Palestine Journal of Mathematics

Vol. 12(2)(2023) , 30–53

On Conformable First and Second Painlevé Equations

Basema Asaad AlTayyan

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 20M99, 13F10; Secondary 13A15, 13M05.

Keywords and phrases: Fractional derivative, Conformable calculus , Conformable Painlevé equations, Generalized Painlevé test.

Abstract In this paper, a generalization of the Painlevé test is constructed to investigate the sufficient condition of the generalized Painlevé property (GPP). The analysis is successfully extended to investigate the GPP of the $2\alpha-$ order $(0 < \alpha \leq 1)$ conformable ordinary differential equations. Thenceforward, applying the analysis to the conformable first and second Painlevé equations $(CP_I$ and $CP_{II})$ complete the study of GPP of these equations. This procedure parameterizes general solutions of the CP_I and CP_{II} in terms of the relevant serieses and shows that the general solutions are $\alpha-$meromorphic in z to its critical points. In particular, it is shown that a special choice of the parameter in the CP_{II} admits a special solution in term of Airy function. In addition, some properties of CP_I and CP_{II} are discussed.

1 Introduction

Fractional calculus (FC) is regarded as a generalization of the classical differentiation and integration for arbitrary non-integer (real or complex) order. FC is almost as old as the classical calculus and goes back to times when Leibniz and Newton invented differential calculus. After 1974, the interest in studing the fractional calculus has been rapidly growing. Fractional derivatives and integrals have many uses and they themselves have arisen from certain requirements in applications. Some of known fractional derivatives are conformable, Riemann-Liouville, modified Riemann-Liouville, Caputo, Hadmard, Erdélyi-Kober, Riesz, Grünwald-Letnikov, Marchaud, and others; see [1]–[9]. The first work devoted exclusively to the subject of conformable calculus was published in 2014 by Khalil, Alhorani, Yousef, and Sababheh [9]. Unlike other definitions, this definition prominently compatible with the classical derivative and it seems to satisfy all the requirements of the standard derivative. The importance of the conformable derivative lies in satisfying the product and quotient formulas. Moreover, it has a simple formula for the chain rule. After Khalil's definition, abundant articles have devoted entirely the conformable calculus for its effectiveness on other mathematical disciplines [10]–[40].

The classification of Painlevé equations originated by Painlevé [41], Gambier [42] and Fuchs [43] around the beginning of the twentieth century, while they were studying problems posed by Picard [44]. A differential equation is said to have the Painlevé property if its solutions have no movable branch points; that is, the locations of multi-valued singularities of any of the solutions are independent of the particular solution chosen and so are dependent only on the equation. Painlevé, Gambier and their colleagues showed that, within the Möbius transformation, there were fifty canonical equations of the form $w'' = F(z, w, w')$ with this property. Among all these equations, six of them are irreducible and can not be solved by known functions; thus they define new functions known as Painlevé transcendents and denoted by $P_I,\ P_{II}, ...,\ P_{VI}$. The other forty-four equations are either integrable in terms of previously known functions or reducible to one of the six Painlevé transcendents [45, 46]. Although the Painlevé equations were discovered from strictly mathematical considerations, they have frequently appeared in many physical problems, and possess rich internal structure. The Painlevé equations play an important role for completely integrable partial differential equations (PDE) [46].

The Painlevé test plays a significant role in the analysis of nonlinear differential equations. It is the most widely used and the most successful technique for detecting integrable differential equations [47]–[50]. The test has been applied to many differential equations and, for those passing the Painlevé test, the indicators of integrability such as the existence of enough conservation laws, the Lax pair, the Daraboux transform, the Bäcklund transform can always be found. However, despite the overwhelming evidence that the integrability of a differential equation should be a closely related to the behavior of its solution near movable singularities, the rigorous study of such a relation has been lacking [51].

It is worth mentioning that this approach is not studied yet with fractional differential equations. The main object of this paper is to develop the method of the analysis of the generalized Painlevé test to investigate the generalized Painlevé Property. This method which utilizes the development of the Painlevé test is applied successfully to to the conformable first and second Painlevé equations (CP_I and CP_{II}). The CP_{II} can be obtained as the similarity reduction of the conformable Korteweg-de Vries $(CKdV)$ and modified Korteweg-de Vries $(CmKdV)$ equations [40]. Furthermore, for a certain choice of the parameter, CP_{II} admits a one-parameter family of solutions in terms of Airy function. Also, P_I and CP_I can be obtained from CP_{II} by the process of contraction. Moreover, many properties which the conformable fractional Painlevé equations possess are illustrated as: Isomonodromy Problems, Generalized Hirota Bilinear Form, Hamiltonian Structure, Generalized Bäcklund Transform, and others.

2 Conformable Calculus

We begin by recalling a brief introduction on the basic definitions and theorems in the conformable calculus that we shall frequently use throughout the paper.

Definition 2.1. [9] Given a function $f : [0, \infty) \to \mathbb{R}$, the conformable derivative of order α of f is defined by

$$D_\alpha[f(z)] = \lim_{\varepsilon \to 0} \frac{f(z + \varepsilon z^{1-\alpha}) - f(z)}{\varepsilon}, \tag{2.1}$$

for all $z > 0$, $\alpha \in (0, 1]$. If $D_\alpha[f(z)]$ exists for z in some interval $(0, a)$, $a > 0$, and $\lim_{z\to 0^+} D_\alpha[f(z)]$ exists, then $D_\alpha[f(0)] = \lim_{z\to 0^+} D_\alpha[f(z)]$.

If, in addition, f is differentiable, then $D_\alpha f(z) = z^{1-\alpha} \frac{df(z)}{dz}$.

Definition 2.2. [9] $I_\alpha[f(z)] = I[z^{\alpha-1} f(z)] = \int_0^z \frac{f(\zeta)}{\zeta^{1-\alpha}} d\zeta$, where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1]$.

Theorem 2.3[9] Let $\alpha \in (0, 1]$ and $f,\ g$ be α-differentiable at a point $z > 0$, then

(i) $D_\alpha[af(z) + bg(z)] = a[D_\alpha f(z)] + b[D_\alpha g(z)]$, for all $a, b \in \mathbb{R}$.

(ii) If $f(z) = z^k$, then $D_\alpha[f(z)] = kz^{k-\alpha}$, for all $k \in \mathbb{R}$.
In particular:

- If $f(z) = \frac{z^\alpha}{\alpha}$, then $D_\alpha[f(z)] = 1$.
- If f is the constant function defined by $f(z) = c$, then $D_\alpha[f(z)] = 0$.

(iii) $D_\alpha[f(z)g(z)] = f(z)D_\alpha[g(z)] + g(z)D_\alpha[f(z)]$.

(iv) $D_\alpha\left[\frac{f(z)}{g(z)}\right] = \frac{g(z)D_\alpha[f(z)] - f(z)D_\alpha[g(z)]}{[g(z)]^2}$.

Lemma 2.4[15] Let $0 < \alpha \le 1$, f be α-differentiable at $g(z) > 0$, and g be α-differentiable at $z > 0$, then $D_\alpha[(fog)(z)] = D_\alpha[f(g(z))]D_\alpha[g(z)][g(z)]^{\alpha-1}$.

Corollary 2.5[22] Let $0 < \alpha \le 1$, f be differentiable at $g(z)$, and g be α-differentiable at $z > 0$, then $D_\alpha[(fog)(z)] = [f'(g(z))]D_\alpha[g(z)]$.

Definition 2.6[15] Let f be a function with n variables $z_1, ...,\ z_n$, and the conformable partial derivative of f of order $0 < \alpha \le 1$ in z_i is defined as follows

$$\frac{\partial^\alpha}{\partial {z_i}^\alpha} f(z_1, ...,\ z_n) = \lim_{\varepsilon \to 0} \frac{f(z_1, ...,\ z_{i-1},\ z_i + \varepsilon z_i^{1-\alpha},\ z_{i+1}, ...,\ z_n) - f(z_1, ...,\ z_n)}{\varepsilon}.$$

Theorem 2.7[15] The Clairaut's theorem for partial derivatives of conformable fractional orders. Assume that $f(t,s)$ is function for which $\frac{\partial^\alpha}{\partial t^\alpha}\left[\frac{\partial^\beta}{\partial s^\beta}f(t,s)\right]$ and $\frac{\partial^\beta}{\partial s^\beta}\left[\frac{\partial^\alpha}{\partial t^\alpha}f(t,s)\right]$ exist and are continuous over the domain $D\subset\mathbb{R}^2$ then

$$\frac{\partial^\alpha}{\partial t^\alpha}\left[\frac{\partial^\beta}{\partial s^\beta}f(t,s)\right]=\frac{\partial^\beta}{\partial s^\beta}\left[\frac{\partial^\alpha}{\partial t^\alpha}f(t,s)\right].$$

In the remainder of this section we propose usefulness concepts which have been used in our study. We refer to the literatures [29]-[36] for basic structures of these concepts.

Definition 2.8. Let $\alpha\in(0,\ 1]$ and $z,z_0\in[0,\ \infty)$. An $\alpha-$power series about z_0 is an infinite series of the form

$$\sum_{n=0}^{\infty}a_n\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^n,\quad a_n\in\mathbb{R},$$

which converges for all z in the domain such that $|z-z_0|<\delta$ $(\delta>0)$ and diverges otherwise, where δ is called the radius of convergent of the given series.

Definition 2.9. Let $\alpha\in(0,\ 1]$ and $z,z_0\in[0,\ \infty)$. If f is an infinitely $\alpha-$differentiable at z_0, then the $\alpha-$Taylor series for the function f at z_0 is

$$f(z)=\sum_{n=0}^{\infty}\frac{1}{n!}D_\alpha^n f(z_0)\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^n,$$

for all $|z-z_0|<\delta$ $(\delta>0)$, δ is the radius of convergent of the given series, and $D_\alpha^n f(z_0)$ denotes the sequential $\alpha-$derivatives on $f(z)$ determined at the point z_0; that is,

$$D_\alpha^2 f(z)=D_\alpha\left(D_\alpha f(z)\right),\quad D_\alpha^n f(z)=D_\alpha\left(D_\alpha^{(n-1)}f(z)\right),\quad n=3,4,\cdots.$$

Definition 2.10. Let $\alpha\in(0,\ 1]$ and $z,z_0\in[0,\ \infty)$. A complex valid function $f(z)$ is said to be an $\alpha-$analytic function at a point z_0 if $f(z)$ possesses a convergent $\alpha-$power series

$$f(z)=\sum_{n=0}^{\infty}a_n\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^n,\quad a_n\in\mathbb{R},$$

for all $|z-z_0|<\delta$ $(\delta>0)$, δ is the radius of convergent of the given series.

Remark 2.11. A function $f(z)$ is an $\alpha-$analytic (or an $\alpha-$holomorghic) if it is an $\alpha-$analytic at each point in the domain.

Remark 2.12. Every $\alpha-$analytic is an infinitely $\alpha-$differentiable.

Remark 2.13. $f(z)$ is an $\alpha-$analytic function if and only if $f(z)$ possesses an $\alpha-$Taylor expansion.

3 Generalized Painlevé Test

(Throughout this paper, we let $\frac{d^{n\alpha}}{dx^{n\alpha}}$, for $n=1,2,...,$ $\alpha\in(0,\ 1]$ denote the conformable derivatives.)
In this section we will construct the theory of generalized Painlevé test. We begin by proposing rigorously some basic concepts in the generalized Painlevé property.

Definition 3.1. The generalized Painlevé property:
A conformable ordinary differential equation $(CODE)$ in the complex domain is said to be of generalized Painlevé type (or has the generalized Painlevé property) if the only movable singularities of its solutions are poles.

Theorem 3.2. A necessary condition that an $n\alpha-$order conformable ordinary differential equation of the form

$$\frac{d^{n\alpha}w(z)}{dz^{n\alpha}} = F\left(z, w, ..., \frac{d^{(n-1)\alpha}w}{dz^{(n-1)\alpha}}\right), \quad 0 < \alpha \leq 1, \tag{3.1}$$

where F is rational in $w, ...,$ and $\frac{d^{(n-1)\alpha}w}{dz^{(n-1)\alpha}}$ and $\alpha-$analytic in z, has the generalized Painlevé property;that is, w has a Laurent expansion about z_0 of the form

$$w(z) = \left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^m \sum_{j=0}^{\infty} a_j \left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^j, \tag{3.2}$$

with $(n-1)$ arbitrary expansion coefficients, besides the pole position which is arbitrary.

