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SOME FIXED POINT THEOREMS FOR INTEGRAL TYPE CONTRACTIONS ON COMPLETE S-METRIC SPACES

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Abstract In this work, we prove some fixed point theorems for integral type contractive conditions in the setting of complete S-metric spaces and give some consequences of the main results. Also we give some examples in support of the results. Our results extend and generalize several results from the existing literature. Specially our results generalize the results of Özgür and Tas [13].

1 Introduction

Banach contraction principle [1] is one of the milestones in the development of fixed point theory. Its significance lies in the vast applicability to a great number of branches of mathematical sciences, for example, theory of existence of solutions for nonlinear differential, integral and functional equations, variational inequalities and optimization and approximation theory.

A mapping $T\colon X \to X$, where X is a nonempty set and (X, d) is a metric space, is said to be a contraction if there exists $b \in [0, 1)$ such that for all $x, y \in X$,

$$d(T(x), T(y)) \leq b\, d(x, y). \tag{1.1}$$

If the metric space (X, d) is complete then the mapping satisfying (1.1) has a unique fixed point. Inequality (1.1) implies continuity of T.

There are many generalization of this principle. These generalizations are made either by using different contractive conditions or by imposing some additional condition on the ambient spaces. On the other hand, a number of generalizations of metric spaces have been done and one of such generalization is an S-metric space.

In 2012, the concept of an S-metric has been introduced and studied as a generalization of a metric. This concept has been given by Sedghi et al. [24] as follows.

Definition 1.1. ([24]) Let X be a nonempty set and $S\colon X^3 \to [0, \infty)$ be a function satisfying the following conditions for all $x,\, y,\, z,\, t \in X$:

$(S1)$ $S(x, y, z) = 0$ if and only if $x = y = z$;

$(S2)$ $S(x, y, z) \leq S(x, x, t) + S(y, y, t) + S(z, z, t)$.

Then the function S is called an S-metric on X and the pair (X, S) is called an S-metric space (in short SMS).

Example 1.2. ([24]) Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on X, then $S(x, y, z) = \|y + z - 2x\| + \|y - z\|$ is an S-metric on X.

Example 1.3. ([24]) Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on X, then $S(x, y, z) = \|x - z\| + \|y - z\|$ is an S-metric on X.

Example 1.4. ([25]) Let $X = \mathbb{R}$ be the real line. Then $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$ is an S-metric on X. This S-metric on X is called the usual S-metric on X.

In literature, several fixed point theorems have been given for self mappings satisfying various contractive conditions on an S-metric space (see [3, 4, 6, 10, 11, 12, 21, 22, 23, 24, 25]). One of the celebrated results among these works is the Banach contraction principle on a complete S-metric space which was generalized by Sedghi et al. [24]. The statement of the result is as follows.

Theorem 1.5. *([24]) Let (X, S) be a complete S-metric space and let $T\colon X \to X$ be a self-mapping of X such that*

$$S(Tx, Ty, Tz) \leq k\, S(x, y, z), \tag{1.2}$$

for all $x, y, z \in X$, where $k \in (0, 1)$ is a constant. Then T has a unique fixed point in X.

On the other hand some generalizations of the well-known fixed point theorems obtained on S-metric spaces via some new fixed point results (see [11, 12, 24, 25] for more details).

In 2014, Mlaiki [7] introduced the notion of complex valued S-metric space and showed the existence and uniqueness of a common fixed point of two self-mappings in such spaces and also illustrated some examples to validate the results. In [8], the same author introduced $\alpha - \psi$ contractive mapping in S-metric spaces and proved the existence of fixed point for such mapping under some conditions imposed on self-mapping T. In 2016, Souayah and Mlaiki [26] have introduced an extension of S-metric spaces called S_b-metric spaces and proved the existence of fixed point for a self-mapping defined on such spaces and also proved some results on the topology of the S_b-metric spaces (see, also [9]).

Later, different applications of some contractive conditions have been obtained on S-metric space such as differential equations, complex valued functions etc. (see [7, 10, 14, 15, 16]).

Now a days, fixed point theory has been examined for various contractive conditions. Indeed, one of those is integral type contraction which was introduced by Branciari [2] in 2002 and proved a fixed point result for mappings defined on a complete metric space satisfying a general contractive type condition of integral type. The Branciari's [2] result is as follows.

Theorem 1.6. *([2]) Let (X, d) be a complete metric space, $h \in (0, 1)$, $\varphi\colon [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, \infty)$, nonnegative and such that for each $\varepsilon > 0$,*

$$\int_0^{\varepsilon} \varphi(t)dt > 0,$$

and let $T\colon X \to X$ be a self-mapping of X such that

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq h \int_0^{d(x,y)} \varphi(t)dt,$$

for all $x, y \in X$. Then T has a unique fixed point $u \in X$ such that $\lim_{n\to\infty} T^n w = u$ for each $w \in X$.

After Theorem 1.6, a lot of research works have been carried out on generalizing contractive conditions of integral type for different contractive mappings satisfying many known properties (see [18, 19]). Affine work has been done by Rhoades [20] extending the result of Theorem 1.6 ([2]) by replacing the condition (1.3) by the following condition:

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq c \int_0^{m(x,y)=\max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}} \varphi(t)dt, \tag{1.3}$$

for each $c \in [0, 1)$ and $x, y \in X$.

In 2016, Rahman et al. [17] have established a common fixed point result of Altman integral type for four self-mappings in the context of S-metric spaces and gave an example in support of the result.

Recently, Özgür and Tas [13] have studied new contractive conditions of integral type on S-metric spaces and established some fixed point theorems for various contractive conditions of integral type and gave examples in support of the established results. Also they obtained an application to the Fredholm integral equation.

Motivated by Özgür and Tas [13] and some others, we investigate some fixed point theorems for contractive conditions of integral type in the framework of S-metric spaces and give some examples in support of the results. The results presented in this paper extend and generalize several results in the existing literature.

2 Preliminaries

At first we recall some basic results about S-metric spaces.

