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A study on trans-Sasakian manifolds with respect to generalized Tanaka Webster Okumura connection

Ali Akbar

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Abstract The main object of the present paper is to study trans-Sasakian manifolds with respect to generalized Tanaka Webster Okumura connection. We have studied locally ϕ-Ricci symmetric trans-sasakian manifolds with respect to generalized Tanaka Webster Okumura connection. We also have studied projectively locally recurrent and projectively locally ϕ-recurrent trans-sasakian manifolds with respect to generalized Tanaka Webster Okumura connection.

1 Introduction

Let M be an n-dimensional, $n \geq 3$, connected smooth Riemannian manifold endowed with the Riemannian metric g. Let ∇, R, S and r be the Levi-Civita connection, curvature tensor, Ricci tensor and the scalar curvature of M respectively.

In 1985 J. A. Oubina [14] introduced a new class of almost contact metric manifolds, called trans-Sasakian manifolds, which includes Sasakian, Kenmotso and Cosymplectic structures. The authors in the paper [2],[4] and [7] studied such manifolds and obtained some interesting results. In the paper [13] the author studied conformally flat ϕ-recurrent trans-Sasakian manifolds. It is known that [11] trans-Sasakian structure of type (0, 0), (0, β) and (α, 0) are Cosymplectic, β-Kenmotsu and α-Sasakian respectively, where α, $\beta \in R$. In [12] J. C. Marrero has shown that a trans-Sasakian manifold of dimension n $\geq$ 5 is either Cosymplectic or α-Sasakian or β-Kenmotsu manifold. The notion of generalized Tanaka Webster Okumura connection was introduced and studied by the authors in the paper [10]. In the present paper we have studied trans-Sasakian manifolds with generalized Tanaka Webster Okumura connection. The present paper is organized as follows. After introduction in Section 1, we give some preliminaries in Section 2. In section 3 we have studied locally ϕ-Ricci symmetric trans-sasakian manifolds with respect to generalized Tanaka Webster Okumura connection. Section 4 is devoted to the study of Projectively locally recurrent and Projectively locally ϕ-recurrent trans-sasakian manifolds with respect to generalized Tanaka Webster Okumura connection.

2 Preliminaries

Let M be a $(2n+1)$-dimensional connected differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is an 1-form and g is a Riemannian metric on M such that [3]

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1. \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y). \quad X, Y \in T(M) \tag{2.2}$$

Then also

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(X) = g(X, \xi). \tag{2.3}$$

$$g(\phi X, X) = 0. \tag{2.4}$$

An almost contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be a trans-Sasakian manifold [14] if $(M^{2n+1} \times R, J, G)$ belongs to the class W_4 [9] of the Hermitian manifolds, where J is the almost complex structure on $M^{2n+1} \times R$ defined by [8]

$$J(Z, f\frac{d}{dt}) = (\phi Z - f\xi, \eta(Z)\frac{d}{dt}), \tag{2.5}$$

for any vector field Z on M^{2n+1} and smooth function f on $M^{2n+1} \times R$ and G is the Hermitian metric on the product $M^{2n+1} \times R$.This may be expressed by the condition [14]

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.6}$$

for some smooth functions α and β on M^{2n+1}, and we say that the trans-Sasakian structure is of type (α, β). It follows from equation (2.6)

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{2.7}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y)\xi + \beta g(\phi X, \phi Y). \tag{2.8}$$

In a $(2n+1)$-dimensional trans-Sasakian manifold, from (2.6), (2.7) and (2.8), we can write [7]

$$\begin{aligned} R(X,Y)\xi &= (\alpha^2 - \beta^2)\{\eta(Y)X - \eta(X)Y\} + 2\alpha\beta\{\eta(Y)\phi X - \eta(X)\phi Y\} \\ &- (X\alpha)\phi Y + (Y\alpha)\phi X - (X\beta)\phi^2 Y + (Y\beta)\phi^2 X. \end{aligned} \tag{2.9}$$

$$S(X, \xi) = \{2n(\alpha^2 - \beta^2) - \xi\beta\}\eta(X) - (2n-1)X\beta - (\phi X)\alpha, \tag{2.10}$$

