a) = 0 \text{ for some } p(x) \in \mathbb{Z}[x]\} \tag{1.1}$$

and call it as the set of **algebraic numbers**. We consider the set T such that

$$A \cup T = \mathbb{C} \text{ and } A \cap T = \emptyset \tag{1.2}$$

The elements of the set T are called the **transcendental numbers**.

We define a subset of A as follows

$$A_R = \{a \in \mathbb{R} : p(a) = 0 \text{ for some } p(x) \in \mathbb{Z}[x]\} \tag{1.3}$$

and call the elements of A_R real algebraic numbers and denote its complement in $\mathbb{R}$ by T_R and the elements of T_R are called real transcendental numbers.

Clearly, the set of rational number $\mathbb{Q}$ is a subset A_R. Therefore, we need only to categorize irrational numbers for T_R.

Therefore, the following questions that will follow immediately have high historical importance in the theory of transcendental numbers:

(i) Is T_R a non-empty set ?

(ii) What is the cardinality of T_R ?

Similar questions for A and T will follow automatically.

The subject of transcendental numbers started in 1844 by Liouville, by launching his discovery Liouville' s approximation theorem. Liouville's theorem enabled him to give first proof on the existence of transcendental numbers. George Cantor taken totally a different approach in 1874 and published a spectacular paper on countability of algebraic numbers.

It is one thing to show the existence of transcendental numbers, and another thing to construct explicitly the transcendental numbers. Apart from these two things there is much more difficult thing is to investigate the specific number is transcendental.

The aim of this article is to take the reader to this three fold nature displayed in the literature on transcendental numbers. An undergraduate student who has been introduced to measure theory, after reading the statement "Algebraic numbers are countable" (whose measure is 0) desperately look for hand full of transcendental numbers! After reading this article one will be in a position to write as many transcendental numbers as one wishes.

Hermite proved the transcendence of e in 1873, and Lindemann proved the transcendence of π in 1882. These proofs were considered among the greatest achievements in the nineteenth century mathematics.

Cantor's methods are non-constructive type so it raised questions about its validity at that time. But the result, set of Algebraic numbers A, is countable must have shocked the mathematical community at that time. A qualitative analysis that every number is almost transcendental opened a path to study more in depth about the transcendental numbers.

One may ask an interesting question here which connects to the linear continuum hypothesis. Cantor proved that

No set S has bijection with its power set $\mathcal{P}(S)$

So Is there any bijection between the $\mathcal{P}(A)$ to $\mathbb{C}$?

2 Rational approximation of an irrational number

We need to approximate an irrational number by rational numbers for practical purposes. Many of the standard books contain this literature. One of the main theorems is that there are infinitely many rational numbers p/q corresponding to any irrational number α such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$$

This can be obtained as a corollary to the following simple result derived by Dirichlet in 1842, by using Pigeon-hole principle.

For any real α and any integer $N > 1$, there exist integers p, q with $0 < q < N$ such that $|q\alpha - p| \leq 1/Q$.

There is a better approximation to this, which is known as The classical Hurwitz-Borel theorem:

For any irrational number α, there are infinitely many rational numbers, p/q such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}$$

and note that here $\sqrt{5}$ is the best possible constant in the above inequality. It means if $\sqrt{5}$ is replaced by any larger constant there are irrational numbers α, which do not satisfy the above inequality for infinitely many rational numbers. One such example seen in textbooks is $\frac{1}{2}(\sqrt{5} - 1)$. In fact, this is an important statement to motivate to find the irrational numbers which can be approximated by infinitely many rational numbers within $1/q^3$, $1/q^4, \ldots, 1/q^n$ etc.

Also note that if α is a rational number and $\alpha \neq p/q$ there are only finitely many rationals p/q such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$$

The value two in the power of q in the above inequality is defined by some authors as approximation exponent or the order of approximation.

Definition 2.1. A real number α has approximation exponent (or order of approximation) $\tau(\alpha)$ if $\tau(\alpha)$ is the smallest number such that for all $n > \tau(\alpha)$

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^n} \tag{2.1}$$

has only finitely many solutions.

One can see that an approximation exponent of a rational number is one. From the theorem of Dirichlet, the approximation exponent of an irrational number is at least two.

When we approximate an irrational number by a rational number we wish that the error is small and the denominator is also as small as possible to use it for practical purposes. Therefore, an approximation index is defined to be the product of these two factors, namely the error and the denominator. The smaller of this product may be the better approximation of this number α. For example,

$$10\left|\sqrt{3}-\frac{17}{10}\right| \approx 0.32$$
$$3\left|\sqrt{3}-\frac{5}{3}\right| \approx 0.195$$

So, we may say, $5/3$ is a better approximation than $17/10$. Since all rational numbers are algebraic, the question comes to categorize only the irrational numbers.