Definition 2.1. ([24]) Let (X, S) be an S-metric space.
(1) A sequence $\{u_n\}$ in X converges to $u \in X$ if and only if $S(u_n, u_n, u) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $S(u_n, u_n, u) < \varepsilon$. We denote this by $\lim_{n\to\infty} u_n = u$ or $u_n \to u$ as $n \to \infty$.
(2) A sequence $\{u_n\}$ in X is called a Cauchy sequence if $S(u_n, u_n, u_m) \to 0$ as $n, m \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $S(u_n, u_n, u_m) < \varepsilon$.
(3) The S-metric space (X, S) is called complete if every Cauchy sequence in X is convergent in X.

Lemma 2.2. *([24], Lemma 2.5) Let (X, S) be an S-metric space. Then, we have $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.*

The above Lemma 2.2 can be considered as a symmetry condition on an S-metric space.

Lemma 2.3. *([24], Lemma 2.12) Let (X, S) be an S-metric space. If $x_n \to x$ and $y_n \to y$ as $n \to \infty$ then $S(x_n, x_n, y_n) \to S(x, x, y)$ as $n \to \infty$.*

In the following lemma we see the relationship between a metric and S-metric.

Lemma 2.4. *([4]) Let (X, d) be a metric space. Then the following properties are satisfied:*
(1) $S_d(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in X$ is an S-metric on X.
(2) $u_n \to u$ in (X, d) if and only if $u_n \to u$ in (X, S_d).
(3) $\{u_n\}$ is Cauchy in (X, d) if and only if $\{u_n\}$ is Cauchy in (X, S_d).
(4) (X, d) is complete if and only if (X, S_d) is complete.

We call the function S_d defined in Lemma 2.4 (1) as the S-metric generated by the metric d. It can be found an example of an S-metric which is not generated by any metric in [4, 12].

3 Main Results

In this section, we shall prove some unique fixed point theorems for integral type contractive conditions in the setting of complete S-metric spaces.

Throughout this paper we assume that $\varphi\colon [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, \infty)$, nonnegative and such that for each $\varepsilon > 0$,

$$\int_0^{\varepsilon} \varphi(t)dt > 0. \tag{3.1}$$

Theorem 3.1. *Let (X, S) be a complete S-metric space, and the function $\varphi\colon [0, \infty) \to [0, \infty)$ be defined as in (3.1) and let $T\colon X \to X$ be a self-mapping of X such that*

$$\begin{aligned}
\int_0^{S(Tx,Tx,Ty)} \varphi(t)dt \quad \leq \quad & q_1 \int_0^{S(x,x,y)} \varphi(t)dt + q_2 \int_0^{S(x,x,Tx)} \varphi(t)dt \\
& + q_3 \int_0^{S(y,y,Ty)} \varphi(t)dt + q_4 \int_0^{S(x,x,Ty)} \varphi(t)dt \\
& + q_5 \int_0^{S(y,y,Tx)} \varphi(t)dt \\
& + q_6 \int_0^{\max\{S(x,x,y),S(y,y,Ty),S(y,y,Tx)\}} \varphi(t)dt \\
& + q_7 \int_0^{\frac{S(x,x,Tx)S(x,x,Ty)}{2S(x,x,Ty)+S(y,y,Tx)+S(y,y,Ty)}} \varphi(t)dt,
\end{aligned} \tag{3.2}$$

for all $x, y \in X$, where $q_1, q_2, q_3, q_4, q_5, q_6, q_7$ are nonnegative reals such that $q_1 + q_2 + q_3 + 3q_4 + 2q_6 + q_7 < 1$. Then T has a unique fixed point $u \in X$ and we have $T^n w = u$ for each $w \in X$.

Proof. Let $x_0 \in X$ and the sequence $\{x_n\}$ be defined as $T^n x_0 = x_n$ for $n = 1, 2, \ldots$. Suppose that $x_{n+1} \neq x_n$ for all n. Using the inequality (3.2), the conditions $(S1)$, $(S2)$ and Lemma 2.2, we have

$$\begin{aligned}
\int_0^{S(x_n,x_n,x_{n+1})} \varphi(t)dt &= \int_0^{S(Tx_{n-1},Tx_{n-1},Tx_n)} \varphi(t)dt \\
&\leq q_1 \int_0^{S(x_{n-1},x_{n-1},x_n)} \varphi(t)dt + q_2 \int_0^{S(x_{n-1},x_{n-1},Tx_{n-1})} \varphi(t)dt \\
&\quad + q_3 \int_0^{S(x_n,x_n,Tx_n)} \varphi(t)dt + q_4 \int_0^{S(x_{n-1},x_{n-1},Tx_n)} \varphi(t)dt \\
&\quad + q_5 \int_0^{S(x_n,x_n,Tx_{n-1})} \varphi(t)dt \\
&\quad + q_6 \int_0^{\max\{S(x_{n-1},x_{n-1},Tx_{n-1}),S(x_n,x_n,Tx_n),S(x_n,x_n,Tx_{n-1})\}} \varphi(t)dt \\
&\quad + q_7 \int_0^{\frac{S(x_{n-1},x_{n-1},Tx_{n-1})S(x_{n-1},x_{n-1},Tx_n)}{2S(x_{n-1},x_{n-1},Tx_n)+S(x_n,x_n,Tx_{n-1})+S(x_n,x_n,Tx_n)}} \varphi(t)dt \\
&= q_1 \int_0^{S(x_{n-1},x_{n-1},x_n)} \varphi(t)dt + q_2 \int_0^{S(x_{n-1},x_{n-1},x_n)} \varphi(t)dt \\
&\quad + q_3 \int_0^{S(x_n,x_n,x_{n+1})} \varphi(t)dt + q_4 \int_0^{S(x_{n-1},x_{n-1},x_{n+1})} \varphi(t)dt \\
&\quad + q_5 \int_0^{S(x_n,x_n,x_n)} \varphi(t)dt \\
&\quad + q_6 \int_0^{\max\{S(x_{n-1},x_{n-1},x_n),S(x_n,x_n,x_{n+1}),S(x_n,x_n,x_n)\}} \varphi(t)dt \\
&\quad + q_7 \int_0^{\frac{S(x_{n-1},x_{n-1},x_n)S(x_{n-1},x_{n-1},x_{n+1})}{2S(x_{n-1},x_{n-1},x_{n+1})+S(x_n,x_n,x_n)+S(x_n,x_n,x_{n+1})}} \varphi(t)dt \\
&\leq (q_1+q_2+q_4+q_6+q_7) \int_0^{S(x_{n-1},x_{n-1},x_n)} \varphi(t)dt \\
&\quad + (q_3+2q_4+q_6) \int_0^{S(x_n,x_n,x_{n+1})} \varphi(t)dt, \qquad (3.3)
\end{aligned}$$

which implies

$$\int_0^{S(x_n,x_n,x_{n+1})} \varphi(t)dt \leq \left(\frac{q_1+q_2+q_4+q_6+q_7}{1-q_3-2q_4-q_6}\right) \int_0^{S(x_{n-1},x_{n-1},x_n)} \varphi(t)dt. \qquad (3.4)$$