where S is the Ricci tensor. Further we have

$$2\alpha\beta + \xi\alpha = 0. \tag{2.11}$$

The generalized Tanaka Webster Okumura connection [10] $\tilde{\nabla}$ and the Levi-Civita connection ∇ are related by

$$\tilde{\nabla}_X Y = \nabla_X Y + A(X,Y) \tag{2.12}$$

for all vectors fields X, Y on M.
Here

$$A(X,Y) = \alpha\{g(X, \phi Y)\xi + \eta(Y)\phi X\} + \beta\{g(X,Y)\xi - \eta(Y)X\} - l\eta(X)\phi Y, \tag{2.13}$$

where l is a real constant.
The Torsion $\tilde{T}$ of the gTWO-connection $\tilde{\nabla}$ is given by

$$\tilde{T}(X,Y) = \alpha\{2g(X.\phi Y)\xi - \eta(X)\phi Y + \eta(Y)\phi X\} + \eta(X)(\beta Y - l\phi Y) - \eta(Y)(\beta X - l\phi X). \tag{2.14}$$

The relation between the curvature tensors $\tilde{R}$ and R with respect to the generalized Tanaka

Webster Okumura connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ respectively is given by [1]

$$\begin{aligned}
& \tilde{R}(X,Y)Z = R(X,Y)Z + \alpha\{g(Y,\nabla_X\phi Z)\xi - g(X,\nabla_Y\phi Z)\xi + g(X,\phi\nabla_Y Z)\xi \\
+ & \eta(\nabla_Y Z)\phi X - g(Y,\phi\nabla_X Z)\xi - \eta(\nabla_X Z)\phi Y - \eta(Z)\phi[X,Y]\} + \beta\{\eta(\nabla_X Z)Y \\
- & \eta(\nabla_Y Z)X + g([X,Y],Z)\xi\} - l\{\eta(X)\phi\nabla_Y Z + \eta(Y)\phi\nabla_X Z + \eta([X,Y])\phi Z\} \\
+ & \{\alpha g(Y,\phi Z) + \beta g(Y,Z)\}\{\nabla_X\xi + \alpha\phi X + \beta(\eta(X)\xi - X)\} \\
- & \{\alpha g(X,\phi Z) + \beta g(X,Z)\}\{\nabla_Y\xi + \alpha\phi Y + \beta(\eta(Y)\xi - Y)\} \\
+ & (\alpha\phi Y - \beta Y)[\nabla_X\eta(Z) + \alpha\{g(X,\phi\eta(Z))\xi + \eta(\eta(Z))\phi X\} + \beta\{g(X,\eta(Z))\xi \\
- & \eta(\eta(Z))X\} - l\eta(X)\phi\eta(Z)] - (\alpha\phi X - \beta X)[\nabla_Y\eta(Z) + \alpha\{g(Y,\phi\eta(Z))\xi \\
+ & \eta(\eta(Z))\phi Y\} + \beta\{g(Y,\eta(Z))\xi - \eta(\eta(Z))Y\} - l\eta(Y)\phi\eta(Z)] \\
+ & l\eta(X)[\nabla_Y\phi Z + \alpha\{\eta(Y)\eta(Z) - g(Y,Z)\}\xi + \beta g(Y,\phi Z)\xi + l\eta(Y)\{Z - \eta(Z)\xi\}] \\
- & l\eta(Y)[\nabla_X\phi Z + \alpha\{\eta(X)\eta(Z) - g(X,Z)\}\xi + \beta g(X,\phi Z)\xi + l\eta(X)\{Z - \eta(Z)\xi\}] \\
+ & \alpha\eta(Z)[\nabla_X\phi Y - \nabla_Y\phi X + 2\beta g(X,\phi Y)\xi + l\{\eta(X)Y - \eta(Y)X\}] \\
- & \beta\eta(Z)[\alpha\{2g(X,\phi Y)\xi + \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(Y)X - \eta(X)Y\} \\
+ & l\{\eta(Y)\phi X - \eta(X)\phi Y\}] - l[\nabla_X\eta(Y) - \nabla_Y\eta(X) + \alpha\{g(X,\phi\eta(Y))\xi \\
+ & \eta(\eta(Y))\phi X) - g(Y,\phi\eta(X))\xi - \eta(\eta(X))\phi Y\} + \beta\{g(X,\eta(Y))\xi \\
- & \eta(\eta(Y))X - g(Y,\eta(X))\xi + \eta(\eta(X))Y\} + l\{\eta(Y)\phi\eta(X) - \eta(X)\phi\eta(Y)]\phi Z.
\end{aligned} \tag{2.15}$$

