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A Commutative Ring which is not Strongly n-Stable for any Positive Integer n

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Abstract We recall some definitions and results related to (strongly) n-stable rings and by applying the almost division algorithm over a class of subrings of $K[x]$ (the ring of polynomials over a field K), construct an example of a commutative ring which is not strongly n-stable for any positive integer n.

Dedicated to the Memory of Marion E.Moore

1 Introduction

The main goal of this work is to provide an example of a commutative ring which is not *strongly n-stable* (Example 2.5), simply, by applying [3, Proposition 7]. In this section, we focus only on some terminologies and definitions and in the next section, will provide the required theorems before constructing our example and, finally, end the paper with a brief comments (mainly taken from [10]) related to the *significant power* of the strongly stable range (or rank) in commutative ring theory.

• The notion of *strongly stable range in commutative rings and unitary modules* was initiated by Rahimi (the second author of this paper) ([18] and [19]) and in [10] there is a detailed discussion related to the significant advantage of the notion of strongly stable range (or rank) for the study of *outer product rings and very strongly completable rings* in comparison to the other (original) previous methods. .

The concept of *stable range* was initiated by H. Bass in his investigation of *the stability properties of the general linear group in algebraic K-theory* [2]. In ring theory, stable range provides an arithmetic invariant for rings that is related to interesting issues such as cancelation, substitution, and exchange. The simplest case of stable range 1 has especially proved to be important in the study of many ring-theoretic topics.

• In this note *a ring R*, unless otherwise indicated, is commutative with identity $1 \neq 0$. Also by a sequence of elements of R, we mean *a finite sequence* and will use it implicitly without any confusion in the context.

Definition 1.1. Let R be a commutative ring and $s \geq 1$ an integer. A sequence $(a_1, a_2, \ldots, a_s, a_{s+1})$ of elements of R is said to be *stable* if $(a_1, a_2, \ldots, a_s, a_{s+1}) = (a_1 + b_1 a_{s+1}, a_2 + b_2 a_{s+1}, \ldots, a_s + b_s a_{s+1})$ for some $b_1, b_2, \ldots, b_s \in R$. A sequence $(a_1, a_2, \ldots, a_s, a_{s+1})$ of elements of R is said to be a *unimodular sequence* if 1 is in the ideal $(a_1, a_2, \ldots, a_s, a_{s+1})$.

Remark 1.2. As in [5], we use $(a_1, a_2, \ldots, a_s, a_{s+1})$, $s \geq 1$, to denote both a sequence and the ideal generated by the elements of the sequence; but the context will always make our meaning clear. Also, we follow [5] for the term “unimodular sequence" instead of “primitive vector" as used in [14]. For a detailed study of stable range in commutative rings, semirings, and (unitary) modules; see [5], [10], [14], [15], [16], [17], [18], [19], and [22].

Definition 1.3. For any fixed integer $n \geq 1$, a commutative ring R is said to be strongly n-stable [resp., n-stable] provided that any [resp., unimodular] sequence of elements in R of size larger than n is stable. For convenience, a strongly n-stable (resp., n-stable) ring is called strongly stable [resp., stable] whenever $n = 1$. It follows that if R is a strongly n-stable [resp., an n-stable] ring, then also R is strongly m-stable [resp., m-stable] for any fixed integer $m \geq n$.

In the following, we write the definition of the stable range in a commutative ring for unimodular sequences as defined in [5] for the sake of completeness. Note that this definition is exactly the same as our definition that defined above.

Definition 1.4. Let R be a commutative ring and $s \geq 1$ an integer. An integer $n \geq 1$ is said to be in the *stable range* of R (or simply, R is n-stable) if every unimodular sequence $(a_1, a_2, \ldots, a_s, a_{s+1})$, $s \geq n$, of elements of R is stable.

Remark 1.5. It is clear that if R is n-stable, then it is m-stable for any integer $m \geq n$. Note that the term "R is n-stable" is used in [15] (for convenience) and is exactly the same as the statement "n is in the stable range of R", which is used by D. Estes and J. Ohm [5, page 345].

We now close this section with two facts related to the dimension of a commutative ring and its stable range for the sake of reference and completeness.

Remark 1.6. Theorem 3.4 in [9] states that any n-dimensional commutative integral domain is $(n+1)$-stable and if R is an arbitrary n-dimensional commutative ring, then it is $(n+2)$-stable. Also, Theorem 2.3 in [5] provides a sharp upper bound for the stable range of a commutative ring by its j-Noetherian dimension.

2 Constructing the Claimed Example

In this section, we discuss some facts that are necessary for the construction of our claimed example (Example 2.5) and, finally, end the section with a brief comments (mainly taken from [10]) related to the significant power of the strongly stable range (or rank) for matrix completions over different types of commutative rings.