Definition 2.2. An irrational number α is called well-approximable if for all positive integers N, n there is a rational number p/q such that

$$\left|\alpha-\frac{p}{q}\right| < \frac{1}{Nq^n}$$

It is also possible to construct as many well-approximable numbers as you want. See Niven [1]. One of the main theorems is due to Joseph Liouville, which says that a well-approximable number can not be algebraic. In fact, this theorem enables us to give a concrete example to show the existence of a transcendental number

Theorem 2.3. *(Liouville, 1844) If α is an algebraic number with degree $n > 1$ then there exists a number $c = c(\alpha) > 0$ such that the inequality*

$$\left|\alpha-\frac{p}{q}\right| > \frac{c}{q^n} \tag{2.2}$$

holds for all rational numbers $p/q(q > 0)$.

We can use Liouville's theorem to construct transcendental numbers. For example, let

$$\xi = \sum_{n=1}^{\infty} 10^{-n!}.$$

Write

$$p_j = 10^{j!}\sum_{n=1}^{j} 10^{-n!}, \quad q_j = 10^{j!} \quad (j = 1, 2, \dots)$$

then p_j and q_j are relatively prime rational integers and we have

$$\left|\xi-\frac{p_j}{q_j}\right| = \sum_{n=j+1}^{\infty} 10^{-n!} < 10^{-(j+1)!}\left(1+10^{-1}+10^{-2}+\dots\right)$$
$$= \frac{10}{9}q_j^{-j-1} < q_j^{-j}.$$

The transcendental numbers which are shown by using this technique are called Liouville numbers.

Definition 2.4. An irrational number α is called a Liouville number if for no pair $c > 0$, $n \geq 2$ the inequality 2.2 in Theorem 2.3 holds for all rational numbers p/q.

Clearly Liouville numbers are transcendental and it is also known that the set of Liouville numbers is of measure zero. Don't get tempted to show that π is a Liouville number !

Mahler[4] proved the following theorem to show π is not a Liouville number.

Theorem 2.5. *If p and $q \geq 2$ are two positive integers, then*

$$\left|\pi-\frac{p}{q}\right| > \frac{1}{q^{42}}$$

3 Construction of transcendental numbers

3.1 Construction of transcendental numbers using polynomials with rational coefficients

Another interesting method of constructing transcendental numbers that are not Liouville's numbers is due to Mahler. He proved a fascinating result choosing the values of a specific non-constant polynomial and writing its values as follows: See [3].

Consider a non- constant polynomial $p(x)$ with rational coefficients such that $p(n) > 0$ for all $n \in \mathbb{N}$. Then the number

$$0.p(1)p(2)p(3)\ldots$$

is a transcendental number but is not a Livoullie's number. For example, for $p(x) = 2x$ we have the following number

$$0.24681012161820\ldots$$

is a transcendental number but not a Livouville's number. By using the method given above a reader can easily see that the number

$$0.123456789101112\ldots$$

is a transcendental number.

3.2 Lindemann's theorem and Transcendence of e and π

There are several theorems of constructing transcendental numbers. In 1882, F. Lindemann sketched in his memoir a more general theorem from which one can derive e and π are transcendental. We state the Lindemann's theorem below:

Theorem 3.1. *For any distinct algebraic numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ and any non-zero algebraic numbers $\beta_1, \beta_2, \ldots, \beta_n$ we have*

$$\beta_1 e^{\alpha_1} + \beta_2 e^{\alpha_2} + \cdots + \beta_n e^{\alpha_n} \neq 0 \tag{3.1}$$

From Theorem 3.1, as a special case we can see e is transcendental: For any rational numbers $\beta_1, \beta_2, \ldots, \beta_n$ we have the following relation

$$\beta_1 e^n + \beta_2 e^{n-1} + \cdots + \beta_n e^0 \neq 0$$

Another way, in the Equation 3.1 for $n = 1$, we have e^α is transcendental for any non-zero algebraic α. For $\alpha = 1$, it follows that e is a transcendental number.

Transcendence of π follows from the following Equation:

$$e^{\pi i} + 1 = 0 \tag{3.2}$$

Euler considered π is one of the five primary numbers in Mathematics and he felt himself the existence of such a simple relation in Eqn. 3.2 with other four primary numbers is a proof of the existence of God!

Also by writing,

$$\cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2}$$

it follows that $\cos\alpha$ is transcendental for any algebraic $\alpha \neq 0$. Similarly one can verify that $\sin\alpha$ and $\tan\alpha$ are transcendental for algebaraic α.

3.3 Hilbert's 7 th problem

In 1900, at the International Congress of Mathematicians, held in Paris, Hilbert raised a list of 23 problems. One of the formulations of Hilbert's 7 th problem was the following:

Is α^β is transcendental for any algebraic number $\alpha \neq 0, 1$ and for any algebraic irrational β?