Theorem 3.3. Let $f_j(w_1, w_2, ..., w_m, z)$, $(j = 1, 2, ..., m)$ be analytic functions of the variables $w_1, w_2, ..., w_m$ with $w_1 = w_1^0, w_2 = w_2^0, ..., w_m = w_m^0$ for $z = z_0$. Then there exists one and only one system of functions $w_j(z) = w_j$, $(j = 1, 2, ..., m)$ $\alpha-$analytic at the point $z = z_0$, and satisfying the system of conformable ordinary differential equations $\frac{d^\alpha w_j}{dz^\alpha} = f_j(w_1, w_2, ..., w_m, z)$ with the conditions $w_j(z_0) = w_j^0$, where $j = 1, 2, ..., m$.

The generalized Painlevé test to the following form of $2\alpha-$order conformable ordinary differential equation

$$\frac{d^{2\alpha}w}{dz^{2\alpha}} = F\left(w, \frac{d^\alpha w}{dz^\alpha}, z\right), \tag{3.3}$$

where F is $\alpha-$analytic in z, and rational in w and $\frac{d^\alpha w}{dz^\alpha}$ will be treated.
The key step to derive the sufficient condition for equation (3.3) to be possessing the generalized Painlevé property (GPP) is the creation of a series solution around z_0 (z_0 arbitrary point) of the form

$$\begin{aligned} w(z) = \beta\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^k + \textstyle\sum_{j=1}^{l-1} a_{k+j}\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^{k+j} + c_1\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^{k+l} \\ + \textstyle\sum_{j=l+1}^{\infty} a_{k+j}\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^{k+j}. \end{aligned} \tag{3.4}$$

The computation of the $\alpha-$derivative for $w(z)$ is given by

$$\begin{aligned} \tfrac{d^\alpha w}{dz^\alpha} = \beta k\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^{k-1} + \textstyle\sum_{j=1}^{l-1}(k+j)a_{k+j}\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^{k+j-1} + \\ (k+l)c_1\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^{k+l-1} + \textstyle\sum_{j=l+1}^{\infty}(k+j)a_{k+j}\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^{k+j-1}. \end{aligned} \tag{3.5}$$

It is of some interest to set

$$w(z) = v(z)^k, \tag{3.6}$$

from which we obtain

$$v(z) = \epsilon_k w(z)^{\frac{1}{k}}, \quad \text{with} \quad \epsilon_k = \begin{cases} \pm 1, & \text{for k even;} \\ 0, & \text{for k odd.} \end{cases} \tag{3.7}$$

Also, it is convenient to write $\frac{d^\alpha w}{dz^\alpha}$ as

$$\frac{d^\alpha w}{dz^\alpha} = \sum_{j=0}^{\infty} b_{j+k-1}(z)v(z)^{j+k-1}. \tag{3.8}$$

With the help of equation (3.7) equation (3.8) becomes

$$\frac{d^\alpha w}{dz^\alpha} = b_{k-1}\epsilon_k^{k-1}w^{1-\frac{1}{k}} + b_k\epsilon_k^k w + b_{k+1}\epsilon_k^{k+1}w^{1+\frac{1}{k}} + \cdots \tag{3.9}$$

Using equation (3.4) and neglecting terms of $O\left[\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{k+l+1}\right]$, then equation (3.9) can be written as

$$\begin{aligned}\frac{d^\alpha w}{dz^\alpha} &= b_{k-1}\epsilon_k^{k-1}\left\{\beta\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^k+\sum_{j=1}^{l-1}a_{k+j}\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{k+j}+c_1\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{k+l}\right\}^{1-\frac{1}{k}}\\ &+b_k\epsilon_k^k\left\{\beta\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^k+\sum_{j=1}^{l-1}a_{k+j}\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{k+j}+c_1\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{k+l}\right\}\\ &+b_{k+1}\epsilon_k^{k+1}\left\{\beta\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^k+\sum_{j=1}^{l-1}a_{k+j}\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{k+j}+c_1\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{k+l}\right\}^{1+\frac{1}{k}}\\ &+O\left[\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{k+l+1}\right].\end{aligned} \tag{3.10}$$

We can simplify equation (3.5) at once in the suitable form

$$\begin{aligned}\frac{d^\alpha w(z)}{dz^\alpha} &= \beta k\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{k-1}+(k+1)a_{k+1}\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{k}+(k+2)a_{k+2}\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{k+1}\\ &+\cdots+(k+l-1)a_{k+l-1}\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{k+l-2}+(k+l)c_1\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{k+l-1}\\ &+O\left[\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{k+l}\right].\end{aligned} \tag{3.11}$$

The comparison between the two equations (3.10) and (3.11) allows to solve for the coefficients $b_{k-1},\ b_k,\cdots,\ b_{k+l-1}$, henceforward, we can exhibit an expansion of the form

$$\frac{d^\alpha w(z)}{dz^\alpha}=\sum_{j=0}^{l}b_{k+j-1}(z)v(z)^{k+j-1}+O\left[\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{k+l}\right]. \tag{3.12}$$

Next, consider the transformation

$$w(z)=v(z)^k, \tag{3.13a}$$

$$\frac{d^\alpha w(z)}{dz^\alpha}=b_{k-1}(z)v(z)^{k-1}+b_k(z)v(z)^k+\cdots+u(z)v(z)^{k+l-1}, \tag{3.13b}$$

the α derivative of $w(z)$ is given by

$$\frac{d^\alpha w}{dz^\alpha}=\frac{dw}{dv}\frac{d^\alpha v}{dz^\alpha},$$

which can be written as

$$\frac{d^\alpha v}{dz^\alpha}=\frac{1}{k}v^{1-k}\frac{d^\alpha w}{dz^\alpha}. \tag{3.14}$$

Hence,

$$\frac{d^\alpha v}{dz^\alpha}=\frac{1}{k}\left[b_{k-1}+b_kv+\cdots+uv^l\right]. \tag{3.15}$$

Consequently, equation (3.3) can be converted to be as

$$\frac{d^{2\alpha} w}{dz^{2\alpha}}=\frac{d^\alpha}{dz^\alpha}\left[b_{k-1}v^{k-1}+b_kv^k+\cdots+uv^{k+l-1}\right], \tag{3.16}$$

which simplifies at once to the form

$$\begin{aligned}\frac{d^{2\alpha} w}{dz^{2\alpha}} &= \frac{d^\alpha b_{k-1}}{dz^\alpha}v^{k-1}+(k-1)b_{k-1}v^{k-2}\frac{d^\alpha v}{dz^\alpha}+\frac{d^\alpha b_k}{dz^\alpha}v^k+(k)b_kv^{k-1}\frac{d^\alpha v}{dz^\alpha}\\ &+\cdots+\frac{d^\alpha u}{dz^\alpha}v^{k+l-1}+(k+l-1)uv^{k+l-2}\frac{d^\alpha v}{dz^\alpha}.\end{aligned} \tag{3.17}$$

Substitution of $\frac{d^\alpha v}{dz^\alpha}$ from equation (3.15) into equation (3.17), leads to

$$\begin{aligned}\frac{d^{2\alpha} w}{dz^{2\alpha}} &= \frac{d^\alpha b_{k-1}}{dz^\alpha}v^{k-1}+\left(\frac{k-1}{k}\right)b_{k-1}v^{k-2}\left[b_{k-1}+b_kv+\cdots+uv^l\right]\\ &+\frac{d^\alpha b_k}{dz^\alpha}v^k+b_kv^{k-1}\left[b_{k-1}+b_kv+\cdots+uv^l\right]+\cdots\\ &+\frac{d^\alpha u}{dz^\alpha}v^{k+l-1}+\frac{(k+l-1)}{k}uv^{k+l-2}\left[b_{k-1}+b_kv+\cdots+uv^l\right].\end{aligned} \tag{3.18}$$

However, $\frac{d^{2\alpha}w}{dz^{2\alpha}}$ must satisfy the equation

$$\frac{d^{2\alpha}w}{dz^{2\alpha}} = F\left(w, \frac{d^{\alpha}w}{dz^{\alpha}}, z\right) = F\left(v^k, b_{k-1}v^{k-1} + b_k v^k + \cdots + uv^{k+l-1}, z\right). \tag{3.19}$$

Now, by equating the right hand sides of the two equations (3.18) and (3.19), one can achieve the following relation

$$\begin{aligned}\frac{d^{\alpha}u}{dz^{\alpha}} &= v^{1-k-l}F\left(v^k, b_{k-1}v^{k-1} + b_k v^k + \cdots + uv^{k+l-1}, z\right)\\ &- \left[b_{k-1} + b_k v + \cdots + uv^l\right]\left[(1-\tfrac{1}{k})b_{k-1}v^{-1-l} + b_k v^{-l} + \cdots + u(1+\tfrac{l-1}{k})v^{-1}\right]\\ &- \left[\frac{d^{\alpha}b_{k-1}}{dz^{\alpha}}v^{-l} + \cdots + \frac{d^{\alpha}b_{k+l-2}}{dz^{\alpha}}v^{-1}\right].\end{aligned}$$

Finally, we can conclude that the given $2\alpha-$order conformable ordinary differential equation (3.3) is equivalent to the system of $\alpha-$order conformable ordinary differential equations (3.15) and (3.20). If the right hand sides of equations (3.15) and (3.20) are $\alpha-$analytic functions of the variables u, and v with the initial values $u(z_0) = u^0$, $v(z_0) = v^0$, hence, the conditions of **Thr 3** will be obtained, and so, in the neighborhood of z_0 there exists one and only one system of $\alpha-$analytical functions $v = v(z)$ and $u = u(z)$ which satisfy the second order system of differential equations with the initial conditions $u(z_0) = u^0$, $v(z_0) = v^0$. Henceforth, the given $2\alpha-$order conformable ordinary differential equation has an $\alpha-$analytic solution in the neighborhood of z_0, and so, the $2\alpha-$order conformable ordinary differential equation has the generalized Painlevé property (GPP).

4 Conformable First and Second Painlevé Equations and the Generalized Painlevé Test

This section is an application to the methodology which has been developed in Section 3. The application will be treated each of the conformable first and second Painlevé equations.

4.1 Conformable First Painlevé Equation

Consider the following conformable first Painlevé equation (CP_I)

$$\frac{d^{2\alpha}w}{dz^{2\alpha}} = 6w^2 + \frac{z^{\alpha}}{\alpha}. \tag{4.1}$$

The essence of the generalized Painlevé test is to establish the $\alpha-$analytic structure of $w(z)$ with respect to z in the entire complex $z-$plane.

Claim: The only algebraic singularities of equation (4.1) are movable double poles. In addition, equation (4.1) has a unique $\alpha-$holomorphic solution $w(z, z_0, w_0, \frac{d^{\alpha}w_0}{dz^{\alpha}})$ in some neighborhood of $z = z_0$ with $w(z_0) = w_0$, and $\frac{d^{\alpha}w}{dz^{\alpha}}(z_0) = \frac{d^{\alpha}w_0}{dz^{\alpha}}$.

According to the theory of the generalized Painlevé test, the algorithm is constructed from three steps.

(i) Finding the dominant behavior:

For this aim we consider

$$w \sim \sigma\left(\frac{z^{\alpha}}{\alpha} - \frac{z_0^{\alpha}}{\alpha}\right)^k, \tag{4.2}$$

from which we will obtain

$$\frac{d^{2\alpha}w}{dz^{2\alpha}} = \sigma k(k-1)\left(\frac{z^{\alpha}}{\alpha} - \frac{z_0^{\alpha}}{\alpha}\right)^{k-2}. \tag{4.3}$$

Direct substitution of w and $\frac{d^{2\alpha}w}{dz^{2\alpha}}$ into equation (4.1), gives

$$\sigma k(k-1)\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{k-2}\sim 6\sigma^2\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{2k}+\frac{z^\alpha}{\alpha}, \tag{4.4}$$

Equation (4.4) can be rewritten in an alternative form as

$$\sigma k(k-1)\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{k-2}\sim 6\sigma^2\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{2k}+\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)+\frac{z_0^\alpha}{\alpha}. \tag{4.5}$$

Next, we need to calculate the possible values of k for which there is a balance between two or more than two terms in the equation, here we find $k=-2$. A successful ansatz for the dominant behavior is

$$\frac{d^{2\alpha}w}{dz^{2\alpha}}\sim 6w^2. \tag{4.6}$$

Substitution of equation (4.2) into equation (4.6), gives

$$\sigma(1-\sigma)=0,$$

this implies $\sigma=0$ or $\sigma=1$, we neglect $\sigma=0$ and take $\sigma=1$.