Theorem 2.1. *Let R be a ring and $n \geq 1$ a fixed integer. Then R is strongly n-stable if and only if any sequence of size $n+1$ is stable.*

Proof. A proof by induction is given for the sufficient part. Assume, $a_1, a_2, \ldots, a_n, a_{n+1}, a_{n+2}$ is a sequence in the ring R. Thus,

$$a_{n+2} \in (a_1, a_2, \ldots, a_n, a_{n+1}, a_{n+2})$$

implies

$$a_{n+2} = \sum_{i=1}^{n+2} a_i x_i = \sum_{i=1}^{n} a_i x_i + l$$

for some $x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2} \in R$, where $l = a_{n+1}x_{n+1} + a_{n+2}x_{n+2}$.
Consequently,

$$a_{n+2} \in (a_1, a_2, \ldots, a_n, l)$$

and for appropriate $r_1, r_2, \ldots, r_n \in R$,

$$a_{n+2} \in (a_1 + r_1 l, a_2 + r_2 l, \ldots, a_n + r_n l)$$

$$\subseteq (a_1 + r_1 x_{n+2} a_{n+2}, a_2 + r_2 x_{n+2} a_{n+2}, \ldots, a_n + r_n x_{n+2} a_{n+2}, a_{n+1} + 0 a_{n+2}).$$

□

Corollary 2.2. *If all unimodular sequences of size $n+1$ ($n \geq 1$ a fixed integer) of a commutative ring R are stable, then any unimodular sequence of size larger than n is stable. That is, R is n-stable.*

We need the following result for construction of our example (Example 2.5).

Theorem 2.3. *The homomorphic image of a strongly n-stable ring is strongly n-stable.*

Proof. By virtue of the above result, it suffices to consider only the sequences of size $n+1$. Let B be an ideal of a strongly n-stable ring A and $a_1+B, a_2+B, \ldots, a_n+B, a_{n+1}+B$ a sequence in A/B. Thus,

$$a_{n+1}+B \in (a_1+B, a_2+B, \ldots, a_n+B, a_{n+1}+B)$$

implies

$$a_{n+1} = \sum_{i=1}^{n+1} r_i a_i + b$$

$$= \sum_{i=1}^{n} r_i a_i + r_{n+1}a_{n+1} + b$$

$$= \sum_{i=1}^{n} r_i a_i + l$$

for some $r_1, r_2, \ldots, r_n, r_{n+1} \in A$ and $b \in B$ with $l = r_{n+1}a_{n+1} + b$. Consequently, for appropriate $s_1, s_2, \ldots, s_n \in A$,

$$a_{n+1} \in (a_1 + s_1 l, a_2 + s_2 l, \ldots, a_n + s_n l)$$

implies

$$a_{n+1}+B \in (a_1 + s_1 r_{n+1}a_{n+1} + B, a_2 + s_2 r_{n+1}a_{n+1} + B, \ldots, a_n + s_n r_{n+1}a_{n+1} + B).$$

proof In [3], chapman defines and applies a weaker form of the division algorithm, namely *the almost division algorithm of index* m (m a positive integer) over a natural class of subrings R of $K[X]$ containing the field K and then shows that *the number of generators of an ideal* I of R can not be only bounded, but also provides examples of ideals that can be generated by n, but not $n-1$ elements. Further, besides [3], [4], and [21], a more general approach to rings and semirings satisfying an almost division algorithm can be found in [13] and [20].

($*$) We now in order to complete our work for the construction of our example (Example 2.5), recall some literature related to the notion of *numerical monoids and commutative semigroup rings* and will take it (exactly) from [3] as follows.

Let $\mathbb{N}_0$ represent the nonnegative integers. An *additive submonoid* S of $\mathbb{N}_0$ is called a *numerical monoid*. Using elementary number theory, it is easy to show that there is a finite set of positive integers $n_1, \ldots, n_k$ such that if $s \in S$, then

$$s = x_1 n_1 + \cdots + x_k n_k,$$

where each x_i is a nonnegative integer. To represent that $n_1, \ldots, n_k$ is a *generating set* for S, we use the notation

$$S = (n_1, \ldots, n_k)$$

$$= \{x_1 n_1 + \cdots + x_k n_k \mid x_i \in \mathbb{N}_0\}.$$

If the generators $n_1, \ldots, n_k$ are *relatively prime*, then S is called *primitive* and in [3, Proposition 2], it is stated that S is always isomorphic to a primitive numerical semigroup; and S has a *unique minimal cardinality generating set*.

Now, if K is a field and S a numerical semigroup, then set

$$K[X;S] = \{f(X) \mid f(X) \in K[X] \; and \; f(X) = \sum_{\sigma \in S} a_i X^{\sigma}\},$$

where it is understood that the sum above is finite. The rings $K[X;S]$ are known as *semigroup rings*, and [6] is a good general reference on the subject. Under our hypotheses, the rings $K[X;S]$ consist of all polynomials with exponents coming from the numerical monoid S.