If we take $\mu = \left(\frac{q_1+q_2+q_4+q_6+q_7}{1-q_3-2q_4-q_6}\right)$, then we find $\mu < 1$ since $q_1+q_2+q_3+3q_4+2q_6+q_7 < 1$. Using the inequality (3.4) again, we obtain

$$\int_0^{S(x_n,x_n,x_{n+1})} \varphi(t)dt \leq \mu^n \int_0^{S(x_0,x_0,x_1)} \varphi(t)dt. \qquad (3.5)$$

Taking the limit as $n \to \infty$, in inequality (3.5), we get

$$\lim_{n\to\infty} \int_0^{S(x_n,x_n,x_{n+1})} \varphi(t)dt = 0, \qquad (3.6)$$

since $0 < \mu < 1$. The condition (3.6) implies

$$\lim_{n\to\infty} S(x_n,x_n,x_{n+1}) = 0. \qquad (3.7)$$

Now, we show that the sequence $\{x_n\}$ is a Cauchy sequence. Assume that $\{x_n\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ and subsequences $\{m_i\}$ and $\{n_i\}$ such that $m_i < n_i < m_{i+1}$ with

$$S(x_{m_i}, x_{m_i}, x_{n_i}) \geq \varepsilon, \tag{3.8}$$

and

$$S(x_{m_i}, x_{m_i}, x_{n_i-1}) < \varepsilon. \tag{3.9}$$

Now, using Lemma 2.2, we have

$$\begin{aligned} S(x_{m_i-1}, x_{m_i-1}, x_{n_i-1}) &\leq 2S(x_{m_i-1}, x_{m_i-1}, x_{m_i}) \\ &\quad + S(x_{n_i-1}, x_{n_i-1}, x_{m_i}) \\ &< 2S(x_{m_i-1}, x_{m_i-1}, x_{m_i}) + \varepsilon, \end{aligned} \tag{3.10}$$

and

$$\lim_{i\to\infty} \int_0^{S(x_{m_i-1}, x_{m_i-1}, x_{n_i-1})} \varphi(t)dt \leq \int_0^{\varepsilon} \varphi(t)dt. \tag{3.11}$$

Using the inequality (3.2), (3.4), (3.8) and (3.11), we obtain

$$\begin{aligned} \int_0^{\varepsilon} \varphi(t)dt &\leq \int_0^{S(x_{m_i}, x_{m_i}, x_{n_i})} \varphi(t)dt \\ &= \int_0^{S(Tx_{m_i-1}, Tx_{m_i-1}, Tx_{n_i-1})} \varphi(t)dt \\ &\leq \mu \int_0^{S(x_{m_i-1}, x_{m_i-1}, x_{n_i-1})} \varphi(t)dt \\ &\leq \mu \int_0^{\varepsilon} \varphi(t)dt, \end{aligned} \tag{3.12}$$

which is a contradiction with our assumption that $0 < \mu < 1$. Thus, the sequence $\{x_n\}$ is a Cauchy sequence in X. Hence by completeness hypothesis of X, there exists $u \in X$ such that

$$T^n x_0 = u.$$

From the given inequality (3.2), we find

$$\begin{aligned}
\int_0^{S(Tu,Tu,x_{n+1})}\varphi(t)dt &= \int_0^{S(Tu,Tu,Tx_n)}\varphi(t)dt \\
&\leq q_1\int_0^{S(u,u,x_n)}\varphi(t)dt+q_2\int_0^{S(u,u,Tu)}\varphi(t)dt \\
&\quad +q_3\int_0^{S(x_n,x_n,Tx_n)}\varphi(t)dt+q_4\int_0^{S(u,u,Tx_n)}\varphi(t)dt \\
&\quad +q_5\int_0^{S(x_n,x_n,Tu)}\varphi(t)dt \\
&\quad +q_6\int_0^{\max\{S(u,u,x_n),S(x_n,x_n,Tx_n),S(x_n,x_n,Tu)\}}\varphi(t)dt \\
&\quad +q_7\int_0^{\frac{S(u,u,Tu)S(u,u,Tx_n)}{2S(u,u,Tx_n)+S(x_n,x_n,Tu)+S(x_n,x_n,Tx_n)}}\varphi(t)dt \\
&= q_1\int_0^{S(u,u,x_n)}\varphi(t)dt+q_2\int_0^{S(u,u,Tu)}\varphi(t)dt \\
&\quad +q_3\int_0^{S(x_n,x_n,x_{n+1})}\varphi(t)dt+q_4\int_0^{S(u,u,x_{n+1})}\varphi(t)dt \\
&\quad +q_5\int_0^{S(x_n,x_n,Tu)}\varphi(t)dt \\
&\quad +q_6\int_0^{\max\{S(u,u,x_n),S(x_n,x_n,x_{n+1}),S(x_n,x_n,Tu)\}}\varphi(t)dt. \\
&\quad +q_7\int_0^{\frac{S(u,u,Tu)S(u,u,x_{n+1})}{2S(u,u,x_{n+1})+S(x_n,x_n,Tu)+S(x_n,x_n,x_{n+1})}}\varphi(t)dt.
\end{aligned} \tag{3.13}$$

Taking the limit as $n \to \infty$ in (3.13) and using the condition $(S1)$ and Lemma 2.2, we get

$$\int_0^{S(Tu,Tu,u)}\varphi(t)dt \leq (q_2+q_5+q_6)\int_0^{S(Tu,Tu,u)}\varphi(t)dt, \tag{3.14}$$

which implies $S(Tu,Tu,u)=0$, that is, $Tu=u$ since $q_2+q_5+q_6<1$. This shows that u is a fixed point of T.