(ii) Finding the resonances:
The next step in the algorithm is to determine the resonances, for this purpose we need to define $w(z)$ as follows:

$$w=\sigma\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{-2}+\rho\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{r-2}. \tag{4.7}$$

Using the definition of w equation (4.7) into the dominant equation (4.6), leads to

$$\begin{aligned}&6\sigma\left(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha}\right)^{-4}+(r-2)(r-3)\rho\left(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha}\right)^{r-4}\sim 6\sigma^2\left(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha}\right)^{-4}+\\&12\sigma\rho\left(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha}\right)^{r-4}+6\rho^2\left(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha}\right)^{2r-4},\end{aligned} \tag{4.8}$$

from which one can achieve

$$(r-2)(r-3)\rho=12\sigma\rho. \tag{4.9}$$

For $\rho\neq 0$ and $\sigma=1$, $(r+1)(r-6)=0$, thus, the resonances are $r=-1,6$.

(iii) Finding the constant of integration: The key step for finding the constant of integration is by assuming w to be in the form

$$w(z)=\sum_{j=0}^{\infty}a_j\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{j-2}. \tag{4.10}$$

One can obtain,

$$\frac{d^\alpha w}{dz^\alpha}=\sum_{j=0}^{\infty}(j-2)a_j\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{j-3}, \tag{4.11a}$$

$$\frac{d^{2\alpha}w}{dz^{2\alpha}}=\sum_{j=0}^{\infty}(j-2)(j-3)a_j\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{j-4}, \tag{4.11b}$$

$$w^2=\sum_{j=0}^{\infty}\sum_{k=0}^{j}a_{j-k}a_k\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{j-4}. \tag{4.11c}$$

Direct Substitution of equations (4.11) into equation (4.1), and collecting similar terms, leads to

$$\sum_{j=0}^{\infty}\left[(j-2)(j-3)a_j-6\sum_{k=0}^{j}a_{j-k}a_k\right]\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{j-4}-\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)-\frac{z_0^\alpha}{\alpha}=0. \quad (4.12)$$

Equating the coefficients of the various powers of $\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)$ to zero, one can obtain the coefficients $a_j's$ for $(j\geq 0)$.
Henceforward, equation (4.10) becomes as

$$\begin{aligned} w(z)=&\left(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha}\right)^{-2}-\tfrac{1}{10}\tfrac{z_0^\alpha}{\alpha}\left(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha}\right)^{2}-\tfrac{1}{6}\left(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha}\right)^{3}\\ &+c\left(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha}\right)^{4}+\textstyle\sum_{j=8}^{\infty}a_j\left(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha}\right)^{j-2}, \end{aligned} \quad (4.13)$$

with z_0 and c are arbitrary constants, and the coefficients a_j, for $j\geq 8$, are uniquely given by the relation

$$a_j=\frac{6}{(j+1)(j-6)}\sum_{k=0}^{j-8}a_{k+2}a_{j-k-6},\quad j\geq 8. \quad (4.14)$$

The resulting series (4.13) is a convergent series in a neighborhood of z_0.
It is more convenient to rewrite equation (4.13) in the following alternative form

$$\begin{aligned} w(z)=&\left(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha}\right)^{-2}-\tfrac{1}{10}\tfrac{z^\alpha}{\alpha}\left(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha}\right)^{2}-\tfrac{1}{15}\left(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha}\right)^{3}\\ &+c\left(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha}\right)^{4}+O\left[\left(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha}\right)^{5}\right]. \end{aligned} \quad (4.15)$$

The $\alpha-$derivative for $w(z)$ is given by

$$\begin{aligned} \tfrac{d^\alpha w}{dz^\alpha}=&-2\left(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha}\right)^{-3}-\tfrac{1}{10}\left(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha}\right)^{2}-\tfrac{1}{5}\tfrac{z^\alpha}{\alpha}(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha})\\ &-\tfrac{1}{5}\left(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha}\right)^{2}+4c\left(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha}\right)^{3}+O\left[\left(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha}\right)^{4}\right]. \end{aligned} \quad (4.16)$$

The requirement for studying asymptotically of $w\sim\sigma\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{-2}$, is in considering a transformation

$$w=v^{-2}, \quad (4.17)$$

such that, v vanishes at z_0 and $\frac{d^\alpha v}{dz^\alpha}$ is finite. Furthermore, we need to show that $v(z)$ is $\alpha-$analytic at z_0 from its $CODE$. Thus, w has a branch point of order -2 at the point z_0. It follows immediately that

$$v=\epsilon w^{\frac{-1}{2}}\text{ with }\epsilon=\pm 1. \quad (4.18)$$

Moreover, we need to define the given formal expansion

$$\frac{d^\alpha w}{dz^\alpha}=\sum_{j=0}^{\infty}b_{j-3}v^{j-3}. \quad (4.19)$$

Corresponding to the relation (4.18), equation (4.19) can be expressed as

$$\frac{d^\alpha w}{dz^\alpha}=b_{-3}\epsilon w^{\frac{3}{2}}+b_{-2}w+b_{-1}\epsilon w^{\frac{1}{2}}+b_0+b_1\epsilon w^{\frac{-1}{2}}+b_2w^{-1}+b_3w^{\frac{-3}{2}}+\cdots \quad (4.20)$$

Substituting equation (4.15) into equation (4.20) and neglecting terms of $O\left[\left(\frac{z^\alpha}{\alpha}-\frac{z_0^\alpha}{\alpha}\right)^{5}\right]$, gives

$$\begin{aligned} \tfrac{d^\alpha w}{dz^\alpha}=&\,b_{-3}\epsilon\left(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha}\right)^{-3}Y^{\frac{3}{2}}+b_{-2}\left(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha}\right)^{-2}Y+b_{-1}\epsilon\left(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha}\right)^{-1}Y^{\frac{1}{2}}\\ &+b_0+b_1\epsilon\left(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha}\right)Y^{\frac{-1}{2}}+b_2\left(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha}\right)^{2}Y^{-1}+b_3\epsilon\left(\tfrac{z^\alpha}{\alpha}-\tfrac{z_0^\alpha}{\alpha}\right)^{3}Y^{\frac{-3}{2}}\\ &+\cdots, \end{aligned} \quad (4.21)$$

where

$$Y = 1 + \left[\frac{-1}{10}\frac{z^\alpha}{\alpha}\left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^4 - \frac{1}{15}\left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^5 + c\left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^6\right]. \tag{4.22}$$

Using the expansion

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots, \quad -1 < x \le 1, \tag{4.23}$$

and collecting similar terms, equation (4.21) reduces to

$$\begin{aligned} \frac{d^\alpha w}{dz^\alpha} &= b_{-3}\epsilon\left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^{-3} + b_{-2}\left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^{-2} + b_{-1}\epsilon\left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^{-1} + b_0 \\ &+ \left[\frac{-3}{20}b_{-3}\frac{z^\alpha}{\alpha} + b_1\right]\epsilon\left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right) + \left[\frac{-1}{10}b_{-3}\epsilon - \frac{1}{10}\frac{z^\alpha}{\alpha}b_{-2} + b_2\right]\left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^2 \\ &+ \left[\frac{3}{2}b_{-3}c - \frac{1}{15}b_{-2} - \frac{1}{20}b_{-1} + b_3\right]\epsilon\left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^3 + O\left[\left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^4\right]. \end{aligned} \tag{4.24}$$

In order to find the values of $b_j' s$, we need to compare between the two equations (4.16) and (4.24), henceforth, the following values of the $b_j' s$ will be obtained

$$b_{-3} = -2\epsilon,\ b_{-2} = 0,\ b_{-1} = 0,\ b_0 = 0,\ b_1 = \frac{-1}{2}\epsilon\frac{z^\alpha}{\alpha},\ b_2 = \frac{-1}{2}, b_3 = 7\epsilon c,\ \cdots \tag{4.25}$$

Thus,

$$\frac{d^\alpha w}{dz^\alpha} = -2\epsilon v^{-3} - \frac{1}{2}\epsilon\frac{z^\alpha}{\alpha}v - \frac{1}{2}v^2 + 7\epsilon c v^3 + \cdots \tag{4.26}$$

Next, we will use the two transformation formulas:

$$w(z) = v(z)^{-2}, \tag{4.27a}$$

$$\frac{d^\alpha w(z)}{dz^\alpha} = -2\epsilon v(z)^{-3} - \frac{1}{2}\epsilon\frac{z^\alpha}{\alpha}v(z) - \frac{1}{2}v(z)^2 + \epsilon u(z)v(z)^3. \tag{4.27b}$$

Hence, we get

$$\frac{d^\alpha w}{dz^\alpha} = -2v^{-3}\frac{d^\alpha v}{dz^\alpha}. \tag{4.28}$$

Equation (4.28) can be reduced to

$$\frac{d^\alpha v}{dz^\alpha} = \frac{-1}{2}v^3\frac{d^\alpha w}{dz^\alpha}. \tag{4.29}$$

Substituting equation (4.27b) into equation (4.29), gives

$$\frac{d^\alpha v}{dz^\alpha} = \epsilon + \frac{1}{4}\epsilon\frac{z^\alpha}{\alpha}v^4 + \frac{1}{4}v^5 - \frac{1}{2}\epsilon u v^6. \tag{4.30}$$

The $\alpha-$differentiation of equation (4.27b), leads to

$$\frac{d^{2\alpha} w}{dz^{2\alpha}} = 6\epsilon v^{-4}\frac{d^\alpha v}{dz^\alpha} - \frac{1}{2}\epsilon v - \frac{1}{2}\epsilon\frac{z^\alpha}{\alpha}\frac{d^\alpha v}{dz^\alpha} - v\frac{d^\alpha v}{dz^\alpha} + \epsilon v^3\frac{d^\alpha u}{dz^\alpha} + 3\epsilon u v^2\frac{d^\alpha v}{dz^\alpha}. \tag{4.31}$$

With the help of equation (4.30), equation (4.31) becomes as follows

$$\begin{aligned} \frac{d^{2\alpha} w}{dz^{2\alpha}} &= 6v^{-4} + \frac{z^\alpha}{\alpha} - \frac{1}{8}\left(\frac{z^\alpha}{\alpha}\right)^2 v^4 - \frac{3}{8}\epsilon\frac{z^\alpha}{\alpha}v^5 + \frac{z^\alpha}{\alpha}uv^6 - \frac{1}{4}v^6 + \frac{5}{4}\epsilon u v^7 \\ &- \frac{3}{2}u^2v^8 + \epsilon v^3\frac{d^\alpha u}{dz^\alpha}. \end{aligned} \tag{4.32}$$

Finally, the original differential equation (4.1) and first equation (4.27a), implies

$$\frac{d^{2\alpha} w}{dz^{2\alpha}} = 6v^{-4} + \frac{z^\alpha}{\alpha}. \tag{4.33}$$

Thus, the comparison between the two equations (4.32) and (4.33), leads to

$$\frac{d^\alpha u}{dz^\alpha} = \epsilon\left(\frac{z^\alpha}{\alpha}\right)^2 v + \frac{3}{8}\frac{z^\alpha}{\alpha}v^2 - \epsilon\frac{z^\alpha}{\alpha}uv^3 + \frac{1}{4}\epsilon v^3 - \frac{5}{4}uv^4 + \frac{3}{2}\epsilon u^2 v^5. \tag{4.34}$$

In summary, it is apparently from the structure of the two equations (4.30) and (4.33) that this system of equations has a unique solution which is $\alpha-$analytic in the neighborhood of z_0 and satisfies the initial conditions $u(z_0) = u^0,\ v(z_0) = 0$. So we can say that CP_I equation (4.1) possess the generalized Painlevé property (GPP).

4.2 Conformable Second Painlevé Equation

In order to study the generalized Painlevé property of the conformable second Painlevé CP_{II}

$$\frac{d^{2\alpha}w}{dz^{2\alpha}} = 2w^3 + \frac{z^\alpha}{\alpha}w + \gamma, \tag{4.35}$$

we apply the methodology which is derived in Section 3.

Claim: The only algebraic singularities of equation (4.35) are movable poles of order one. Furthermore, there is a unique solution $w(z, z_0, w_0, \frac{d^\alpha w_0}{dz^\alpha})$ satisfies equation (4.35). This solution is $\alpha-$holomorphic in some neighborhood of $z = z_0$, where it takes on the value w_0 while its derivatives equals $\frac{d^\alpha w_0}{dz^\alpha}$.