Let $n > 1$ be a positive integer and set $S = (n, n+1, \ldots, 2n-1)$. Notice that S consists of 0 along with all positive integers greater than or equal to n. Thus, a typical element in $K[X; < n, n+1, \ldots, 2n-1 >]$ is of the form

$$f(X) = a_0 + \sum_{i=n}^{k} a_i X^i$$

where $k \geq n$ and again each a_i is in K.

If $S = (n_1, \ldots, n_k)$ is a numerical semigroup, then the semigroup ring $K[X;S]$ is equivalent to the extension of K by the *monomial terms* $X^{n_1}, \ldots, X^{n_k}$ (i.e., $K[X;S] = K[X^{n_1}, \ldots, X^{n_k}]$).

$(**)$ A *Noetherian integral domain* in which the ideals can be *n-generated* is said to have the *n-generator* property. If an integral domain D has the n-generator property for some $n \in \mathbb{N}$, then it has the m-generator property for some *minimal value* $m \in \mathbb{N}$. Dedekind domains are generally not *principal ideal domains*, but they always have the 2-generator property (a proof of this can be found in [12, Theorem 17]). Actually, any ring with n-generator property ($n \geq 2$ a minimal integer) can not be a strongly $(n-1)$-stable. Otherwise, there exists an $(n-1)$-generated ideal in R which contradicts the minimality of n. Hence, Dedekind domains are not generally strongly stable.

We will use the following remark in the next example.

Remark 2.4. In Proposition 7 of [3], it is shown that the ideal

$$I = (X^n, X^{n+1}, \ldots, X^{2n-1})$$

is not an $(n-1)$-generated ideal in $K[X;S]$, where K is a field, $n > 1$ a positive integer, and $S = (n, n+1, \ldots, 2n-1)$ a numerical semigroup.

We now construct our example by applying Proposition 7 in [3].

Example 2.5. Let $n \geq 2$ be a fixed integer, K a field, and $R_n = K[X;S_n]$, where $S_n = (n, n+1, \ldots, 2n-1)$ a numerical monoid. Clearly, by [3, Proposition 7], R_n is not strongly $(n-1)$-stable (see the preceding remark). Now, let $R = \prod_{n \geq 2} R_n$ be the direct product of R_n's as defined above. Then R is not strongly $(n-1)$-stable since the homomorphic image of a strongly m-stable ring is again strongly m-stable (Theorem 2.3) for any positive integer $m \geq 1$.

• We close by recalling some brief notes (mainly) from [10] related to the significant advantage of the notion of strongly stable range (or rank) for the study of outer product rings and very strongly completable rings in comparison to the other classical approaches.

In [10], there is a discussion of matrix completions over different types of rings with many references related to this context. Completable rings have been extensively studied, largely in connection with Serre's Problem (now the Quillen-Suslin Theorem), which can be phrased as: polynomial rings in finitely many variables over fields are completable [11]. In 1981 Gustafson, Moore, and Reiner [7] extended Hermite's classic result along a different course, showing that (or more generally any Dedekind domain) is *very strongly completable*, i.e., given an $m \times n$ matrix A ($m < n$) and an element d of the ideal generated by its $m \times m$ minors, we can extend A to an $n \times n$ matrix with determinant d. Nearly thirty years later, Gustafson, Robinson, Richter, and Wardlaw [8] returned to the topic, using a similar technique to show that principal ideal rings are very strongly completable.

The literature on outer product rings and very strongly completable rings (as described in [10]) has focused almost exclusively on the Noetherian case. These results are often deep, with proofs that do not typically generalize to non-Noetherian rings at all, so it is likely to be extremely difficult to achieve the same level of understanding of the general case. However, Juett and Williams in [10] achieve a significant expansion of the theory of outer product rings and very strongly completable rings by providing non-Noetherian generalizations of some of the examples given in the introduction of their paper [10].

Their generalizations involve sufficient conditions in terms of the notion of strongly stable rank which was introduced by Rahimi (the second author of this paper) in [18]. Their method involving strongly stable rank provides a completely different approach to that used in the original proofs, and it is arguably simpler. Because strongly stable rank is a previously relatively unexplored concept, which holds some interest in its own right and has the potential to find other applications, they spent a good deal of time developing its theory.

They mentioned that the first study of strongly stable rank conditions was as recent as Rahimi's papers in 2003 [18] and 2005 [19]. This notion is still relatively unexplored and they claimed that they could not find any mention of it outside of Rahimi's work. So they developed its basic properties and found upper bounds on the strongly stable ranks of certain modules. These bounds will eventually lead to new examples of outer product rings and very strongly completable rings.

Also, in [1], there is a discussion of matrix completions over J-stable rings and in [1, Theorem 4.11], it is shown that every J-stable ring is strongly completable. The authors in the paragraph preceding [1, Corollary 4.2] refer to [15, Corollary 2.1], which is a typo and should be "[15, Corollary 2.11]" that states every 2-stable ring is completable.

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