A study on trans-Sasakian manifolds with respect to generalized Tanaka Webster Okumura connection

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Communicated by Zafar Ahsan


Keywords and phrases: Projectively flat, generalized Tanaka Webster Okumura connection, Levi-Civita connection.

Abstract The main object of the present paper is to study trans-Sasakian manifolds with respect to generalized Tanaka Webster Okumura connection. We have studied locally \(\phi\)-Ricci symmetric trans-sasakian manifolds with respect to generalized Tanaka Webster Okumura connection. We also have studied projectively locally recurrent and projectively locally \(\phi\)-recurrent trans-sasakian manifolds with respect to generalized Tanaka Webster Okumura connection.

1 Introduction

Let \(M\) be an \(n\)-dimensional, \(n \geq 3\), connected smooth Riemannian manifold endowed with the Riemannian metric \(g\). Let \(\nabla, R, S\) and \(r\) be the Levi-Civita connection, curvature tensor, Ricci tensor and the scalar curvature of \(M\) respectively.

In 1985 J. A. Oubina \([14]\) introduced a new class of almost contact metric manifolds, called trans-Sasakian manifolds, which includes Sasakian, Kenmotsu and Cosymplectic structures. The authors in the paper \([2],[4]\) and \([7]\) studied such manifolds and obtained some interesting results. In the paper \([13]\) the author studied conformally flat \(\phi\)-recurrent trans-Sasakian manifolds. It is known that \([11]\) trans-Sasakian structure of type \((0, 0), (0, \beta)\) and \((\alpha, 0)\) are Cosymplectic, \(\beta\)-Kenmotsu and \(\alpha\)-Sasakian respectively, where \(\alpha, \beta \in \mathbb{R}\). In \([12]\) J. C. Marrero has shown that a trans-Sasakian manifold of dimension \(n \geq 5\) is either Cosymplectic or \(\alpha\)-Sasakian or \(\beta\)-Kenmotsu manifold. The notion of generalized Tanaka Webster Okumura connection was introduced and studied by the authors in the paper \([10]\). In the present paper we have studied trans-Sasakian manifolds with generalized Tanaka Webster Okumura connection. The present paper is organized as follows. After introduction in Section 1, we give some preliminaries in Section 2. In section 3 we have studied locally \(\phi\)-Ricci symmetric trans-sasakian manifolds with respect to generalized Tanaka Webster Okumura connection. Section 4 is devoted to the study of Projectively locally recurrent and Projectively locally \(\phi\)-recurrent trans-sasakian manifolds with respect to generalized Tanaka Webster Okumura connection.

2 Preliminaries

Let \(M\) be a \((2n + 1)\)-dimensional connected differentiable manifold endowed with an almost contact metric structure \((\phi, \xi, \eta, g)\), where \(\phi\) is a tensor field of type \((1, 1)\), \(\xi\) is a vector field, \(\eta\) is an 1-form and \(g\) is a Riemannian metric on \(M\) such that \([3]\)

\[
\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1. \quad (2.1)
\]

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \quad X, Y \in T(M) \quad (2.2)
\]

Then also

\[
\phi \xi = 0, \quad \eta(\phi X) = 0, \quad \eta(X) = g(X, \xi). \quad (2.3)
\]

\[
g(\phi X, X) = 0. \quad (2.4)
\]
An almost contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be a trans-Sasakian manifold [14] if $(M^{2n+1} \times R, J, G)$ belongs to the class $W_q$ [9] of the Hermitian manifolds, where $J$ is the almost complex structure on $M^{2n+1} \times R$ defined by [8]

$$J(Z, f \frac{d}{dt}) = (\phi Z - f\xi, \eta(Z) \frac{d}{dt}),$$

(2.5)

for any vector field $Z$ on $M^{2n+1}$ and smooth function $f$ on $M^{2n+1} \times R$ and $G$ is the Hermitian metric on the product $M^{2n+1} \times R$. This may be expressed by the condition [14]

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X),$$

(2.6)

for some smooth functions $\alpha$ and $\beta$ on $M^{2n+1}$, and we say that the trans-Sasakian structure is of type $(\alpha, \beta)$. It follows from equation (2.6)

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi),$$

(2.7)

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y)\xi + \beta g(\phi X, \phi Y).$$

(2.8)

In a $(2n+1)$-dimensional trans-Sasakian manifold, from (2.6), (2.7) and (2.8), we can write [7]

$$R(X, Y)\xi = (\alpha^2 - \beta^2)\{\eta(Y)X - \eta(X)Y\} + 2\alpha\beta\{\eta(Y)\phi X - \eta(\phi Y)\xi\}$$

$$= (X\alpha)\phi Y + (Y\alpha)\phi X - (X\beta)\phi^2 Y + (Y\beta)\phi^2 X,$$

(2.9)

$$S(X, \xi) = 2n(\alpha^2 - \beta^2 - \xi\beta)\eta(X) - (2n - 1)X\beta - (\phi X)\alpha,$$

(2.10)

where $S$ is the Ricci tensor. Further we have

$$2\alpha\beta + \xi\alpha = 0.$$  

(2.11)

The generalized Tanaka Webster Okumura connection [10] $\tilde{\nabla}$ and the Levi-Civita connection $\nabla$ are related by

$$\tilde{\nabla}_X Y = \nabla_X Y + A(X, Y)$$

(2.12)

for all vectors fields $X, Y$ on $M$.