In 1929 Gelfond showed that α^β is transcendental for any algebraic number $\alpha \neq 0, 1$ and for any imaginary quadratic irrational β. In particular, this implies that

$$e^\pi = (-1)^{-i} \tag{3.3}$$

is transcendental. Equation [3.3] can be easily deduced from the Equation 3.2.

4 Roth's theorem and Approximation of π

Friedrich Roth was awarded Field's medal at I.C.M. in Edenburg, in 1958 for proving the following theorem:

Theorem 4.1. ***(Roth's Theorem)*** *If α is any algebraic number, and $\epsilon > 0$, then the inequality*

$$\left|\alpha - \frac{p}{q}\right| > \frac{1}{q^{2+\epsilon}} \tag{4.1}$$

holds for all but finitely many rational numbers p/q.

Note that the ϵ in Roth's theorem can not be dropped as any irrational number can be approximated by infinitely many rational numbers within $1/q^2$.

The reader must note that to show that α is transcendental, one must show that α can be approximated by infinitely many rational p/q within $1/q^{2+\epsilon}$ for some $\epsilon > 0$. That is

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}$$

holds for infinitely many rational numbers.

4.1 Approximation of π

One of the oldest mathematical problems is to determine the area and perimeter of a circle of radius r, Archimedes calculated approximate values of π by approximating the area of regular polygons inscribed in the disc of radius 1. Consider the regular n^{th} polygon of sides $s_n = 3 \times 2^n$ sides. It is easy to find the area a_n of each regular polygon of sides s_n as follows:

By using the area of each isosceles triangle inscribed inside the regular polygon s_n one can compute the area a_n of each regular polygon of sides s_n is equal to

$$a_n = \frac{1}{2} s_n \sin\left(\frac{2\pi}{s_n}\right)$$

for $n = 1, 2, \ldots$.

First note that the area of each isosceles triangle inscribed inside the regular polygon s_n is equal to

$$\frac{1}{2} \sin\left(\frac{2\pi}{s_n}\right).$$

Therefore, the area a_n of each regular polygon of sides s_n is equal to

$$a_n = \frac{1}{2} s_n \sin\left(\frac{2\pi}{s_n}\right)$$

for $n = 1, 2, \ldots$.

Now consider the outer polygons with circle inside. Once again it is not hard to see that the area of each outer isosceles triangle is

$$\tan\left(\frac{\pi}{s_n}\right)$$

and similarly we can find the area of each outer polygon A_n circumscribed the circle is equal to

$$A_n = s_n \tan\left(\frac{\pi}{s_n}\right)$$

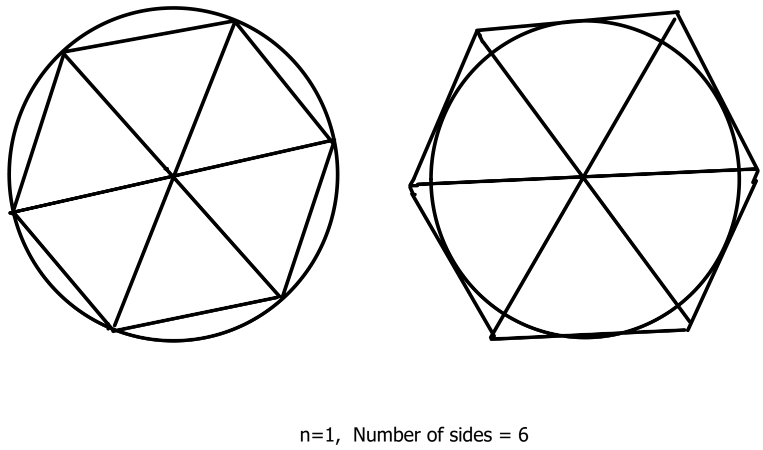

Figure 1. Regular polygons with 6 sides

for $n = 1, 2, \ldots$.

One may verify that both the sequences converge to the same real number which is the area of the unit circle and we define it as π.

Since we know that π is transcendental, by Roth's theorem there is a sequence of rationals $\left(\frac{p_n}{q_n}\right)$, $a_n \leq \frac{p_n}{q_n} \leq A_n$ such that for an $\epsilon > 0$

$$\left|\pi - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{2+\epsilon}}$$

where

$$a_n = 3 \times 2^{n-1} \sin\left(\frac{\pi}{3 \times 2^{n-1}}\right)$$

and

$$A_n = 3 \times 2^n \tan\left(\frac{\pi}{3 \times 2^n}\right)$$

But, the difficulty is to find explicitly a sequence of rational numbers!

References

[1] Ivan Niven, Numbers: Rational And Irrational Numbers. The L.W.Singer Company, Fourth Edition, 1961.

[2] Alan Baker, Transcendental Number Theory, Cambridge University Press,1975.

[3] Joseph Lipman, Transcendental numbers, Queen's University, Kinston, Ontaro, 1966.

[4] Mahler, On the Approximation of π, Proceedings, Series A,56, No. 1 and Indag. Math., 15, No 1, 1953.

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