Now, we show that the uniqueness of the fixed point. Let u_1 be another fixed point of T with

$u_1 \neq u$. Using the inequality (3.2) and Lemma 2.2, we obtain

$$\begin{aligned}
\int_0^{S(u,u,u_1)} \varphi(t)dt &= \int_0^{S(Tu,Tu,Tu_1)} \varphi(t)dt \\
&\leq q_1 \int_0^{S(u,u,u_1)} \varphi(t)dt + q_2 \int_0^{S(u,u,Tu)} \varphi(t)dt \\
&\quad + q_3 \int_0^{S(u_1,u_1,Tu_1)} \varphi(t)dt + q_4 \int_0^{S(u,u,Tu_1)} \varphi(t)dt \\
&\quad + q_5 \int_0^{S(u_1,u_1,Tu)} \varphi(t)dt \\
&\quad + q_6 \int_0^{\max\{S(u,u,Tu),S(u_1,u_1,Tu_1),S(u_1,u_1,Tu)\}} \varphi(t)dt \\
&\quad + q_7 \int_0^{\frac{S(u,u,Tu)S(u,u,Tu_1)}{2S(u,u,Tu_1)+S(u_1,u_1,Tu)+S(u_1,u_1,Tu_1)}} \varphi(t)dt \\
&= q_1 \int_0^{S(u,u,u_1)} \varphi(t)dt + q_2 \int_0^{S(u,u,u)} \varphi(t)dt \\
&\quad + q_3 \int_0^{S(u_1,u_1,u_1)} \varphi(t)dt + q_4 \int_0^{S(u,u,u_1)} \varphi(t)dt \\
&\quad + q_5 \int_0^{S(u_1,u_1,u)} \varphi(t)dt \\
&\quad + q_6 \int_0^{\max\{S(u,u,u),S(u_1,u_1,u_1),S(u_1,u_1,u)\}} \varphi(t)dt \\
&\quad + q_7 \int_0^{\frac{S(u,u,u)S(u,u,u_1)}{2S(u,u,u_1)+S(u_1,u_1,u)+S(u_1,u_1,u_1)}} \varphi(t)dt \\
&= (q_1 + q_4 + q_5 + q_6) \int_0^{S(u,u,u_1)} \varphi(t)dt, \qquad (3.15)
\end{aligned}$$

which implies $S(u, u, u_1) = 0$, that is, $u = u_1$ since $q_1 + q_4 + q_5 + q_6 < 1$. Consequently, T has a unique fixed point in X. This completes the proof. □

If we take $q_1 = h$ and $q_2 = q_3 = q_4 = q_5 = q_6 = q_7 = 0$ in Theorem 3.1, then we obtain the following result.

Corollary 3.2. *([13]) Let (X, S) be a complete S-metric space, and the function $\varphi\colon [0,\infty) \to [0,\infty)$ be defined as in (3.1) and let $T\colon X \to X$ be a self-mapping of X such that*

$$\int_0^{S(Tx,Tx,Ty)} \varphi(t)dt \leq h \int_0^{S(x,x,y)} \varphi(t)dt, \qquad (3.16)$$

for all $x, y \in X$, where $h \in (0,1)$ is a constant. Then T has a unique fixed point $u \in X$ and we have $T^n w = u$ for each $w \in X$.

Remark 3.3. (1) Corollary 3.2 is a generalization of Branciari [2] fixed point result from complete metric space to the setting of complete S-metric space.

(2) In Corollary 3.2, if we set the function $\varphi\colon [0,\infty) \to [0,\infty)$ as $\varphi(t) = 1$ for all $t \in [0,\infty)$, then we obtain Theorem 3.1 of [24] for $n = 1$.

(3) Corollary 3.2 is also a generalization of Sedghi's et al. result [24] to the case of integral type contraction condition.

(4) Theorem 3.1 is a generalization of Theorem 2.4 of Özgür and Tas [13]. Indeed, if we take $q_1 = h$ and $q_2 = q_3 = q_4 = q_5 = q_6 = q_7 = 0$ in Theorem 3.1, then we get Theorem 2.4 of [13].

(5) Since an S-metric space is a generalization of a metric space. Corollary 3.2 is a generalization of the classical Banach fixed point theorem [1].

(6) If we set the S-metric as $S\colon X^3 \to \mathbb{C}$ and take the function $\varphi\colon [0,\infty) \to [0,\infty)$ as $\varphi(t) = 1$ for all $t \in [0,\infty)$ in Corollary 3.2, then we get Theorem 3.1 in [14] and Corollary 2.5 in [7] for $n = 1$.

Example 3.4. Let $X = \mathbb{R}$, $a > 1$ be a fixed real number and the function $S\colon X^3 \to [0,\infty)$ be defined as

$$S(x,y,z) = \frac{a}{a+1}(|y-z| + |y+z-2x|),$$

for all $x,y,z \in \mathbb{R}$. It can be easily seen that the function S is an S-metric on X. Now we show that this S-metric can not be generated by any metric d. On the contrary, we suppose that there exists a metric d such that

$$S(x,y,z) = d(x,z) + d(y,z), \tag{3.17}$$

for all $x,y,z \in \mathbb{R}$. Hence, we find

$$S(x,x,z) = 2d(x,z) = \frac{2a}{a+1}|x-z|,$$

and

$$d(x,z) = \frac{a}{a+1}|x-z|. \tag{3.18}$$

Similarly, we get

$$S(y,y,z) = 2d(y,z) = \frac{2a}{a+1}|y-z|,$$

and

$$d(y,z) = \frac{a}{a+1}|y-z|. \tag{3.19}$$

Using the equations (3.17), (3.18) and (3.19), we get

$$\frac{a}{a+1}(|y-z| + |y+z-2x|) = \frac{a}{a+1}|x-z| + \frac{a}{a+1}|y-z|,$$

which is a contradiction. Consequently, S is not generated by any metric and $(\mathbb{R}, S)$ is a complete S-metric space.