(i) Dominant behavior:
Coinciding with the computation of the dominant behavior, we need to define

$$w \sim \sigma\left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^k, \tag{4.36}$$

and this yields

$$\frac{d^{2\alpha}w}{dz^{2\alpha}} = \sigma k(k-1)\left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^{k-2}. \tag{4.37}$$

Using equations (4.36) and (4.37) to substitute w and $\frac{d^{2\alpha}w}{dz^{2\alpha}}$ in (4.35), gives

$$\sigma k(k-1)\left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^{k-2} \sim 2\sigma^3\left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^{3k} + \sigma\frac{z^\alpha}{\alpha}\left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^k + \gamma. \tag{4.38}$$

Equation (4.38) simplifies at once to the form

$$\sigma k(k-1)\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^{k-2} \sim 2\sigma^3\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^{3k} + \sigma\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^{k+1} + \sigma\tfrac{z_0^\alpha}{\alpha}\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^k + \gamma. \tag{4.39}$$

For which two or more than two terms in the equation may be balancing the value of k must be $k = -1$, hence, the dominant equation will be given by

$$\frac{d^{2\alpha}w}{dz^{2\alpha}} \sim 2w^3. \tag{4.40}$$

Substituting from (4.36) into (4.40), as a result we will obtain

$$\sigma(1 - \sigma^2) = 0,$$

and so, $\sigma = 0$ or $\sigma = \epsilon$, we neglect $\sigma = 0$ and take $\sigma = \epsilon$ with $\epsilon = \pm 1$.

(ii) Resonances:
To find the Resonances, it is convenient to write w in the form

$$w = \sigma\left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^{-1} + \rho\left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^{r-1}. \tag{4.41}$$

Employing w in the dominant equation (4.40), the relevant equation will be given by

$$2\sigma\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^{-3} + (r-1)(r-2)\rho\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^{r-3} \sim 2\sigma^3\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^{-3} + 6\sigma^2\rho\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^{r-3} + 6\sigma\rho^2\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^{2r-3} + 2\rho^3\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^{3r-3}. \tag{4.42}$$

Consequently,

$$(r-1)(r-2)\rho = 6\sigma^2\rho, \tag{4.43}$$

if $\rho \neq 0$, and for $\sigma = \epsilon$, we get $(r+1)(r-4) = 0$, and so, the resonances are $r = -1, 4$.

(iii) Constant of integration:
In order to provide a constant of integration, we suppose there is a series solution around an arbitrary point z_0 in the complex $z-$plane of the form

$$w(z) = \sum_{j=0}^{\infty} a_j \left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^{j-1}. \tag{4.44}$$

From which we will obtain

$$\frac{d^\alpha w}{dz^\alpha} = \sum_{j=0}^{\infty}(j-1)a_j \left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^{j-2}, \tag{4.45a}$$

$$\frac{d^{2\alpha} w}{dz^{2\alpha}} = \sum_{j=0}^{\infty}(j-1)(j-2)a_j (\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha})^{j-3}, \tag{4.45b}$$

$$w^2 = \sum_{j=0}^{\infty}\sum_{k=0}^{j} a_{j-k}a_k \left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^{j-2}, \tag{4.45c}$$

$$w^3 = \sum_{j=0}^{\infty}\sum_{l=0}^{j}\sum_{k=0}^{l} a_{j-l}a_{l-k}a_k \left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^{j-3}, \tag{4.45d}$$

$$\frac{z^\alpha}{\alpha} w = \sum_{j=0}^{\infty} a_j \left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^{j} + \frac{z^\alpha}{\alpha}\left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^{j-1}. \tag{4.45e}$$

The substitution of equations (4.45) into equation (4.35) with some simplification, gives

$$\begin{aligned}&\textstyle\sum_{j=0}^{\infty}\left[(j-1)(j-2)a_j - 2\sum_{l=0}^{j}\sum_{k=0}^{l} a_{j-l}a_{l-k}a_k\right]\left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^{j-3}\\ &-\textstyle\sum_{j=3}^{\infty} a_{j-3}\left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^{j-3} - \frac{z_0^\alpha}{\alpha}\sum_{j=2}^{\infty} a_{j-2}\left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^{j-3} + \gamma = 0.\end{aligned} \tag{4.46}$$

The conventionally treatment of equation (4.46) leads to compute the coefficients $a_j's$. Henceforth, the formal expansion of $w(z)$ near $z = z_0$ can be given by

$$\begin{aligned}w(z) &= \epsilon\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^{-1} - \tfrac{1}{6}\epsilon\tfrac{z_0^\alpha}{\alpha}\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right) - \tfrac{1}{4}(\epsilon+\gamma)\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^2\\ &+ c\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^3 + O\left[\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^4\right].\end{aligned} \tag{4.47}$$

It is of some interest to rewrite equation (4.47) in an equivalent form as:

$$\begin{aligned}w(z) &= \epsilon\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^{-1} - \tfrac{1}{6}\epsilon\tfrac{z^\alpha}{\alpha}\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right) - \tfrac{1}{12}(\epsilon+3\gamma)\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^2\\ &+ c\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^3 + O\left[\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^4\right].\end{aligned} \tag{4.48}$$

However, the $\alpha-$derivative for $w(z)$ is given by

$$\begin{aligned}\tfrac{d^\alpha w}{dz^\alpha} &= -\epsilon\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^{-2} - \tfrac{1}{6}\epsilon\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right) - \tfrac{1}{6}\epsilon\tfrac{z^\alpha}{\alpha} - \tfrac{1}{6}(\epsilon+3\gamma)\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)\\ &+ 3c\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^2 + O\left[\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^3\right].\end{aligned} \tag{4.49}$$

Now, to prove that $w \sim \sigma\left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^{-1}$ is a asymptotic, we define a new variable

$$w = v^{-1}, \tag{4.50}$$

from which one can obtain

$$v = w^{-1}, \tag{4.51}$$

by construction, v vanishes at z_0, whereas, $\frac{d^\alpha v}{dz^\alpha}$ is finite.
Now, we need to show that $v(z)$ is $\alpha-$analytic at z_0 from its $CODE$, and so, it follows from $v = w^{-1}$ that w has a branch point of order -1 at z_0.
First step, we have to set

$$\frac{d^\alpha w}{dz^\alpha} = \sum_{j=0}^{\infty} b_{j-2} v^{j-2}. \tag{4.52}$$

It follows immediately that:

$$\frac{d^\alpha w}{dz^\alpha} = b_{-2} w^2 + b_{-1} w + b_0 + b_1 w^{-1} + b_2 w^{-2} + \cdots \tag{4.53}$$

The successive application of w equation (4.48) into equation (4.53) with neglecting terms of $O\left[\left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^4\right]$, leads to

$$\begin{aligned} \tfrac{d^\alpha w}{dz^\alpha} &= b_{-2}\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^{-2} Y^2 + b_{-1}\epsilon\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^{-1} Y + b_0 \\ &+ b_1\epsilon\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right) Y^{-1} + b_2\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^2 Y^{-2} + \cdots, \end{aligned} \tag{4.54}$$

where

$$Y = 1 + \left[\frac{-1}{6}\frac{z^\alpha}{\alpha}\left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^2 - \frac{1}{12}\epsilon(3\gamma+\epsilon)(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha})^3 + c\epsilon\left(\frac{z^\alpha}{\alpha} - \frac{z_0^\alpha}{\alpha}\right)^4\right]. \tag{4.55}$$

With the help of the expansion (4.23), and by usual simplifications equation, (4.54) can be reduced to the relation

$$\begin{aligned} \tfrac{d^\alpha w}{dz^\alpha} &= b_{-2}\left[\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^{-2} - \tfrac{1}{3}\tfrac{z^\alpha}{\alpha} - \tfrac{1}{6}\epsilon(3\gamma+\epsilon)\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right) + 2c\epsilon\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^2\right] \\ &+ b_{-1}\left[\epsilon\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^{-1} - \tfrac{1}{6}\epsilon\tfrac{z^\alpha}{\alpha}\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right) - \tfrac{1}{12}(3\gamma+\epsilon)\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^2\right] \\ &+ b_0 + b_1\epsilon\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right) + b_2\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^2 + O\left[\left(\tfrac{z^\alpha}{\alpha} - \tfrac{z_0^\alpha}{\alpha}\right)^3\right]. \end{aligned} \tag{4.56}$$

Corresponding to the two equations (4.49) and (4.56), the values of the $b_j's$ will be given as:

$$b_{-2} = -\epsilon,\ b_{-1} = 0,\ b_0 = \frac{-1}{2}\epsilon\frac{z^\alpha}{\alpha},\ b_1 = \frac{-1}{2} - \gamma\epsilon,\ b_2 = 5c,\ \cdots \tag{4.57}$$

In this case, $\frac{d^\alpha w}{dz^\alpha}$ equation (4.52) reads

$$\frac{d^\alpha w}{dz^\alpha} = -\epsilon v^{-2} - \frac{1}{2}\epsilon\frac{z^\alpha}{\alpha} - \left(\frac{1}{2} + \gamma\epsilon\right)v + 5cv^2 + \cdots \tag{4.58}$$

As a next step, we will use the two transformation formulas:

$$w(z) = v(z)^{-1}, \tag{4.59a}$$

$$\frac{d^\alpha w}{dz^\alpha} = -\epsilon v^{-2} - \frac{1}{2}\epsilon\frac{z^\alpha}{\alpha} - \left(\frac{1}{2} + \gamma\epsilon\right)v + u(z)v(z)^2. \tag{4.59b}$$

The $\alpha-$derivative of equation (4.59a), gives

$$\frac{d^\alpha w}{dz^\alpha} = -v^{-2}\frac{d^\alpha v}{dz^\alpha}, \tag{4.60}$$

from which we can obtain

$$\frac{d^\alpha v}{dz^\alpha} = -v^2\frac{d^\alpha w}{dz^\alpha}. \tag{4.61}$$

Using the definition of $\frac{d^\alpha w}{dz^\alpha}$ given in equation (4.59b), then $\frac{d^\alpha v}{dz^\alpha}$ has the expression

$$\frac{d^\alpha v}{dz^\alpha} = \epsilon + \frac{1}{2}\epsilon\frac{z^\alpha}{\alpha}v^2 + \left(\frac{1}{2} + \gamma\epsilon\right)v^3 - uv^4. \tag{4.62}$$

The $\alpha-$derivative of equation (4.59b), leads to

$$\frac{d^{2\alpha}w}{dz^{2\alpha}} = 2\epsilon v^{-3}\frac{d^{\alpha}v}{dz^{\alpha}} - \frac{1}{2}\epsilon - \left(\frac{1}{2}+\gamma\epsilon\right)\frac{d^{\alpha}v}{dz^{\alpha}} + v^{2}\frac{d^{\alpha}u}{dz^{\alpha}} + 2uv\frac{d^{\alpha}v}{dz^{\alpha}}. \tag{4.63}$$

Direct substitution of $\frac{d^{\alpha}v}{dz^{\alpha}}$ equation (4.62) into equation (4.63), yields

$$\begin{aligned}\frac{d^{2\alpha}w}{dz^{2\alpha}} &= 2v^{-3} + \frac{z^{\alpha}}{\alpha}v^{-1} + 2\epsilon\left(\frac{1}{2}+\gamma\epsilon\right) - 2\epsilon uv - \frac{1}{2}\epsilon - (\frac{1}{2}+\gamma\epsilon)\epsilon \\ &- \frac{1}{2}\epsilon\left(\frac{1}{2}+\gamma\epsilon\right)\frac{z^{\alpha}}{\alpha}v^{2} - \left(\frac{1}{2}+\gamma\epsilon\right)^{2}v^{3} - \left(\frac{1}{2}+\gamma\epsilon\right)uv^{4} + v^{2}\frac{d^{\alpha}u}{dz^{\alpha}} \\ &+ 2\epsilon uv + \epsilon\frac{z^{\alpha}}{\alpha}uv^{3} + 2\left(\frac{1}{2}+\gamma\epsilon\right)v^{4} - 2u^{2}v^{5}.\end{aligned} \tag{4.64}$$

Now, the CP_{II} equation (4.35) and equation (4.59a), implies that

$$\frac{d^{2\alpha}w}{dz^{2\alpha}} = 2v^{-3} + \frac{z^{\alpha}}{\alpha}v^{-1} + \gamma. \tag{4.65}$$

In this case, the two equations (4.64) and (4.65), give

$$\begin{aligned}\frac{d^{\alpha}u}{dz^{\alpha}} &= \frac{\epsilon}{2}\left(\frac{1}{2}+\gamma\epsilon\right)\frac{z^{\alpha}}{\alpha} + \left(\frac{1}{2}+\gamma\epsilon\right)^{2}v + \left(\frac{1}{2}+\gamma\epsilon\right)uv^{2} \\ &-\epsilon\frac{z^{\alpha}}{\alpha}uv - 2\left(\frac{1}{2}+\gamma\epsilon\right)v^{2} + 2u^{2}v^{3}.\end{aligned} \tag{4.66}$$

It follows that the system of equations (4.62) and (4.66) has an unique solution which is $\alpha-$analytic in the neighborhood of z_0 and satisfies the initial conditions $u(z_0) = u^0,\ v(z_0) = 0$. This can be shown that equation (4.35) is possessing the generalized Painlevé property.