Here

$$A(X, Y) = \alpha\{g(X, \phi Y)\xi + \eta(Y)\phi X\} + \beta\{g(X, Y)\xi - \eta(Y)X\} - \eta(X)\phi Y,$$

(2.13)

where $l$ is a real constant.

The Torsion $\tilde{T}$ of the gTWO-connection $\tilde{\nabla}$ is given by

$$\tilde{T}(X, Y) = \alpha\{2g(X, \phi Y)\xi - \eta(X)\phi Y + \eta(Y)\phi X\} + \eta(X)(\beta Y - l\phi Y) - \eta(Y)(\beta X - l\phi X).$$

(2.14)

The relation between the curvature tensors $\tilde{R}$ and $R$ with respect to the generalized Tanaka
Webster Okumura connection $\nabla$ and the Levi-Civita connection $\nabla$ respectively is given by [1]

$$R(X, Y)Z = R(X, Y)Z + \alpha\{g(Y, \nabla_X \phi Z)\xi - g(X, \nabla_Y \phi Z)\xi + g(X, \phi \nabla_Y Z)\xi + \eta(\nabla_Y Z)\phi X - g(Y, \phi \nabla_X Z)\xi - \eta(\nabla_X Z)\phi Y - \eta(Z)\phi [X, Y]\} + \beta\{\eta(\nabla_Z Y)\phi X - \eta(\nabla_X Y)\phi Z - \eta(\nabla_Y X)\phi Z\phi\}$$

$$+ \{\alpha g(X, \phi Z) + \beta g(Y, Z)\}\{\nabla_X \xi + \alpha \phi X + \beta\eta(\xi)\phi Z\}$$

$$- \{\alpha g(X, \phi Z) + \beta g(Y, Z)\}\{\nabla_Y \xi + \alpha \phi Y + \beta(\eta(\xi)\phi Y - \eta(\xi)Y)\} - \{\alpha g(X, \phi Z) + \beta g(Y, Z)\}\{\nabla_X \xi + \alpha \phi X + \beta\eta(\xi)\phi Z\}/2$$

$$\phi^2(\nabla_W Q)X = 0 \quad \text{(3.1)}$$

3 Locally $\phi$-Ricci symmetric trans-sasakian manifolds with respect to generalized Tanaka Webster Okumura connection

**Definition 3.1.** A $(2n+1)$-dimensional $(n>1)$ trans-sasakian manifold will be called locally $\phi$-Ricci symmetric with respect to generalized Tanaka Webster Okumura connection if

$$\phi^2(\nabla_W Q)X = 0 \quad \text{(3.1)}$$

; where the vector fields $X$ and $W$ are orthogonal to $\xi$. The notion of locally $\phi$-Ricci symmetry was introduced by U. C. De and A. Sarkar [6].

Suppose $X$, $Y$ and $Z$ are orthogonal to $\xi$. Then in view of (2.15) the relation between the curvature tensors $R$ and $\tilde{R}$ with respect to the generalized Tanaka Webster Okumura connection $\nabla$ and the Levi-Civita connection $\nabla$ respective is given by

$$\tilde{R}(X, Y)Z = R(X, Y)Z + \alpha g(Y, \nabla_X \phi Z) - g(X, \nabla_Y \phi Z) + g(X, \phi \nabla_Y Z)\xi + \eta(\nabla_Y Z)\phi X - \eta(\nabla_X Z)\phi Y - \eta(Z)\phi [X, Y]\} + \beta\{\eta(\nabla_Z Y)\phi X - \eta(\nabla_X Y)\phi Z - \eta(\nabla_Y X)\phi Z\phi\}$$

Taking inner product on both side of (3.2) by $W$ we get

$$g(R(X, Y)Z, W) = g(\tilde{R}(X, Y)Z, W). \quad \text{(3.3)}$$

From relation (3.3) we have

$$S(X, W) = S(X, W) \quad \text{(3.4)}$$

Again we know that for a trans-sasakian manifolds the Ricci tensor $S$ with respect to Levi Civita connection is

$$S(X, Y) = (\frac{\xi}{2} + \xi\beta - (\alpha^2 - \beta^2))g(X, Y), \quad \text{(3.5)}$$