Let us define the self-mapping $T\colon \mathbb{R} \to \mathbb{R}$ as $T(x) = \frac{x}{2}$ for all $x \in \mathbb{R}$ and the function $\varphi\colon [0,\infty) \to [0,\infty)$ as $\varphi(t) = 2t$ for all $t \in [0,\infty)$. Then we get

$$\int_0^{\varepsilon} \varphi(t)dt = \int_0^{\varepsilon} 2tdt = \varepsilon^2 > 0,$$

for each $\varepsilon > 0$. Therefore T satisfies the inequality (3.16) of Corollary 3.2 for $h = \frac{1}{2}$. Indeed, we have

$$\frac{a^2}{(a+1)^2}|x-y|^2 \le \frac{2a^2}{(a+1)^2}|x-y|^2,$$

for all $x,y \in \mathbb{R}$. Thus, T has a unique fixed point $x = 0$ in $\mathbb{R}$.

Theorem 3.5. *Let (X,S) be a complete S-metric space, and the function $\varphi\colon [0,\infty) \to [0,\infty)$ be defined as in (3.1) and let $T\colon X \to X$ be a self-mapping of X such that*

$$\begin{aligned}\int_0^{S(Tx,Tx,Ty)} \varphi(t)dt \quad &\le \quad a_1 \int_0^{S(x,x,y)} \varphi(t)dt \\ &\quad + a_2 \int_0^{\max\{S(x,x,y),S(x,x,Tx),S(y,y,Ty)\}} \varphi(t)dt \\ &\quad + a_3 \int_0^{\max\{S(x,x,y),S(y,y,Ty),S(y,y,Tx)\}} \varphi(t)dt,\end{aligned} \tag{3.20}$$

for all $x, y \in X$, where a_1, a_2, a_3 are nonnegative reals such that $a_1 + 2a_2 + 2a_3 < 1$. Then T has a unique fixed point $u \in X$ and we have $T^n w = u$ for each $w \in X$.

Proof. Let $x_0 \in X$ and the sequence $\{x_n\}$ be defined as $T^n x_0 = x_n$ for $n = 1, 2, \ldots$. Suppose that $x_{n+1} \neq x_n$ for all n. Using the inequality (3.20), the conditions $(S1)$, $(S2)$ and Lemma 2.2, we have

$$\begin{aligned}
\int_0^{S(x_n,x_n,x_{n+1})} \varphi(t)dt &= \int_0^{S(Tx_{n-1},Tx_{n-1},Tx_n)} \varphi(t)dt \\
&\leq a_1 \int_0^{S(x_{n-1},x_{n-1},x_n)} \varphi(t)dt \\
&\quad + a_2 \int_0^{\max\{S(x_{n-1},x_{n-1},x_n),S(x_{n-1},x_{n-1},Tx_{n-1}),S(x_n,x_n,Tx_n)\}} \varphi(t)dt \\
&\quad + a_3 \int_0^{\max\{S(x_{n-1},x_{n-1},x_n),S(x_n,x_n,Tx_n),S(x_n,x_n,Tx_{n-1})\}} \varphi(t)dt \\
&= a_1 \int_0^{S(x_{n-1},x_{n-1},x_n)} \varphi(t)dt \\
&\quad + a_2 \int_0^{\max\{S(x_{n-1},x_{n-1},x_n),S(x_{n-1},x_{n-1},x_n),S(x_n,x_n,x_{n+1})\}} \varphi(t)dt \\
&\quad + a_3 \int_0^{\max\{S(x_{n-1},x_{n-1},x_n),S(x_n,x_n,x_{n+1}),S(x_n,x_n,x_n)\}} \varphi(t)dt \\
&\leq (a_1 + a_2 + a_3) \int_0^{S(x_{n-1},x_{n-1},x_n)} \varphi(t)dt \\
&\quad + (a_2 + a_3) \int_0^{S(x_n,x_n,x_{n+1})} \varphi(t)dt, \qquad (3.21)
\end{aligned}$$

which implies

$$\int_0^{S(x_n,x_n,x_{n+1})} \varphi(t)dt \leq \left(\frac{a_1 + a_2 + a_3}{1 - a_2 - a_3}\right) \int_0^{S(x_{n-1},x_{n-1},x_n)} \varphi(t)dt. \qquad (3.22)$$

If we take $d = \left(\frac{a_1+a_2+a_3}{1-a_2-a_3}\right)$, then we find $d < 1$ since $a_1 + 2a_2 + 2a_3 < 1$. Using the inequality (3.22) again, we obtain

$$\int_0^{S(x_n,x_n,x_{n+1})} \varphi(t)dt \leq d^n \int_0^{S(x_0,x_0,x_1)} \varphi(t)dt. \qquad (3.23)$$

Taking the limit as $n \to \infty$, in inequality (3.23), we get

$$\lim_{n\to\infty} \int_0^{S(x_n,x_n,x_{n+1})} \varphi(t)dt = 0, \qquad (3.24)$$

since $0 < d < 1$. The condition (3.24) implies

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0. \qquad (3.25)$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. By the similar arguments used in the proof of Theorem 3.1, we see that the sequence $\{x_n\}$ is Cauchy. Then there exists $u \in X$ such that

$$T^n x_0 = u,$$

since by hypothesis (X,S) is a complete S-metric space. From the given inequality (3.20), we find

$$\begin{aligned}
\int_0^{S(Tu,Tu,x_{n+1})}\varphi(t)dt &= \int_0^{S(Tu,Tu,Tx_n)}\varphi(t)dt \\
&\leq a_1\int_0^{S(u,u,x_n)}\varphi(t)dt \\
&\quad +a_2\int_0^{\max\{S(u,u,x_n),S(u,u,Tu),S(x_n,x_n,Tx_n)\}}\varphi(t)dt \\
&\quad +a_3\int_0^{\max\{S(u,u,x_n),S(u,u,Tx_n),S(x_n,x_n,Tu)\}}\varphi(t)dt \\
&= a_1\int_0^{S(u,u,x_n)}\varphi(t)dt \\
&\quad +a_2\int_0^{\max\{S(u,u,x_n),S(u,u,Tu),S(x_n,x_n,x_{n+1})\}}\varphi(t)dt \\
&\quad +a_3\int_0^{\max\{S(u,u,x_n),S(u,u,x_{n+1}),S(x_n,x_n,Tu)\}}\varphi(t)dt.
\end{aligned} \tag{3.26}$$

Passing to the limit as $n\to\infty$ in (3.26) and using the condition $(S1)$ and Lemma 2.2, we get

$$\int_0^{S(Tu,Tu,u)}\varphi(t)dt \leq (a_2+a_3)\int_0^{S(Tu,Tu,u)}\varphi(t)dt,$$

which implies $S(Tu,Tu,u)=0$, that is, $Tu=u$ since $a_2+a_3<1$. This shows that u a fixed point of T.