In the reminder of this subsection, we will study some special cases.
For an example, from equation (4.66) if we set $u = 0$, we will have $\gamma = \frac{-1}{2}\epsilon$, henceforth, equation (4.62) becomes as follows:

$$\frac{d^{\alpha}v}{dz^{\alpha}} = \epsilon + \frac{1}{2}\epsilon\frac{z^{\alpha}}{\alpha}v^{2}, \tag{4.67}$$

the resulting equation is the so-called a conformable Ricatti equation (CRicatti), which is given in [40]. Equation (4.67) can be linearized by the transformation

$$v = \frac{-2\alpha}{\epsilon z^{\alpha}\varphi}\frac{d^{\alpha}\varphi}{dz^{\alpha}}, \tag{4.68}$$

to the given $2\alpha-$order conformable linear differential equation

$$\frac{d^{2\alpha}\varphi}{dz^{2\alpha}} - \frac{\alpha}{z^{\alpha}}\frac{d^{\alpha}\varphi}{dz^{\alpha}} + \frac{1}{2}\frac{z^{\alpha}}{\alpha}\varphi = 0. \tag{4.69}$$

Solving this linear equation is equivalent to solve the CP_{II} with $\gamma = \frac{-1}{2}\epsilon$.

On the other hand, the successive application of the transformation $w = v^{-1}$ into equation (4.67), leads to

$$\frac{d^{\alpha}w}{dz^{\alpha}} = -\epsilon w^{2} - \frac{1}{2}\epsilon\frac{z^{\alpha}}{\alpha}, \tag{4.70}$$

which is also a CRicatti equation. By the transformation $w = \frac{\epsilon}{\phi}\frac{d^{\alpha}\phi}{dz^{\alpha}}$, equation (4.70) can be transform to the given conformable fractional Airy (CAiry) equation

$$\frac{d^{2\alpha}\phi}{dz^{2\alpha}} + \frac{1}{2}\frac{z^{\alpha}}{\alpha}\phi = 0, \tag{4.71}$$

when $\alpha = 1$ equation (4.71) is the classical version of Airy equation given in [52].
Coinciding with the theory of conformable Fourier ($C-$Fourier) transform $\left[\Phi(\varpi) = \int_{-\infty}^{\infty}\phi(z)e^{-i\varpi\frac{z^{\alpha}}{\alpha}}z^{\alpha-1}dz\right]$ in [16], the successive application into equation (4.71) leads to the following first order ordinary differential equation

$$(i\varpi)^{2}\Phi + \frac{1}{2}i\frac{d}{d\varpi}\Phi = 0. \tag{4.72}$$

By the usual computations of equation (4.72), One can achieve

$$\Phi(\varpi) = ce^{\frac{-2}{3}i\varpi^3}, \tag{4.73}$$

where c is a constant of integration. Now, applying the inverse $C-$Fourier transform $\left[\phi(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\varpi)e^{i\varpi\right.$ in [16], we obtain

$$\phi(z) = \frac{c}{\pi}\int_{-\infty}^{\infty} e^{i(\frac{-2}{3}\varpi^3+\varpi\frac{z^\alpha}{\alpha})}d\varpi. \tag{4.74}$$

It is convenient to use the change of variables $y = \frac{-1}{\sqrt[3]{2}}\frac{z^\alpha}{\alpha}$ and $k = -\sqrt[3]{2}\varpi$ transforms (4.74) into

$$\varphi(y) = \frac{-c}{\sqrt[3]{2}}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\frac{k^3}{3}+ky)}dk\right\}. \tag{4.75}$$

Apparently, the quantity in curly brackets behaves as Airy function (Ai(y)) [52].

5 Some Properties of Conformable Painlevé Equations

Here we would like to present an additional aspect to the introduction Of the conformable Painlevé equations.

5.1 Isomonodromy Problems Of CP_I and CP_{II}

Conformable Painlevé equations are expressed as the compatibility condition of Lax pairs that can be used to study asymptotics and connection formulae.

Isomonodromy Problems Of CP_I

CP_I can be considered as the isomonodromic condition (the compatibility condition) for the linear system

$$\frac{\partial^\alpha Y(z,t)}{\partial z^\alpha} = \left(A_4\left(\frac{z^\alpha}{\alpha}\right)^4 + A_2\left(\frac{z^\alpha}{\alpha}\right)^2 + A_1\frac{z^\alpha}{\alpha} + A_0 + A_{-1}\left(\frac{z^\alpha}{\alpha}\right)^{-1}\right)Y(z,t), \tag{5.1a}$$

$$\frac{\partial^\beta Y(z,t)}{\partial t^\beta} = \left(B_1\frac{z^\alpha}{\alpha} + B_{-1}\left(\frac{z^\alpha}{\alpha}\right)^{-1}\right)Y(z,t), \tag{5.1b}$$

where $A_i,\ i = 4,2,1,0,-1,$ and $B_j,\ j = 1,-1$ are matrices whose entries depend on the solution $u(t)$ of CP_I equation (4.1), and

$$A_4 = -4i\sigma_3,\ A_2 = 4u\sigma_2,\ A_1 = 2u_t^\beta\sigma_1,\ A_0 = -i\left(2u^2 + \frac{t^\beta}{\beta}\right)(\sigma_3 - i\sigma_2),\ A_{-1} = -\frac{1}{2}\sigma_1, \tag{5.2a}$$

$$B_1 = -i\sigma_3, \quad B_{-1} = iu(\sigma_3 - i\sigma_2). \tag{5.2b}$$

The Pauli matrices $\sigma_j,\ j = 1,2,3$ are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.3}$$

The compatibility condition of equations (5.1a) and (5.1b) is

$$\frac{\partial^\beta}{\partial t^\beta}\frac{\partial^\alpha Y(z,t)}{\partial z^\alpha} = \frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta Y(z,t)}{\partial t^\beta}, \tag{5.4}$$

and this yields the condition:

$$\frac{\partial^\beta A}{\partial t^\beta} - \frac{\partial^\alpha B}{\partial z^\alpha} + [A,\ B] = 0, \tag{5.5}$$

where the commutator $[A,\ B]$ is given by

$$[A,\ B] = AB - BA. \tag{5.6}$$

By using $A = A_4\left(\frac{z^\alpha}{\alpha}\right)^4 + A_2\left(\frac{z^\alpha}{\alpha}\right)^2 + A_1\frac{z^\alpha}{\alpha} + A_0 + A_{-1}\left(\frac{z^\alpha}{\alpha}\right)^{-1}$, and $B = B_1\frac{z^\alpha}{\alpha} + B_{-1}\left(\frac{z^\alpha}{\alpha}\right)^{-1}$, equation (5.5) gives

$$\begin{aligned} &[A_4,\ B_1] = [A_0,\ B_{-1}] = 0,\ \ [A_4,\ B_{-1}] + [A_2,\ B_1] = 0,\ \ -B_{-1} + [A_{-1},\ B_{-1}] = 0,\\ &\tfrac{\partial^\beta A_2}{\partial t^\beta} + [A_1,\ B_1] = 0,\ \ \tfrac{\partial^\beta A_1}{\partial t^\beta} + [A_2,\ B_{-1}] + [A_0,\ B_{-1}] = 0,\\ &\tfrac{\partial^\beta A_0}{\partial t^\beta} - B_1 + [A_1,\ B_{-1}] + [A_{-1},\ B_1] = 0. \end{aligned} \tag{5.7}$$

Substituting $A_i,\ \ i = 4, 2, 1, 0, -1$ and $B_j,\ \ j = 1, -1$, from equations (5.2) into equations (5.7) yields the CP_I.

Isomonodromy Problems Of CP_{II}

The CP_{II} can be written as the compatibility condition of the following linear system of equations:

$$\frac{\partial^\alpha \Phi(y,\tau)}{\partial y^\alpha} = \left(B_1\frac{\tau^\beta}{\beta} + B_0\right)\Phi(y,\tau), \tag{5.8a}$$

$$\frac{\partial^\beta \Phi(y,\tau)}{\partial \tau^\beta} = \left(A_2\left(\frac{\tau^\beta}{\beta}\right)^2 + A_1\frac{\tau^\beta}{\beta} + A_0 + A_{-1}\left(\frac{\tau^\beta}{\beta}\right)^{-1}\right)\Phi(y,\tau), \tag{5.8b}$$

where $A_i,\ i = 2, 1, 0, -1,$ and $B_j,\ j = 1, 0$ are matrices whose entries depend on the solution $w(y)$ of CP_{II} equation (4.35), and

$$A_2 = -4i\sigma_3,\ \ A_1 = 4w\sigma_1,\ \ A_0 = -i\left(2w^2 + \frac{y^\alpha}{\alpha}\right)\sigma_3 - 2\frac{d^\alpha}{dy^\alpha}w,\ \ A_{-1} = -\gamma\sigma_1, \tag{5.9a}$$

$$B_1 = -i\sigma_3,\ \ B_0 = w\sigma_1, \tag{5.9b}$$

The compatibility condition of equations (5.8a) and (5.8b) is given by

$$\frac{\partial^\beta}{\partial \tau^\beta}\frac{\partial^\alpha \Phi(y,\tau)}{\partial y^\alpha} = \frac{\partial^\alpha}{\partial y^\alpha}\frac{\partial^\beta \Phi(y,\tau)}{\partial \tau^\beta}, \tag{5.10}$$

which yields the condition

$$\frac{\partial^\beta B}{\partial \tau^\beta} - \frac{\partial^\alpha A}{\partial y^\alpha} + [B,\ A] = 0, \tag{5.11}$$

Using $A = A_2\left(\frac{\tau^\beta}{\beta}\right)^2 + A_1\frac{\tau^\beta}{\beta} + A_0 + A_{-1}\left(\frac{\tau^\beta}{\beta}\right)^{-1},\ \ B = B_1\frac{\tau^\beta}{\beta} + B_0$, equation (5.11) gives

$$\begin{aligned} &[B_1,\ A_2] = 0,\ \ [B_0,\ A_{-1}] = 0,\ \ [B_1,\ A_1] + [B_0,\ A_2] = 0,\\ &\tfrac{\partial^\alpha A_1}{\partial y^\alpha} = [B_1,\ A_0] + [B_0,\ A_1],\\ &\tfrac{\partial^\alpha A_0}{\partial y^\alpha} = B_1 + [B_1,\ A_{-1}] + [B_0,\ A_0]. \end{aligned} \tag{5.12}$$

Substituting $A_i,\ \ i = 2, 1, 0, -1,$ and $B_j,\ \ j = 1, 0$, from equations (5.9) into equations (5.12) yields the CP_{II}.