3 Locally ϕ-Ricci symmetric trans-sasakian manifolds with respect to generalized Tanaka Webster Okumura connection

Definition 3.1. A (2n+1)-dimensional (n>1) trans-sasakian manifold will be called locally ϕ-Ricci symmetric with respect to generalized Tanaka Webster Okumura connection if

$$\phi^2(\tilde{\nabla}_W Q)X = 0 \tag{3.1}$$

; where the vector fields X and W are orthogonal to ξ. The notion of locally ϕ-Ricci symmetry was introduced by U. C. De and A. Sarkar [6].
Suppose X, Y and Z are orthogonal to ξ. Then in view of (2.15) the relation between the curvature tensors $\tilde{R}$ and R with respect to the generalized Tanaka Webster Okumura connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ respectivel is given by

$$\begin{aligned}
\tilde{R}(X,Y)Z &= R(X,Y)Z + \alpha\{g(Y,\nabla_X\phi Z) - g(X,\nabla_Y\phi Z) + g(X,\phi\nabla_Y Z) \\
&- g(Y,\phi\nabla_X Z)\}\xi + \beta g([X,Y],Z)\xi
\end{aligned} \tag{3.2}$$

Taking inner product on both side of (3.2) by W we get

$$g(\tilde{R}(X,Y)Z,W) = g(R(X,Y)Z,W). \tag{3.3}$$

From relation (3.3) we have

$$\tilde{S}(X,W) = S(X,W) \tag{3.4}$$

Again we know that for a trans-sasakian manifolds the Ricci tensor S with respect to Levi Civita connection is

$$S(X,Y) = (\frac{r}{2} + \xi\beta - (\alpha^2 - \beta^2))g(X,Y), \tag{3.5}$$

;where X and Y are othogonal to ξ. Therefore by (3.4) and (3.5) we get

$$\tilde{S}(X,Y) = (\frac{r}{2} + \xi\beta - (\alpha^2 - \beta^2))g(X,Y), \tag{3.6}$$

Again we know that the Ricci operator $\tilde{Q}$ with generalized Tanaka Webster Okumura connection is given by

$$\tilde{S}(X,Y) = g(\tilde{Q}X,Y). \tag{3.7}$$

Combining (3.6) and (3.7) we get

$$\tilde{Q}X = (\frac{r}{2} + \xi\beta - (\alpha^2 - \beta^2))X. \tag{3.8}$$

Differentiating both side covariantly by W with respect to the generalized Tanaka Webster Okumura connection $\tilde{\nabla}$ we get from (3.8)

$$(\tilde{\nabla}_W \tilde{Q})X = \frac{dr(W)}{2}X.. \tag{3.9}$$

Now applying ϕ^2 on both side of (3.9) and using (2.1) we get

$$\phi^2(\tilde{\nabla}_W \tilde{Q})X = -\frac{dr(W)}{2}X. \tag{3.10}$$

Thus we are in a position to state the following:
Theorem 3.1. A (2n+1)-dimensional (n>1) trans-sasakian manifold of type (α, β) is locally ϕ-Ricci symmetric with respect to generalized Tanaka Webster Okumura connection if and only if the scalar curvature is constant, provided α and β are constant.

4 Projectively locally recurrent and Projectively locally ϕ-recurrent trans-sasakian manifolds with respect to generalized Tanaka Webster Okumura connection

Definition 4.1. For a (2n+1)-dimensional (n>1) trans-sasakian manifold the Weyl projective curvature tensor $\tilde{P}$ with respect to generalized Tanaka Webster Okumura connection will be given by,

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{2n}\{\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y\}. \tag{4.1}$$