;where $X$ and $Y$ are orthogonal to $\xi$. Therefore by (3.4) and (3.5) we get

$$\tilde{S}(X, Y) = (\frac{\xi}{2} + \xi\beta - (\alpha^2 - \beta^2))g(X, Y), \quad \text{(3.6)}$$

Again we know that the Ricci operator $\tilde{Q}$ with generalized Tanaka Webster Okumura connection is given by

$$\tilde{S}(X, Y) = g(\tilde{Q}X, Y). \quad \text{(3.7)}$$
Combining (3.6) and (3.7) we get
\[ \tilde{Q}X = \left( \frac{r}{2} + \xi \beta - (\alpha^2 - \beta^2) \right) X. \] (3.8)

Differentiating both side covariantly by \( W \) with respect to the generalized Tanaka Webster Okumura connection \( \tilde{\nabla} \) we get from (3.8)
\[ (\tilde{\nabla}_W \tilde{Q})X = \frac{dr(W)}{2} X. \] (3.9)

Now applying \( \phi^2 \) on both side of (3.9) and using (2.1) we get
\[ \phi^2(\tilde{\nabla}_W \tilde{Q})X = -\frac{dr(W)}{2} X. \] (3.10)

Thus we are in a position to state the following:

**Theorem 3.1.** A \((2n+1)\)-dimensional \((n>1)\) trans-sasakian manifold of type \((\alpha, \beta)\) is locally \( \phi \)-Ricci symmetric with respect to generalized Tanaka Webster Okumura connection if and only if the scalar curvature is constant, provided \( \alpha \) and \( \beta \) are constant.

### 4 Projectively locally recurrent and Projectively locally \( \phi \)-recurrent trans-sasakian manifolds with respect to generalized Tanaka Webster Okumura connection

**Definition 4.1.** For a \((2n+1)\)-dimensional \((n>1)\) trans-sasakian manifold the Weyl projective curvature tensor \( \bar{P} \) with respect to generalized Tanaka Webster Okumura connection will be given by,
\[ \bar{P}(X,Y)Z = R(X,Y)Z - \frac{1}{2n} \left( S(Y,Z)X - S(X,Z)Y \right). \] (4.1)

**Definition 4.2.** A \((2n+1)\)-dimensional \((n>1)\) trans-sasakian manifold with respect to generalized Tanaka Webster Okumura connection is called Projectively flat if it satisfies
\[ \bar{P}(X,Y)Z = 0 \] (4.2).

; for any vector fields \( X, Y, Z \) on the manifold.

**Definition 4.3.** A \((2n+1)\)-dimensional trans-Sasakian manifold will be called projectively locally recurrent with respect to the generalized Tanaka Webster Okumura connection \( \tilde{\nabla} \) if
\[ (\tilde{\nabla}_W \bar{P})(X,Y)Z = A(W) \bar{P}(X,Y)Z \] (4.3)

; for any vector fields \( X, Y, Z \) and \( W \) orthogonal to \( \xi \) and \( A \) is an 1-form defined by \( A(W) = g(W, \rho) \), for some vector field \( \rho \).

**Definition 4.4.** A \((2n+1)\)-dimensional trans-Sasakian manifold will be called projectively locally \( \phi \)-recurrent with respect to the generalized Tanaka Webster Okumura connection \( \tilde{\nabla} \) if
\[ \phi^2(\tilde{\nabla}_W \bar{P})(X,Y)Z = A(W) \bar{P}(X,Y)Z \] (4.4)

; for any vector fields \( X, Y, Z \) and \( W \) orthogonal to \( \xi \) and \( A \) is an 1-form defined by \( A(W) = g(W, \rho) \), for some vector field \( \rho \).