The matrices A and B which are given by equations (5.9) are derived from the Lax pair of $CmKdV$ equation $\left(\frac{\partial^\beta u}{\partial t^\beta} - 6u^2\frac{\partial^\alpha u}{\partial x^\alpha} + \frac{\partial^{3\alpha} u}{\partial x^{3\alpha}} = 0\right)$,

$$\frac{\partial^\beta \psi(x,t)}{\partial t^\beta} = \begin{pmatrix} -4i\left(\frac{k^\beta}{\beta}\right)^3 - 2i\frac{k^\beta}{\beta}u^2 & 4\left(\frac{k^\beta}{\beta}\right)^2 u + 2i\frac{k^\beta}{\beta}u_x^\alpha - u_{xx}^\alpha + 2u^3 \\ 4\left(\frac{k^\beta}{\beta}\right)^2 u - 2i\frac{k^\beta}{\beta}u_x^\alpha - u_{xx}^\alpha + 2u^3 & 4i\left(\frac{k^\beta}{\beta}\right)^3 + 2i\frac{k^\beta}{\beta}u^2 \end{pmatrix}\psi(x,t), \tag{5.13a}$$

$$\frac{\partial^\alpha \psi(x,t)}{\partial x^\alpha} = \begin{pmatrix} -i\frac{k^\beta}{\beta} & u \\ u & i\frac{k^\beta}{\beta} \end{pmatrix}\psi(x,t). \tag{5.13b}$$

Through the scaling reduction

$$z = xt^{\frac{-\beta}{3\alpha}}, \quad u(x,t) = v(z)t^{\frac{-\beta}{3}}, \quad \lambda = kt^{\frac{1}{3}}, \quad \psi(x,t) = \Psi(z,\lambda), \tag{5.14}$$

equations (5.13) is converted to

$$\begin{aligned}\frac{\partial^\beta \Psi(z,\lambda)}{\partial t^\beta} &= \left[\frac{-12i}{\beta}\sigma_3\left(\frac{\lambda^\beta}{\beta}\right)^2 + \frac{12}{\beta}v\sigma_1\frac{\lambda^\beta}{\beta} + -i\left(\frac{6}{\beta}v^2 + \frac{z^\alpha}{\alpha}\right)\sigma_3 - \frac{6}{\beta}v_z^\alpha\sigma_2\right]\Psi(z,\lambda) \\ &+ \left[\frac{-3}{\beta}v_{zz}^\alpha + \frac{6}{\beta}v^3 + \frac{z^\alpha}{\alpha}v\right]\sigma_1\left(\frac{\lambda^\beta}{\beta}\right)^{-1}\Psi(z,\lambda),\end{aligned} \tag{5.15}$$

$$\frac{\partial^\alpha \Psi(z,\lambda)}{\partial z^\alpha} = \left(-i\frac{\lambda^\beta}{\beta}\sigma_3 + v\sigma_1\right)\Psi(z,\lambda). \tag{5.16}$$

Thenceforward, the scale

$$y = \left(\frac{\beta}{3}\right)^{\frac{1}{3\alpha}} z, \quad v = \left(\frac{\beta}{3}\right)^{\frac{1}{3}} w, \quad \lambda = \left(\frac{\beta}{3}\right)^{\frac{1}{3\beta}} \tau, \quad \Psi(z,\lambda) = Y(y,\tau), \tag{5.17}$$

converts the Lax pair equations (5.13) to the Lax pair of CP_{II} equations (5.8) and (5.9).

5.2 The Generalized Hirota Bilinear Form

The fundamental idea behind Hirota's direct method is changing into new variables in which the solutions have the simplest form. In this part we discuss how the Painlevé equations can be written in terms of entire function, and so, in the generalized Hirota bilinear form

The Generalized Hirota Bilinear Form of CP_I

Let us introduce the transformation

$$y = -\frac{d^{2\alpha}}{dz^{2\alpha}}\left(log\varphi(z)\right) = \frac{-\varphi d^{2\alpha}\varphi + (d^\alpha\varphi)^2}{\varphi^2}, \tag{5.18}$$

where $d^\alpha = \frac{d^\alpha}{dz^\alpha}$, and $d^{2\alpha} = \frac{d^{2\alpha}}{dz^{2\alpha}}$ Henceforth,

$$\frac{d^{2\alpha}y}{dz^{2\alpha}} = \frac{-12\varphi(d^\alpha\varphi)^2 d^{2\alpha}\varphi + 3\varphi^2(d^{2\alpha}\varphi)^2 + 4\varphi^2 d^\alpha\varphi d^{3\alpha}\varphi - \varphi^3 d^{4\alpha}\varphi + 6(d^\alpha\varphi)^4}{\varphi^4}, \tag{5.19a}$$

$$y^2 = \frac{\varphi^2(d^{2\alpha}\varphi)^2 - 2\varphi d^{2\alpha}\varphi(d^\alpha\varphi)^2 + (d^\alpha\varphi)^4}{\varphi^4}. \tag{5.19b}$$

The substitution of equations (5.19) into CP_I equation (4.1), gives

$$\varphi d^{4\alpha}\varphi - 4d^\alpha\varphi d^{3\alpha}\varphi + 3(d^{2\alpha}\varphi)^2 + \frac{z^\alpha}{\alpha}\varphi^2 = 0. \tag{5.20}$$

Now, equation (5.20) can be written in the form

$$\frac{1}{2}D_z^{4\alpha}(\varphi.\varphi)+\frac{z^\alpha}{\alpha}\varphi^2=0. \tag{5.21}$$

Hence, equation (5.21) can be rewritten in an equivalent form

$$\left[D_z^{4\alpha}+2\frac{z^\alpha}{\alpha}\right]\varphi.\varphi=0, \tag{5.22}$$

which is called the generalized Hirota bilinear representation of CP_I. The generalized Hirota operator $D_z^{4\alpha}(\varphi.\varphi)$ is given by

$$\begin{aligned} D_z^{4\alpha}(\varphi.\varphi)&=\left[\tfrac{d^\alpha}{dz_1^\alpha}-\tfrac{d^\alpha}{dz_2^\alpha}\right]^4|_{z_1=z_2=z}(\varphi.\varphi)\\ &=2\varphi\tfrac{d^{4\alpha}\varphi}{dz^{4\alpha}}-8\tfrac{d^\alpha\varphi}{dz^\alpha}\tfrac{d^{3\alpha}\varphi}{dz^{3\alpha}}+6\left(\tfrac{d^{2\alpha}\varphi}{dz^{2\alpha}}\right)^2. \end{aligned} \tag{5.23}$$

The Generalized Hirota Bilinear Form of CP_{II}

Let us introduce the transformation

$$\begin{aligned} w(z)&=\tfrac{d^\alpha}{dz^\alpha}\left\{\ln\left[\tfrac{F(z)}{G(z)}\right]\right\}\\ &=\tfrac{d^\alpha F(z)}{F(z)}-\tfrac{d^\alpha G(z)}{G(z)}. \end{aligned} \tag{5.24}$$

From which we will have

$$\frac{d^{2\alpha}w}{dz^{2\alpha}}=\frac{G^3[-3Fd^\alpha Fd^{2\alpha}F+F^2d^{3\alpha}F+2(d^\alpha F)^3]+F^3[3Gd^\alpha Gd^{2\alpha}G-G^2d^{3\alpha}G-2(d^\alpha G)^3]}{F^3G^3}, \tag{5.25a}$$

$$w^3=\frac{G^3(d^\alpha F)^3-3FG^2(d^\alpha F)^2d^\alpha G+3F^2Gd^\alpha F(d^\alpha G)^2-F^3(d^\alpha G)^3}{F^3G^3}. \tag{5.25b}$$

The substitution from equations (5.25) into CP_{II} equation (4.35) with some simplifications, leads to

$$\begin{aligned} &Gd^{3\alpha}F-3d^{2\alpha}Fd^\alpha G+3d^\alpha Fd^{2\alpha}G-Fd^3G-\tfrac{z^\alpha}{\alpha}\left(Gd^\alpha F-Fd^\alpha G\right)-\gamma FG=\\ &\tfrac{3(Gd^\alpha F-Fd^\alpha G)}{FG}\left[Gd^{2\alpha}F-2d^\alpha Fd^\alpha G+Fd^{2\alpha}G\right]. \end{aligned} \tag{5.26}$$

If we use a separate function $\lambda(z)$, then equation (5.26) can be written in a decoupling form as:

$$Gd^{2\alpha}F-2d^\alpha Fd^\alpha G+Fd^{2\alpha}G=-\lambda(z)FG, \tag{5.27a}$$

$$Gd^{3\alpha}F-3d^{2\alpha}Fd^\alpha G+3d^\alpha Fd^{2\alpha}G-Fd^3G=\left[\frac{z^\alpha}{\alpha}-3\lambda(z)\right]\left[Gd^\alpha F-Fd^{3\alpha}G\right]+\gamma FG. \tag{5.27b}$$

By using the generalized Hirota D_z^α operator

$$D_z^\alpha(F.G)=\left(\frac{d^\alpha}{dz_1^\alpha}-\frac{d^\alpha}{dz_2^\alpha}\right)[F(z_1)G(z_2)]\,|_{z_1=z_2=z}=Gd^\alpha F-Fd^\alpha G,$$

equations (5.27a) and (5.27b) can be written in a condensed form as follows:

$$[D_z^{2\alpha}+\lambda(z)](F.G)=0, \tag{5.28a}$$

$$\left\{D_z^{3\alpha}-\left[\frac{z^\alpha}{\alpha}-3\lambda(z)\right]D_z^\alpha-\gamma\right\}(F.G)=0. \tag{5.28b}$$

The generalized Hirota operators $D_z^{2\alpha}$ and $D_z^{3\alpha}$ are given by

$$\begin{aligned} D_z^{2\alpha}(F.G) &= \left[\frac{d^\alpha}{dz_1^\alpha} - \frac{d^\alpha}{dz_2^\alpha}\right]^2 |_{z_1=z_2=z}(F.G) \\ &= Gd^{2\alpha}F - 2d^\alpha F d^\alpha G + Fd^{2\alpha}G, \end{aligned} \tag{5.29}$$

$$\begin{aligned} D_z^{3\alpha}(F.G) &= \left[\frac{d^\alpha}{dz_1^\alpha} - \frac{d^\alpha}{dz_2^\alpha}\right]^3 |_{z_1=z_2=z}(F.G) \\ &= Gd^{3\alpha}F - 3d^{2\alpha}Fd^\alpha G + 3d^\alpha Fd^{2\alpha}G - Fd^3G. \end{aligned} \tag{5.30}$$

5.3 Hamiltonian Structure

Conformable Painlevé equations can be written as a Hamiltonian system.

Hamiltonian Structure of CP_I

CP_I can be written as the Hamiltonian system

$$\frac{d^\alpha q}{dz^\alpha} = \frac{\partial H_I}{\partial p} = p, \tag{5.31a}$$

$$\frac{d^\alpha p}{dz^\alpha} = -\frac{\partial H_I}{\partial p} = 6q^2 + \frac{z^\alpha}{\alpha}, \tag{5.31b}$$

where $H_I(q,p)$ is the Hamiltonian defined by

$$H_I = \frac{1}{2}p^2 - 2q^3 - \frac{z^\alpha}{\alpha}q. \tag{5.32}$$

If we eliminate p from the equations (5.31) then it is easily to show that q satisfies CP_I, and p is defined by first equation of (5.31). However, the elimination of q from the equations (5.31), leads to

$$q = \pm\frac{1}{6}\left(\frac{d^\alpha p}{dz^\alpha} - \frac{z^\alpha}{\alpha}\right)^{\frac{1}{2}}, \tag{5.33}$$

from which we will obtain

$$\frac{d^{2\alpha}p}{dz^{2\alpha}} = 1 \pm 12p\left(\frac{d^\alpha p}{dz^\alpha} - \frac{z^\alpha}{\alpha}\right)^{\frac{1}{2}}. \tag{5.34}$$

Therefore, if q satisfies CP_I, then p which is given by equation (5.31a) satisfies equation (5.34), and conversely, if p satisfies equation (5.34), then q which is given by equation (5.33) satisfies CP_I.

Hamiltonian Structure of CP_{II}

CP_{II} can be written as the Hamiltonian system

$$\frac{d^\alpha q}{dz^\alpha} = \frac{\partial H_{II}}{\partial p} = p - q^2 - \frac{1}{2}\frac{z^\alpha}{\alpha}, \tag{5.35a}$$

$$\frac{d^\alpha p}{dz^\alpha} = -\frac{\partial H_{II}}{\partial p} = 2pq + \gamma + \frac{1}{2}, \tag{5.35b}$$

where $H_{II}(q,p,\gamma)$ is the Hamiltonian defined by

$$H_{II} = \frac{1}{2}p^2 - \left(q^2 + \frac{1}{2}\frac{z^\alpha}{\alpha}\right)p - \left(\gamma + \frac{1}{2}\right)q. \tag{5.36}$$

Eliminating p from equations (5.35) then q satisfies CP_{II}, detail as follows: The $\alpha-$derivative of equation (5.35), gives

$$\frac{d^{2\alpha}q}{dz^{2\alpha}} = \frac{d^\alpha p}{dz^\alpha} - 2q\frac{d^\alpha q}{dz^\alpha} - \frac{1}{2}. \tag{5.37}$$

Substituting $\frac{d^\alpha p}{dz^\alpha}$ from equation (5.35b) into equation (5.37), leads to

$$\frac{d^{2\alpha}q}{dz^{2\alpha}} = 2qp + \gamma - 2q\frac{d^\alpha q}{dz^\alpha}, \tag{5.38}$$

For q to be satisfying CP_{II}, one can find

$$p = \frac{d^\alpha q}{dz^\alpha} + q^2 + \frac{1}{2}\frac{z^\alpha}{\alpha}. \tag{5.39}$$

Whereas, the elimination of q from equations (5.35), leads to

$$q = \frac{1}{2}p^{-1}\left[\frac{d^\alpha p}{dz^\alpha} - \gamma - \frac{1}{2}\right]. \tag{5.40}$$

Henceforth, one can obtain

$$p\frac{d^{2\alpha}p}{dz^{2\alpha}} = \frac{1}{2}\left(\frac{d^\alpha p}{dz^\alpha}\right)^2 + 2p^3 - \frac{z^\alpha}{\alpha}p^2 - \frac{1}{2}\left(\gamma + \frac{1}{2}\right)^2, \tag{5.41}$$

the resulting equation is the conformable P_{34} (CP_{34}) which is given in [40]. Furthermore, if q satisfies CP_{II}, then p which is given by equation (5.39) satisfies (CP_{34}), and conversely, if p satisfies (CP_{34}), then q which is given by equation (5.40) satisfies CP_{II}. Thus, there is one-to-one correspondence between solutions of CP_{II} and (CP_{34}).