Definition 4.2. A (2n+1)-dimensional (n>1) trans-sasakian manifold with respect to generalized Tanaka Webster Okumura connection is called Projectively flat if it satisfies

$$\tilde{P}(X,Y)Z = 0 \tag{4.2}$$

; for any vector fields X, Y, Z on the manifold .
Definition 4.3. A (2n+1)-dimensional trans-Sasakian manifold will be called projectively locally recurrent with respect to the generalized Tanaka Webster Okumura connection $\tilde{\nabla}$ if

$$(\tilde{\nabla}_W \tilde{P})(X,Y)Z = A(W)\tilde{P}(X,Y)Z \tag{4.3}$$

; for any vector fields X, Y, Z and W orthogonal to ξ and A is an 1-form defined by $A(W) = g(W,\rho)$, for some vector field ρ.
Definition 4.4. A (2n+1)-dimensional trans-Sasakian manifold will be called projectively locally ϕ-recurrent with respect to the generalized Tanaka Webster Okumura connection $\tilde{\nabla}$ if

$$\phi^2(\tilde{\nabla}_W \tilde{P})(X,Y)Z = A(W)\tilde{P}(X,Y)Z \tag{4.4}$$

; for any vector fields X, Y, Z and W orthogonal to ξ and A is an 1-form defined by $A(W) = g(W,\rho)$, for some vector field ρ.
In this connection it should be mentioned that the notion of locally ϕ- recurrent manifolds was introduced in the paper [5] in context of Sasakian geometry. In view of (3.2) and (3.7) we get from (4.1)

$$\begin{aligned} \tilde{P}(X,Y)Z &= R(X,Y)Z + \alpha\{g(Y,\nabla_X\phi Z) - g(X,\nabla_Y\phi Z) \\ &+ g(X,\phi\nabla_Y Z) - g(Y,\phi\nabla_X Z)\}\xi + \beta g([X,Y],Z)\xi \\ &- \tfrac{1}{2n}\{g(Y,Z)X - g(X,Z)Y\}\{\tfrac{r}{2} + \xi\beta - (\alpha^2 - \beta^2)\}. \end{aligned} \tag{4.5}$$

Now differentiating both side covariantly by W with respect to the generalized Tanaka Webster Okumura connection $\tilde{\nabla}$ we obtain from (4.5)

$$\begin{aligned}(\tilde{\nabla}_W\tilde{P})(X,Y)Z &= (\tilde{\nabla}_W R)(X,Y)Z + \alpha\{g(Y,\nabla_X\phi Z) - g(X,\nabla_Y\phi Z)\\ &+ g(X,\phi\nabla_Y Z) - g(Y,\phi\nabla_X Z)\}\tilde{\nabla}_W\xi + \beta g([X,Y],Z)\tilde{\nabla}_W\xi\\ &+ \alpha\{g(X,\tilde{\nabla}_W\phi\nabla_Y Z) - g(Y,\tilde{\nabla}_W\phi\nabla_X Z)\}\xi\\ &- \frac{dr(W)}{4n}\{g(Y,Z)X - g(X,Z)Y\},\end{aligned} \tag{4.6}$$

where α and β are considered as a constant. Again using (22) we get

$$(\tilde{\nabla}_W R)(X,Y)Z = (\nabla_W R)(X,Y)Z + \{\alpha g(W,\phi R(X,Y)Z) + \beta g(W.R(X,Y)Z)\}\xi. \tag{4.7}$$

and

$$\tilde{\nabla}_W\xi = \nabla_W\xi. \tag{4.8}$$

Using (4.7) and (4.8) in (4.6) we get

$$\begin{aligned}(\tilde{\nabla}_W\tilde{P})(X,Y)Z &= (\nabla_W R)(X,Y)Z + \{\alpha g(W,\phi R(X,Y)Z) + \beta g(W.R(X,Y)Z)\}\xi\\ &+ \alpha\{g(Y,\nabla_X\phi Z) - g(X,\nabla_Y\phi Z)\\ &+ g(X,\phi\nabla_Y Z) - g(Y,\phi\nabla_X Z)\}\nabla_W\xi + \beta g([X,Y],Z)\nabla_W\xi\\ &+ \alpha\{g(X,\tilde{\nabla}_W\phi\nabla_Y Z) - g(Y,\tilde{\nabla}_W\phi\nabla_X Z)\}\xi\\ &- \frac{dr(W)}{4n}\{g(Y,Z)X - g(X,Z)Y\},\end{aligned} \tag{4.9}$$