In this connection it should be mentioned that the notion of locally \( \phi \)- recurrent manifolds was introduced in the paper [5] in context of Sasakian geometry. In view of (3.2) and (3.7) we get from (4.1)
\[
\bar{P}(X,Y)Z = R(X,Y)Z + \alpha \{ g(Y, \nabla_X \phi Z) - g(X, \nabla_Y \phi Z) \\
+ g(X, \phi \nabla_Y Z) - g(Y, \phi \nabla_X Z) \} \xi + \beta \{ [X,Y], Z \} \xi \\
- \frac{1}{2n} \{ g(Y,Z)X - g(X,Z)Y \} \left( \frac{r}{2} + \xi \beta - (\alpha^2 - \beta^2) \right).
\] (4.5)
Now differentiating both side covariantly by \( W \) with respect to the generalized Tanaka Webster Okumura connection \( \tilde{\nabla} \) we obtain from (4.5)
\[
(\tilde{\nabla}_W \bar{P})(X, Y, Z) = (\tilde{\nabla}_W R)(X, Y, Z) + \alpha \{ g(Y, \nabla_X \phi Z) - g(X, \nabla_Y \phi Z) \\
+ g(X, \phi \nabla_Y Z) - g(Y, \phi \nabla_X Z) \} \tilde{\nabla}_W \xi + \beta g([X, Y], Z) \tilde{\nabla}_W \xi \tag{4.6}
\]
where \( \alpha \) and \( \beta \) are considered as a constant. Again using (22) we get
\[
(\tilde{\nabla}_W R)(X, Y, Z) = (\tilde{\nabla}_W R)(X, Y, Z) + \{ \alpha g(W, \phi R(X, Y) Z) + \beta g(W.R(X, Y) Z) \} \xi. \tag{4.7}
\]
and
\[
\tilde{\nabla}_W \xi = \nabla_W \xi. \tag{4.8}
\]
Using (4.7) and (4.8) in (4.6) we get
\[
(\tilde{\nabla}_W \bar{P})(X, Y, Z) = (\tilde{\nabla}_W R)(X, Y, Z) + \alpha \{ g(Y, \nabla_X \phi Z) - g(X, \nabla_Y \phi Z) \\
+ g(X, \phi \nabla_Y Z) - g(Y, \phi \nabla_X Z) \} \tilde{\nabla}_W \xi + \beta g([X, Y], Z) \tilde{\nabla}_W \xi \tag{4.9}
\]
Suppose that the manifold is projectively locally recurrent. Then the equation (4.9) becomes
\[
A(W) \bar{P}(X, Y, Z) = (\tilde{\nabla}_W R)(X, Y, Z) + \alpha \{ g(Y, \nabla_X \phi Z) - g(X, \nabla_Y \phi Z) \\
+ g(X, \phi \nabla_Y Z) - g(Y, \phi \nabla_X Z) \} \tilde{\nabla}_W \xi + \beta g([X, Y], Z) \tilde{\nabla}_W \xi \tag{4.10}
\]
Taking inner product with respect to \( W \) in both side of (4.10) and considered \( r \) as constant we get
\[
A(W) g(\bar{P}(X, Y, Z), W) = g((\tilde{\nabla}_W R)(X, Y, Z), W) + \alpha \{ g(Y, \nabla_X \phi Z) - g(X, \nabla_Y \phi Z) \\
+ g(X, \phi \nabla_Y Z) - g(Y, \phi \nabla_X Z) \} g(W, \tilde{\nabla}_W \xi) + \beta g([X, Y], Z) g(W, \tilde{\nabla}_W \xi) \tag{4.11}
\]
Suppose \( \alpha \) is a constant, then by (2.11) we get \( \beta = 0 \). In such a case from equation (4.11) we get
\[
A(W) g(\bar{P}(X, Y, Z), W) = g((\tilde{\nabla}_W R)(X, Y, Z), W). \tag{4.12}
\]
Thus we are in a position to state the following:

**Theorem 4.1.** A \((2n+1)\)-dimensional Projectively locally recurrent trans-Sasakian manifold of type \((\alpha, \beta)\) with \( \alpha \) as a constant is projectively flat with respect to generalized Tanaka Webster Okumura connection \( \tilde{\nabla} \) if and only if it is locally symmetric with respect to Levi-Civita connection \( \nabla \) and the scalar curvature is constant. Again applying \( \phi^2 \) on both side of (4.9) we get,
Suppose that the manifold is projectively locally $\phi$-recurrent. Then the equation (4.13) becomes

$$
A(W)\tilde{P}(X,Y)Z = \phi^2(\nabla_W R)(X,Y)Z + \alpha\{g(Y,\nabla_X \phi Z) - g(X,\nabla_Y \phi Z)
+ g(X,\phi \nabla_Y Y) - g(Y,\phi \nabla_X X)\}(\alpha \phi W - \beta W)
+ \beta g([X,Y],Z)(\alpha \phi W - \beta W)
+ \frac{dr(W)}{4n}\{g(Y,Z)X - g(X,Z)Y\},
$$

(4.14)

Now taking inner product on both side of (4.14) by $\xi$ we get,

$$
A(W)g(\tilde{P}(X,Y)Z,\xi) = g(\phi^2(\nabla_W R)(X,Y)Z,\xi),
$$

(4.15)

Thus we are in a position to state the following:

**Theorem 4.2.** A $(2n+1)$-dimensional $(n>1)$ locally $\phi$-recurrent trans-sasakian manifold of type $(\alpha, \beta)$ is projectively flat with respect to generalized Tanaka Webster Okumura connection if and only if it is locally $\phi$-symmetric with respect to Levi-Civita connection.

**References**


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Received: 2022-04-16
Accepted: 2022-06-09