Now, we show that the fixed point of T is unique. For this, assume that u_1 is another fixed point of T with $u_1\neq u$. Using the inequality (3.20), the condition $(S1)$ and Lemma 2.2, we obtain

$$\begin{aligned}
\int_0^{S(u,u,u_1)}\varphi(t)dt &= \int_0^{S(Tu,Tu,Tu_1)}\varphi(t)dt \\
&\leq a_1\int_0^{S(u,u,u_1)}\varphi(t)dt \\
&\quad +a_2\int_0^{\max\{S(u,u,u_1),S(u,u,Tu),S(u_1,u_1,Tu_1)\}}\varphi(t)dt \\
&\quad +a_3\int_0^{\max\{S(u,u,u_1),S(u,u,Tu_1),S(u_1,u_1,Tu)\}}\varphi(t)dt \\
&= a_1\int_0^{S(u,u,u_1)}\varphi(t)dt \\
&\quad +a_2\int_0^{\max\{S(u,u,u_1),S(u,u,u),S(u_1,u_1,u_1)\}}\varphi(t)dt \\
&\quad +a_3\int_0^{\max\{S(u,u,u_1),S(u,u,u_1),S(u_1,u_1,u)\}}\varphi(t)dt \\
&= (a_1+a_2+a_3)\int_0^{S(u,u,u_1)}\varphi(t)dt,
\end{aligned} \tag{3.27}$$

which implies $S(u,u,u_1)=0$, that is, $u=u_1$ since $a_1+a_2+a_3<1$. Thus, the fixed point of T is unique. This completes the proof. □

Remark 3.6. (1) In Theorem 3.5, if we take $a_1 = h$ and $a_2 = a_3 = 0$, then we get Corollary 3.2 which is a generalization of Branciari [2] fixed point result from complete metric space to the setting of complete S-metric space.

(2) Theorem 3.5 is a generalization of Theorem 2.4 of Özgür and Tas [13]. Indeed, if we take $a_1 = h$ and $a_2 = a_3 = 0$ in Theorem 3.5, then we get Theorem 2.4 of [13].

(3) Since an S-metric space is a generalization of a metric space. Theorem 3.5 is a generalization of the classical Banach fixed point theorem [1].

Theorem 3.7. *Let (X, S) be a complete S-metric space, and the function $\varphi\colon [0,\infty) \to [0,\infty)$ be defined as in (3.1) and let $T\colon X \to X$ be a self-mapping of X such that*

$$\begin{aligned}
\int_0^{S(Tx,Tx,Ty)} \varphi(t)dt \;\leq\;& a\int_0^{S(x,x,y)} \varphi(t)dt + b\int_0^{S(x,x,Ty)} \varphi(t)dt \\
&+c\int_0^{S(y,y,Tx)} \varphi(t)dt + d\int_0^{S(x,x,Tx)} \varphi(t)dt \\
&+e\int_0^{S(y,y,Ty)} \varphi(t)dt \\
&+f\int_0^{\frac{S(y,y,Ty)[1+S(x,x,Tx)]}{[1+S(x,x,y)]}} \varphi(t)dt, \qquad (3.28)
\end{aligned}$$

for all $x, y \in X$, where $a, b, c, d, e, f \geq 0$ are nonnegative reals satisfying $a+3b+d+e+f < 1$. Then T has a unique fixed point $u \in X$ and we have $T^n w = u$ for each $w \in X$.

Proof. Let $x_0 \in X$ and the sequence $\{x_n\}$ be defined as $T^n x_0 = x_n$ for $n = 1, 2, \ldots$. Suppose that $x_{n+1} \neq x_n$ for all n. Using the inequality (3.28), the conditions $(S1)$, $(S2)$ and Lemma 2.2, we have

$$\begin{aligned}
\int_0^{S(x_n,x_n,x_{n+1})} \varphi(t)dt \;=\;& \int_0^{S(Tx_{n-1},Tx_{n-1},Tx_n)} \varphi(t)dt \\
\leq\;& a\int_0^{S(x_{n-1},x_{n-1},x_n)} \varphi(t)dt + b\int_0^{S(x_{n-1},x_{n-1},Tx_n)} \varphi(t)dt \\
&+c\int_0^{S(x_n,x_n,Tx_{n-1})} \varphi(t)dt + d\int_0^{S(x_{n-1},x_{n-1},Tx_{n-1})} \varphi(t)dt \\
&+e\int_0^{S(x_n,x_n,Tx_n)} \varphi(t)dt \\
&+f\int_0^{\frac{S(x_n,x_n,Tx_n)[1+S(x_{n-1},x_{n-1},Tx_{n-1})]}{[1+S(x_{n-1},x_{n-1},x_n)]}} \varphi(t)dt \\
=\;& a\int_0^{S(x_{n-1},x_{n-1},x_n)} \varphi(t)dt + b\int_0^{S(x_{n-1},x_{n-1},x_{n+1})} \varphi(t)dt \\
&+c\int_0^{S(x_n,x_n,x_n)} \varphi(t)dt + d\int_0^{S(x_{n-1},x_{n-1},x_n)} \varphi(t)dt \\
&+e\int_0^{S(x_n,x_n,x_{n+1})} \varphi(t)dt \\
&+f\int_0^{\frac{S(x_n,x_n,x_{n+1})[1+S(x_{n-1},x_{n-1},x_n)]}{[1+S(x_{n-1},x_{n-1},x_n)]}} \varphi(t)dt \\
\leq\;& (a+b+d)\int_0^{S(x_{n-1},x_{n-1},x_n)} \varphi(t)dt \\
&+(2b+e+f)\int_0^{S(x_n,x_n,x_{n+1})} \varphi(t)dt, \qquad (3.29)
\end{aligned}$$

which implies

$$\int_0^{S(x_n,x_n,x_{n+1})} \varphi(t)dt \le \Big(\frac{a+b+d}{1-2b-e-f}\Big) \int_0^{S(x_{n-1},x_{n-1},x_n)} \varphi(t)dt. \tag{3.30}$$