5.4 The generalized Bäcklund transformations

The generalized Bäcklund transformations map solutions of a given conformable Painlevé equation to solutions of the same Painlevé equation, but with different values of the parameters.

- The generalized Bäcklund transformations for CP_{II} are given by
  (i) Suppose that $w(z;-\gamma)$ is a solution of the given CP_{II}

$$\frac{d^{2\alpha}}{dz^{2\alpha}}w(z;-\gamma) = 2w^3 + \frac{z^\alpha}{\alpha}w - \gamma, \tag{5.42}$$

  then

$$\begin{aligned}\tfrac{d^{2\alpha}}{dz^{2\alpha}}w(z;-\gamma) &= -\left[2(-w)^3 + \tfrac{z^\alpha}{\alpha}(-w) + \gamma\right]\\ &= -\tfrac{d^{2\alpha}}{dz^{2\alpha}}\left(-w(z;\gamma)\right).\end{aligned} \tag{5.43}$$

  Thus, $w(z;-\gamma) = -w(z;\gamma)$.
  (ii) Also, if $w(z;\gamma)$ is a solution of the CP_{II} equation then

$$w(z;\gamma \pm 1) = -w(z;\gamma) - \frac{2\gamma \pm 1}{2w^2(z;\gamma) \pm 2\frac{d^\alpha w(z;\gamma)}{dz^\alpha} + \frac{z^\alpha}{\alpha}} \tag{5.44}$$

  are also solutions of CP_{II} with the parameter $\gamma \pm 1$ and provided that

$$2w^2(z;\gamma) \pm 2\frac{d^\alpha w(z;\gamma)}{dz^\alpha} + \frac{z^\alpha}{\alpha} \neq 0.$$

- CP_{II} possess hierarchies of rational and algebraic solutions for special values of the parameters, as we illustrate here.
  (i) For every $\gamma = n \in \mathbf{Z}$ there exists a unique solution of CP_{II}; that is, for $w(z;\gamma)$ is a solution of CP_{II} with $\gamma = n \in \mathbf{Z}$, then Bäcklund transformation (5.44) becomes as:

$$w(z;n+1) = -w(z;n) - \frac{2n+1}{2w^2(z;n) + 2\frac{d^\alpha w(z;n)}{dz^\alpha} + \frac{z^\alpha}{\alpha}}, \tag{5.45}$$

  generates a hierarchy of rational solutions of CP_{II} from the "seed solution" $w(z;0) = 0$. For instance

when $n = 0, \quad w(z;1) = -\left(\frac{z^\alpha}{\alpha}\right)^{-1},$

when $n = 1, \quad w(z;2) = \left(\frac{z^\alpha}{\alpha}\right)^{-1} - \frac{3(\frac{z^\alpha}{\alpha})^2}{4+(\frac{z^\alpha}{\alpha})^3}.$

(ii) For every $\gamma = n+\frac{1}{2}$ with $n \in \mathbf{Z}$, there exists a unique one-parameter family of classical solutions of CP_{II} generates from the "seed solution" $w(z;\frac{1}{2}) = \frac{1}{\phi}\frac{d^\alpha \phi}{dz^\alpha}$, where ϕ is the solution of CAiry equation (4.71). By Bäcklund transformation (5.44) each of which is rationally written in terms of CAiry functions.

(iii) For all other values of α, the solution of CP_{II} is nonclassical (transcendental).

- The following special Bäckland transformation of CP_{II}

$$W\left(\zeta;\frac{1}{2}\varepsilon\right) = 2^{\frac{-1}{3}}\varepsilon w^{-1}(z;0)\frac{d^\alpha}{dz^\alpha}w(z;0), \tag{5.46a}$$

$$w^2(z;0) = 2^{\frac{-1}{3}}\left\{W^2\left(\zeta;\frac{1}{2}\varepsilon\right) - \varepsilon\frac{d^\alpha}{d\zeta^\alpha}W\left(\zeta;\frac{1}{2}\varepsilon\right) + \frac{1}{2}\frac{\zeta^\alpha}{\alpha}\right\}, \tag{5.46b}$$

where $\zeta = (-2)^{\frac{1}{3\alpha}}z, \;\; \varepsilon = \pm 1$, maps between solutions for $\gamma = 0$ and solutions for $\gamma = \frac{1}{2}\varepsilon$, the detail as follows

$$\begin{aligned}\frac{d^\alpha W}{d\zeta^\alpha} &= -(2)^{\frac{-1}{3}}\varepsilon w^{-2}\left(\frac{d^\alpha w}{dz^\alpha}\right)^2\frac{d^\alpha z}{d\zeta^\alpha}z^{\alpha-1} + 2^{\frac{-1}{3}}\varepsilon w^{-1}\frac{d^{2\alpha}w}{dz^{2\alpha}}\frac{d^\alpha z}{d\zeta^\alpha}z^{\alpha-1}\\ &= (-2)^{\frac{-2}{3}}\varepsilon w^{-2}\left(\frac{d^\alpha w}{dz^\alpha}\right)^2 - (-2)^{\frac{-2}{3}}\varepsilon w^{-1}\frac{d^{2\alpha}w}{dz^{2\alpha}}\\ &= \varepsilon W^2 + (-2)^{\frac{1}{3}}\varepsilon w^2 - (-2)^{\frac{-2}{3}}\varepsilon\frac{z^\alpha}{\alpha},\end{aligned} \tag{5.47}$$

and

$$\frac{d^{2\alpha}W}{d\zeta^{2\alpha}} = 2\varepsilon W\frac{d^\alpha W}{d\zeta^\alpha} + 2(-2)^{\frac{1}{3}}\varepsilon w\frac{d^\alpha w}{dz^\alpha}\frac{d^\alpha z}{d\zeta^\alpha}z^{\alpha-1} - (-2)^{-1}\varepsilon, \tag{5.48}$$

which simplifies at once to the form

$$\frac{d^{2\alpha}W}{d\zeta^{2\alpha}} = 2W^3 + \frac{\zeta^\alpha}{\alpha}W + \frac{1}{2}\varepsilon. \tag{5.49}$$

Conversely, by solving equation (5.46a) for $\frac{d^\alpha w}{dz^\alpha}$

$$\frac{d^\alpha w}{dz^\alpha} = 2^{\frac{1}{3}}\varepsilon wW, \tag{5.50}$$

one can actually differentiate equation (5.50) once to be as

$$\frac{d^{2\alpha}w}{dz^{2\alpha}} = 2^{\frac{1}{3}}\varepsilon\frac{d^\alpha w}{dz^\alpha}W + 2^{\frac{1}{3}}\varepsilon w\frac{d^\alpha W}{d\zeta^\alpha}\frac{d^\alpha\zeta}{d\alpha}\zeta^{\alpha-1}. \tag{5.51}$$

Solving equation (5.46b) for $\frac{d^\alpha W}{d\zeta^\alpha}$, then substituting the result and equation (5.50) into equation (5.51), leads to

$$\frac{d^{2\alpha}w}{dz^{2\alpha}} = 2w^3 + \frac{z^\alpha}{\alpha}w. \tag{5.52}$$

Briefly we can say that, the combination of Bäcklund transformation equation (5.44) with the transformation (5.46) provides a relation between two CP_{II} equations whose parameters γ are either integers or half odd-integers. In other word, there is a mapping between the rational solutions of CP_{II}, which arise when $\gamma = n$ for $n \in \mathbf{Z}$, and the one-parameter CAiry function solutions, which arise when $\gamma = n + \frac{1}{2}$ for $n \in \mathbf{Z}$.

- CP_{II} has associated **Affine Weyl group**. An affine Weyl group is essentially a group of translations and reflections on a lattice. For the Painlevé equations, this lattice is in the parameter space [46].
Whereas, the composition of two Bäcklund transformations is a Bäcklund transformation, the affine Weyl group $\mathcal{W} =< \mathcal{S},\, \mathcal{T}_+ >$ of generalized Bäcklund transformations is generated by

a reflection $\mathcal{S}: \quad w(z;-\gamma) = -w(z;\gamma) \quad ,\gamma \in \mathbb{C},$
and
a translation $\mathcal{T}_{\pm}: \quad w(z;\gamma \pm 1) = -w(z;\gamma) - \frac{2\gamma \pm 1}{2w^2(z;\gamma) \pm 2\frac{d^\alpha w(z;\gamma)}{dz^\alpha} + \frac{z^\alpha}{\alpha}},$
with

$$\mathcal{S}^2 = \mathcal{T}_+\mathcal{T}_- = \mathcal{T}_-\mathcal{T}_+ = \mathcal{I}$$

where $\mathcal{I}$ is the identity transformation.

5.5 Some Other Properties to the Solutions of CP_{II}

In this part, many properties which CP_{II} possess are studied.

- Generic solution of CP_{II} equation (4.47) are $\alpha-$meromorphic functions. These generic solutions have an infinity set of simple poles accumulating at the essential singularity at $z = \infty$.
- CP_{II} admits the finite group of order 6 of scalings

$$w = \varepsilon\lambda^{2\alpha}\phi, \quad z = \lambda\zeta, \quad \gamma = \varepsilon\mu, \quad \text{with } \lambda^3 = 1, \text{ and } \varepsilon^2 = 1. \tag{5.53}$$

This immediately yields the set of equations

$$\frac{d^{2\alpha}w}{dz^{2\alpha}} = \varepsilon\frac{d^{2\alpha}\phi}{d\zeta^{2\alpha}}, \tag{5.54a}$$

$$w^3 = \varepsilon\phi^3, \tag{5.54b}$$

$$\frac{z^\alpha}{\alpha}w = \varepsilon\frac{\zeta^\alpha}{\alpha}\phi. \tag{5.54c}$$

Henceforth, the substitution of equations (5.54) into CP_{II}, leads to

$$\frac{d^{2\alpha}w}{dz^{2\alpha}} - 2w^3 - \frac{z^\alpha}{\alpha}w - \gamma = \varepsilon\left[\frac{d^{2\alpha}\phi}{d\zeta^{2\alpha}} - 2\phi^3 - \frac{\zeta^\alpha}{\alpha}\phi - \mu\right], \tag{5.55}$$

that is, w is a solution of CP_{II} if and only if ϕ is a solution of CP_{II}.