Suppose that the manifold is projectively locally recurrent. Then the equation (4.9) becomes

$$\begin{aligned}A(W)\tilde{P}(X,Y)Z &= (\nabla_W R)(X,Y)Z + \{\alpha g(W,\phi R(X,Y)Z) + \beta g(W.R(X,Y)Z)\}\xi\\ &+ \alpha\{g(Y,\nabla_X\phi Z) - g(X,\nabla_Y\phi Z)\\ &+ g(X,\phi\nabla_Y Z) - g(Y,\phi\nabla_X Z)\}\nabla_W\xi + \beta g([X,Y],Z)\nabla_W\xi\\ &+ \alpha\{g(X,\tilde{\nabla}_W\phi\nabla_Y Z) - g(Y,\tilde{\nabla}_W\phi\nabla_X Z)\}\xi\\ &- \frac{dr(W)}{4n}\{g(Y,Z)X - g(X,Z)Y\},\end{aligned} \tag{4.10}$$

Taking inner product with respect to W in both side of (4.10) and considered r as constant we get

$$\begin{aligned}A(W)g(\tilde{P}(X,Y)Z,W) &= g((\nabla_W R)(X,Y)Z,W) + \alpha\{g(Y,\nabla_X\phi Z) - g(X,\nabla_Y\phi Z)\\ &+ g(X,\phi\nabla_Y Z) - g(Y,\phi\nabla_X Z)\}g(W,\nabla_W\xi)\\ &+ \beta g([X,Y],Z)g(W,\nabla_W\xi)\\ &.\end{aligned} \tag{4.11}$$

Suppose α is a constant , then by (2.11) we get $\beta = 0$. In such a case from equation (4.11) we get

$$A(W)g(\tilde{P}(X,Y)Z,W) = g((\nabla_W R)(X,Y)Z,W). \tag{4.12}$$

Thus we are in a position to state the following:

Theorem 4.1. A (2n+1)-dimensional Projectively locally recurrent trans-Sasakian manifold of type (α,β) with α as a constant is projectively flat with respect to generalized Tanaka Webster Okumura connection $\tilde{\nabla}$ if and only if it is locally symmetric with respect to Levi-Civita connection ∇ and the scalar curvature is constant.

Again applying ϕ^2 on both side of (4.9) we get,

$$\begin{aligned}\phi^2(\tilde{\nabla}_W\tilde{P})(X,Y)Z &= \phi^2(\nabla_W R)(X,Y)Z + \alpha\{g(Y,\nabla_X\phi Z) - g(X,\nabla_Y\phi Z)\\ &+ g(X,\phi\nabla_Y Z) - g(Y,\phi\nabla_X Z)\}(\alpha\phi W - \beta W)\\ &+ \beta g([X,Y],Z)(\alpha\phi W - \beta W)\\ &+ \frac{dr(W)}{4n}\{g(Y,Z)X - g(X,Z)Y\},\end{aligned} \tag{4.13}$$

Suppose that the manifold is projectively locally ϕ-recurrent. Then the equation (4.13) becomes

$$\begin{aligned} A(W)\tilde{P}(X,Y)Z &= \phi^2(\nabla_W R)(X,Y)Z + \alpha\{g(Y,\nabla_X \phi Z) - g(X,\nabla_Y \phi Z) \\ &+ g(X,\phi\nabla_Y Z) - g(Y,\phi\nabla_X Z)\}(\alpha\phi W - \beta W) \\ &+ \beta g([X,Y],Z)(\alpha\phi W - \beta W) \\ &+ \frac{dr(W)}{4n}\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned} \tag{4.14}$$

Now taking inner product on both side of (4.14) by ξ we get,

$$A(W)g(\tilde{P}(X,Y)Z,\xi) = g(\phi^2(\nabla_W R)(X,Y)Z,\xi), \tag{4.15}$$

Thus we are in a position to state the following:
Theorem 4.2. A (2n+1)-dimensional (n>1) locally ϕ-recurrent trans-sasakian manifold of type (α, β) is projectively flat with respect to generalized Tanaka Webster Okumura connection if and only if it is locally ϕ-symmetric with respect to Levi-Civita connection.

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Author information

Ali Akbar, Department of Mathematics, Rampurhat College, Rampurhat- 741235, West Bengal, India.
E-mail: aliakbar.akbar@rediffmail.com

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