If we take $\beta = \Big(\frac{a+b+d}{1-2b-e-f}\Big)$, then we find $\beta < 1$ since $a + 3b + d + e + f < 1$. Using the inequality (3.30) again, we obtain

$$\int_0^{S(x_n,x_n,x_{n+1})} \varphi(t)dt \le \beta^n \int_0^{S(x_0,x_0,x_1)} \varphi(t)dt. \tag{3.31}$$

Taking the limit as $n \to \infty$, in inequality (3.31), we get

$$\lim_{n\to\infty} \int_0^{S(x_n,x_n,x_{n+1})} \varphi(t)dt = 0, \tag{3.32}$$

since $0 < \beta < 1$. The condition (3.32) implies

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0. \tag{3.33}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. By the similar arguments used in the proof of Theorem 3.1, we see that the sequence $\{x_n\}$ is Cauchy. Then there exists $u \in X$ such that

$$T^n x_0 = u,$$

since by hypothesis (X, S) is a complete S-metric space. From the given inequality (3.28), we find

$$\begin{aligned}
\int_0^{S(Tu,Tu,x_{n+1})} \varphi(t)dt &= \int_0^{S(Tu,Tu,Tx_n)} \varphi(t)dt \\
&\le a \int_0^{S(u,u,x_n)} \varphi(t)dt + b \int_0^{S(u,u,Tx_n)} \varphi(t)dt \\
&\quad +c \int_0^{S(x_n,x_n,Tu)} \varphi(t)dt + d \int_0^{S(u,u,Tu)} \varphi(t)dt \\
&\quad +e \int_0^{S(x_n,x_n,Tx_n)} \varphi(t)dt \\
&\quad +f \int_0^{\frac{S(x_n,x_n,Tx_n)[1+S(u,u,Tu)]}{[1+S(u,u,x_n)]}} \varphi(t)dt \\
&= a \int_0^{S(u,u,x_n)} \varphi(t)dt + b \int_0^{S(u,u,x_{n+1})} \varphi(t)dt \\
&\quad +c \int_0^{S(x_n,x_n,Tu)} \varphi(t)dt + d \int_0^{S(u,u,Tu)} \varphi(t)dt \\
&\quad +e \int_0^{S(x_n,x_n,x_{n+1})} \varphi(t)dt \\
&\quad +f \int_0^{\frac{S(x_n,x_n,x_{n+1})[1+S(u,u,Tu)]}{[1+S(u,u,x_n)]}} \varphi(t)dt.
\end{aligned} \tag{3.34}$$

Passing to the limit as $n \to \infty$ in (3.34) and using the condition $(S1)$ and Lemma 2.2, we get

$$\int_0^{S(Tu,Tu,u)} \varphi(t)dt \le (c+d) \int_0^{S(Tu,Tu,u)} \varphi(t)dt,$$

which implies $S(Tu, Tu, u) = 0$, that is, $Tu = u$ since $c + d < 1$. This shows that u a fixed point of T.

Now, we show that the fixed point of T is unique. For this, assume that u_1 is another fixed point of T with $u_1 \neq u$. Using the inequality (3.28), the condition $(S1)$ and Lemma 2.2, we obtain

$$\begin{aligned}
\int_0^{S(u,u,u_1)} \varphi(t)dt &= \int_0^{S(Tu,Tu,Tu_1)} \varphi(t)dt \\
&\leq a \int_0^{S(u,u,u_1)} \varphi(t)dt + b \int_0^{S(u,u,Tu_1)} \varphi(t)dt \\
&\quad + c \int_0^{S(u_1,u_1,Tu)} \varphi(t)dt + d \int_0^{S(u,u,Tu)} \varphi(t)dt \\
&\quad + e \int_0^{S(u_1,u_1,Tu_1)} \varphi(t)dt \\
&\quad + f \int_0^{\frac{S(u_1,u_1,Tu_1)[1+S(u,u,Tu)]}{[1+S(u,u,u_1)]}} \varphi(t)dt \\
&= a \int_0^{S(u,u,u_1)} \varphi(t)dt + b \int_0^{S(u,u,u_1)} \varphi(t)dt \\
&\quad + c \int_0^{S(u_1,u_1,u)} \varphi(t)dt + d \int_0^{S(u,u,u)} \varphi(t)dt \\
&\quad + e \int_0^{S(u_1,u_1,u_1)} \varphi(t)dt \\
&\quad + f \int_0^{\frac{S(u_1,u_1,u_1)[1+S(u,u,u)]}{[1+S(u,u,u_1)]}} \varphi(t)dt \\
&= (a+b+c) \int_0^{S(u,u,u_1)} \varphi(t), \qquad (3.35)
\end{aligned}$$

which implies $S(u,u,u_1) = 0$, that is, $u = u_1$ since $a+b+c < 1$. Thus, the fixed point of T is unique. This completes the proof. □

Remark 3.8. (1) In Theorem 3.7, if we take $b = c = d = e = f = 0$ and $a = h$ then we obtain Theorem 2.4 of [13].

(2) Theorem 3.7 is another generalization of Theorem 2.4 of [13].

(3) Since Theorem 3.7 is another generalization of Theorem 2.4 of [13], Theorem 3.7 generalizes the classical Banach fixed point theorem.

Example 3.9. Let $X = \mathbb{R}$ be the complete S-metric space with the S-metric defined as $S(x,y,z) = |y-z| + |y+z-2x|$ for all $x,y,z \in X$. Let us define the self-mapping $T\colon X \to X$ as

$$T(x) = \begin{cases} x+2, & \text{if } x \in \{0,2\}, \\ 3, & \text{if otherwise,} \end{cases}$$

for all $x \in X$ and the function $\varphi\colon [0,\infty) \to [0,\infty)$ as $\varphi(t) = 2t$ for all $t \in [0,\infty)$. Then we get

$$\int_0^{\varepsilon} \varphi(t)dt = \int_0^{\varepsilon} 2tdt = \varepsilon^2 > 0,$$

for each $\varepsilon > 0$. Therefore T satisfies: (1) The inequality (3.2) in Theorem 3.1 for (by taking $x = 0$ and $y = 1$)

(i) $q_1 = 0, q_2 = \frac{1}{4}, q_3 = q_4 = q_5 = q_6 = q_7 = 0$;
(ii) $q_1 = q_2 = 0, q_3 = \frac{1}{4}, q_4 = q_5 = q_6 = q_7 = 0$;
(iii) $q_1 = q_2 = q_3 = 0, q_4 = \frac{1}{9}, q_5 = q_6 = q_7 = 0$;
(iv) $q_1 = q_2 = q_3 = q_4 = q_5 = q_7 = 0, q_6 = \frac{1}{4}$.

(2) The inequality (3.20) in Theorem 3.5 for (by taking $x = 0$ and $y = 1$)

(i) $a_1 = 0, a_2 = \frac{1}{4}, a_3 = 0$;
(ii) $a_1 = 0, a_2 = 0, a_3 = \frac{1}{4}$.
(3) The inequality (3.28) in Theorem 3.7 for (by taking $x = 0$ and $y = 1$)
(i) $a = 0, b = \frac{1}{9}, c = d = e = f = 0$;
(ii) $a = b = c = e = f = 0, d = \frac{1}{4}$;
(iii) $a = b = c = d = f = 0, e = \frac{1}{4}$;
(iv) $a = b = c = d = e = 0, f = \frac{9}{50}$.

Hence in all the above cases T has a unique fixed point $x = 3$. But T does not satisfy the inequality (3.16) in Corollary 3.2. Indeed, if we take $x = 0$ and $y = 1$, the we obtain

$$\int_0^2 2tdt = 4 \leq h \int_0^2 2tdt = 4h,$$

which is a contradiction since $h \in (0, 1)$.

4 An application to Fredhlom Integral Equation

In this section, we give an application of the contraction condition (3.16) to the Fredhlom integral equation

$$y(u) = h(u) + \mu \int_a^b K(u, t)y(t)dt, \tag{4.1}$$

where $y\colon [a, b] \to \mathbb{R}$ with $-\infty < a < b < \infty$, $K(u, t)$ is called kernel of the integral equation (4.1) is continuous on the squared region $[a, b] \times [a, b]$ with $|K(u, t)| \leq \mathcal{M}$ ($\mathcal{M} > 1$) and $h(u)$ is continuous on $[a, b]$.

Let $C[a, b] = \{f | f\colon [a, b] \to \mathbb{R}$ is a continuous function$\}$. Now,we define the function $S\colon C[a, b] \times C[a, b] \times C[a, b] \to [0, \infty)$ by

$$S(x, y, z) = \sup_{t \in [a,b]} |y(t) - z(t)| + \sup_{t \in [a,b]} |y(t) + z(t) - 2x(t)|, \tag{4.2}$$

for all $x, y, z \in C[a, b]$. Then the function S is an S-metric. Now, we show that this S-metric can not be generated by and metric d. We assume that this S-metric is generated by any metric d, that is, there exists a metric d such that

$$S(x, y, z) = d(x, z) + d(y, z), \tag{4.3}$$

for all $x, y, z \in C[a, b]$. Then we get

$$S(x, x, z) = 2d(x, z) = 2 \sup_{t \in [a,b]} |x(t) - z(t)|,$$

and

$$d(x, z) = \sup_{t \in [a,b]} |x(t) - z(t)|. \tag{4.4}$$

Similarly, we obtain

$$S(y, y, z) = 2d(y, z) = 2 \sup_{t \in [a,b]} |y(t) - z(t)|,$$

and

$$d(y, z) = \sup_{t \in [a,b]} |y(t) - z(t)|. \tag{4.5}$$

From equations (4.3), (4.4) and (4.5), we get

$$\begin{aligned} \sup_{t \in [a,b]} |y(t) - z(t)| + \sup_{t \in [a,b]} |y(t) + z(t) - 2x(t)| &= \sup_{t \in [a,b]} |x(t) - z(t)| \\ &\quad + \sup_{t \in [a,b]} |y(t) - z(t)|, \end{aligned}$$

which is a contradiction. Hence this S-metric is not generated by any metric d. Thus, $(C[a, b], S)$ is a complete S-metric space.

Theorem 4.1. *(Proposition 3.1, [13]) Let* $(C[a,b], S)$ *be a complete* S*-metric space with the* S*-metric defined in (4.2) and* μ *be a real number such that*

$$|\mu| < \frac{1}{\mathcal{M}(b-a)}.$$

Then the Fredhlom integral equation (4.1) has a unique solution $y\colon [a,b] \to \mathbb{R}$.

Proof. Let us define the function $T\colon C[a,b] \to C[a,b]$ as

$$T(y(u)) = h(u) + \mu \int_a^b K(u,t)y(t)dt.$$

Now we show that T satisfies the contractive condition (3.16). We get

$$\begin{aligned}
S(Ty_1, Ty_1, Ty_2) &= 2 \sup_{u \in [a,b]} |Ty_1(u) - Ty_2(u)| \\
&= 2 \sup_{u \in [a,b]} \left| \mu \int_a^b K(u,t)[y_1(u) - y_2(u)]dt \right| \\
&\leq 2|\mu|\mathcal{M} \sup_{u \in [a,b]} \left| \int_a^b [y_1(u) - y_2(u)]dt \right| \\
&\leq 2|\mu|\mathcal{M} \sup_{u \in [a,b]} |y_1(u) - y_2(u)| \left| \int_a^b dt \right| \\
&\leq |\mu|\mathcal{M}(b-a)S(y_1, y_1, y_2) \\
&< S(y_1, y_1, y_2),
\end{aligned}$$

which implies

$$\int_0^{S(Ty_1,Ty_1,Ty_2)} \varphi(t)dt < \int_0^{S(y_1,y_1,y_2)} \varphi(t)dt.$$

Consequently, the contractive condition (3.16) is satisfied and the Fredhlom integral equation (4.1) has a unique solution y. □

5 Conclusion

In this paper, we prove existence and uniqueness of some fixed point theorems for self-mappings that satisfy an integral type contraction in the setting of complete S-metric spaces and give some consequences of the main results. We provide illustrated examples to validate the results in this paper and also obtained an application to the Fredhlom integral equation. The results presented in this paper generalize and extend several results from the existing literature.

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