- CP_{II} can be obtained by the scaling reduction

$$z = xt^{\frac{-\beta}{3\alpha}}, \quad \psi = t^{\frac{-\beta}{3}}w(z) \tag{5.56}$$

of the conformable modified Korteweg-de Vries $(CmKdV)$ equation

$$\frac{\partial^\beta\psi}{\partial t^\beta} - 6\psi^2\frac{\partial^\alpha\psi}{\partial x^\alpha} + \frac{\partial^{3\alpha}\psi}{\partial x^{3\alpha}} = 0, \tag{5.57}$$

where $0 < \beta,\ \alpha \leq 1$, and $\beta,\ \alpha$ are parameters describing the order of the conformable time and space derivatives, respectively. Then after integrating once, $w(z)$ satisfies CP_{II} with γ the arbitrary constant of integration [40].
Also, CP_{II} can be reduced by the similarity reduction

$$\zeta = xt^{\frac{-\beta}{3\alpha}}, \quad \psi = t^{\frac{-2\beta}{3}}\Psi(\zeta) \tag{5.58}$$

of the conformable Korteweg-de Vries $(CKdV)$ equation

$$\frac{\partial^\beta\psi}{\partial t^\beta} + 6\psi\frac{\partial^\alpha\psi}{\partial x^\alpha} + \frac{\partial^{3\alpha}\psi}{\partial x^{3\alpha}} = 0, \tag{5.59}$$

henceforth, the scale $\omega = \left(\frac{\beta}{3}\right)^{\frac{1}{3\alpha}}\zeta,\ \ \Psi(\zeta) = \omega = \left(\frac{\beta}{3}\right)^{\frac{2}{3}}W(\omega)$ transformed equation (5.59) to

$$\frac{d^{3\alpha}W}{d\omega^{3\alpha}} + 6W\frac{d^\alpha W}{d\omega^\alpha} - \frac{\omega^\alpha}{\alpha}\frac{d^\alpha W}{d\omega^\alpha} - 2W = 0. \tag{5.60}$$

There exist a one-to-one correspondence between solutions of equation (5.60) and those of CP_{II}, given by

$$W = -\frac{d^{\alpha}w}{d\omega^{\alpha}} - w^2, \quad w = \frac{\frac{d^{\alpha}W}{d\omega^{\alpha}} + \gamma}{2W - \frac{\omega^{\alpha}}{\alpha}}, \tag{5.61}$$

for further detail see [40].

- Under the scale $w = \varepsilon y + \frac{1}{\varepsilon^5}$, $\frac{z^{\alpha}}{\alpha} = \varepsilon^2 x - \frac{6}{\varepsilon^{10}}$, $\gamma = \frac{4}{\varepsilon^{15}}$, CFP_{II} can be converted to

$$\frac{d^2y}{dx^2} = 6y^2 + x + \varepsilon^6(2y^3 + xy). \tag{5.62}$$

Letting $\varepsilon \to 0$ in equation (5.62), we find

$$\frac{d^2y}{dx^2} = 6y^2 + x, \tag{5.63}$$

apparently, equation (5.63) is the classical first Painlevé equation (P_I).

Also, it is of some interest to examine the transformation

$$w = \varepsilon y + \frac{1}{\varepsilon^5}, \quad \frac{z^{\alpha}}{\alpha} = \varepsilon^2 \frac{x^{\alpha}}{\alpha} - \frac{6}{\varepsilon^{10}}, \quad \gamma = \frac{4}{\varepsilon^{15}}$$

into CP_{II}. Here we obtain

$$\frac{d^{2\alpha}y}{dx^{2\alpha}} = 6y^2 + \frac{x^{\alpha}}{\alpha} + \varepsilon^6\left(2y^3 + \frac{x^{\alpha}}{\alpha}y\right). \tag{5.64}$$

Letting $\varepsilon \to 0$, gives

$$\frac{d^{2\alpha}y}{dx^{2\alpha}} = 6y^2 + \frac{x^{\alpha}}{\alpha}. \tag{5.65}$$

The resulting equation is the conformable first Painlevé equation CP_I.

6 Conclusion

We proposed a generalization of Painlevé test for conformable fractional ordinary differential equations, and introduced a sufficient condition of the generalized Painlevé property. The differential equations are considered to be in the form

$$\frac{d^{n\alpha}w(z)}{dz^{n\alpha}} = F\left(z, w, ..., \frac{d^{(n-1)\alpha}w}{dz^{(n-1)\alpha}}\right), \quad 0 < \alpha \leq 1$$

where F is $\alpha-$analytic in z and rational in other arguments. The analysis is successfully applied to investigate the generalized Painlevé property of CP_I, also to CP_{II} equations. Furthermore, we gave exact solution to $(CP_I$ and $CP_{II})$ in terms of the Laurent series and shows that the general solution is $\alpha-$meromorphic in z to its critical points. Moreover, we show that for a particular choice of the parameter in the CP_{II} admit a special solution in terms of Airy function.

P_I can be obtained from CP_{II} by the process of contraction. In a similar way, it was possible to obtain the associated transformation for CP_I from the transformation for CP_{II}.

An introduction to some of the fascinating properties which $(CP_I$ and $CP_{II})$ possess are given. The isomondromy problems, Hirota Bilinear Form, Hamiltonian Structure, Bäcklund transformations and others are discussed.

It is interesting to apply the analysis to other conformable Painlevé equations. In addition, there are several very important open problems related to the area of conformable Painlevé equations.

References

[1] S. G. Samko, A. A. Kilbas, O. L. Marichev, “Fractional integrals and derivatives: theory and applications", New York: Gordon and Breach; (1993).

[2] A. A. Kilbas, B. S. Srivastava, J. J. Trujillo, “Theory and applications of fractional differential equations", Amsterdam: Elsevier; (2006).

[3] I. Podlubny, “Fractional differential equations", San Diego: Academic Press; (2006).

[4] K. Diethelm, “The Analysis of Fractional Differential Equations", New York: Springer; (2010).

[5] J. T. Machado, V. Kiryakova, F. Mainardi, “Recent history of fractional calculus", Commun Nonlinear Sci Numer Simulat, **16**, 1140-1153 (2011).

[6] G. Jumarie, “Modified Riemann-Liouville derivative and fractional Taylor series of non-differentiable functions further results", Comput. Math. Appl., **51**, 1367-1376 (2006).

[7] E. C. Olivera, J. A. Tenreiro Machado, “A Review of Definitions for Fractional Derivatives and Integral", Mathematical Problems in Engineering, ID **238459**, 6 pages (2014).

[8] U. Katugampola, “A new fractional derivative with classical properties", ArXiv: 1410.6535v2 (2014).

[9] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, “A new definition of fractional derivative", Journal of Computational and Applied Mathematics, **264**, 65-70 (2014).

[10] M. A. Hammad, R. Khalil, “Abel’s formula and Wronskian for conformable fractional differential equations", International Journal of Differential Equations and Applications, **13**, 177-183 (2014).

[11] I. Abu Hammad, R. Khalil, “Fractional Fourier series with applications", J. ajcam., **4(6)**, 187-191 (2014).

[12] M. Abu Hammad, R. Khalil, “Conformable fractional heat differential equation" J. ijpam. volume **94**, No. **2**, 215-221 (2014).

[13] T. Abdeljawad, M. Al Horani, R. Khalil, “Conformable fractional semigroups of operators", Journal of Semigroup Theory and Applications, (2015) Article-ID.

[14] H. Batarfi, J. Losada, J. Nieto, W. SHammakh, “Three-point boundary value problem for conformable fractional differential equations", J. jfs. Volum 2015, article ID **706383**, 6 pages (2015).

[15] A. Atangana, D. Baleanu, A. Alsaedi, “New properties of conformable derivative", Open Mathematics, **13**, 889-898 (2015).

[16] Y. Çenesiz, A. Kurt, “The solution of time and space conformable fractional heat equations with conformable Fourier transform", J. ausm. **7**, **2**, 130-140 (2015).

[17] D. R. Anderson, D. J. Ulness, “Results for conformable differential equations", preprint (2016).

[18] B. Bayour, D. D. F.M. Toress, “Existence of solution to local fractional nonlinear differential equation", J. jcam, (2016).

[19] O. S. Iyiola, E. R. Nwaeze, “Some new results on the new conformable fractional calculus with application using D’Alambert approach", Progress in Fractional Differentiation and Applications", **2**, 115-122 (2016).

[20] M. Z. Sarikaya, “Gronwall type inequality for conformable fractional integrals". RGMIA Research Report Collection, **19** Article 122 (2016).

[21] F. Usta, M. Z. Sarikaya, “On generalization conformable fractional integral inequalities", RGMIA Research Report Collection, **19**, Article 123 (2016).

[22] P. Michal, L. P. *Škripková*, “Sturm’s theorems for conformable fractional differential equations", Math. Commun., **21**, 273-281 (2016).

[23] U. Fuat, Z. S. Mehmet, “Explicit bounds on certain integral inequalities via conformable fractional calculus", Cogent Mathematics, **4**, 1277505 (2017).

[24] Y. Çenesiz, A. Kurt, O. Tasbozan, “On the new solutions of the conformable time fractional generalized Hirota-Satsuma coupled KdV system", LV, **1**, 37-49 (2017).

[25] D. Zhao, M. Luo,“General conformable fractional derivative and its physical interpretation", Calcolo, 1-15 (2017).

[26] M. Kaplana, A. Akbulu, “Application of two different algorithms to the approximate long water wave equation with conformable fractional derivative", ARAB JOURNAL OF BASIC AND APPLIED SCIENCES, (2018).

[27] A. Ortega, J. J. Rosales, “Newton’s law of cooling with fractional conformable derivative", Revista Mexicana de Fisica, **64**, 172-175 (2018).

[28] H. Y. Karayer, D. Demirhan, F. Buyukkilic, “Solutions of local fractional sine-Gordon equations", Waves in Random and Complex Media, DOI: 10.1080/17455030.2018.1425572 (2018).

[29] T. Abdeljawad, “On conformable fractional calculus", Journal of Computational and Applied Mathematics, **279**, 57-66 (2015).

[30] Anderson, Douglas R. "Taylors formula and integral inequalities for conformable fractional derivatives." Contributions in mathematics and engineering. Springer, Cham, 2016. 25-43.

[31] Al-Zhour, Zeyad, et al. "Series solutions for the Laguerre and Lane-Emden fractional differential equations in the sense of conformable fractional derivative." Alexandria Engineering Journal 58.4 (2019): 1413-1420.

[32] Martinez, Francisco, et al. "Some new results on conformable fractional power series." Authorea Preprints (2020).

[33] Amryeen, Rasha, et al. "Adaptation of residual power series approach for solving time-fractional nonlinear Kline-Gordon equations with conformable derivative." Appl Math Inf Sci 14 (2020): 563-575.

[34] Tariq, Hira, et al. "New travelling wave analytic and residual power series solutions of conformable CaudreyDoddGibbonSawadaKotera equation." Results in Physics 29 (2021): 104591.

[35] Liaqat, Muhammad Imran, Adnan Khan, and Ali Akgl. "Adaptation on power series method with conformable operator for solving fractional order systems of nonlinear partial differential equations." Chaos, Solitons Fractals 157 (2022): 111984.

[36] Shqair, Mohammed, et al. "Adaptation of conformable residual power series scheme in solving nonlinear fractional quantum mechanics problems." Applied Sciences 10.3 (2020): 890.

[37] Ibrahim, Rabha W., and Dumitru Baleanu. "On quantum hybrid fractional conformable differential and integral operators in a complex domain." Revista de la Real Academia de Ciencias Exactas, Fcas y Naturales. Serie A. Matemcas 115.1 (2021): 1-13.

[38] Yokus, Asif, et al. "Numerical comparison of Caputo and Conformable derivatives of time fractional Burgers-Fisher equation." Results in Physics 25 (2021): 104247.

[39] Darvishi, M. T., Mohammad Najafi, and Abdul-Majid Wazwaz. "Conformable space-time fractional nonlinear (1+ 1)-dimensional Schrdinger-type models and their traveling wave solutions." Chaos, Solitons Fractals 150 (2021): 111187.

[40] B. A. Tayyan, A. H. Sakka, "Lie symmetry analysis of some conformable fractional partial differential equations", Arabian Journal of Mathematics, Springer (2018).

[41] P. Painlevé, Bull. Soc. Math. Fr., **28**(1900) 214; Acta Math. **25**(1902).

[42] B. Gambier, Acta Math. **33**(1909).

[43] R. Fuchs, "*Ü*ber lineare homogene differentialgleichungen zweiter ordnung mit drei im endlich gelegene wesentlich singul*ä*ren stellen", Math. Ann., **63** (1907) 301-321.

[44] E. Picard, "Memoire sur la theorie des fonctions algebrique de deux variables", J. de Math., **5**(1889) 135-318.

[45] E.L. Ince, "Ordinary Differential Equations", Dover, New York, (1956).

[46] P.A. Clarkson, "Painlevé Equations - Nonlinear Special Functions", http://math.nist.gov/DLozier/SF21/SF21slides/Clarkson.pdf .

[47] M.J. Ablowitz, P.A. Clarkson, "Solitons, Nonlinear Evolution Equations and Inverse Scattering", L.M.S. Lect. Notes Math., vol. **149**, C.U.P., Cambridge, (1991).

[48] M.J. Ablowitz, H. Segur, "Solitons and the Inverse Scattering Transform", SIAM. Philadelphia, (1981).

[49] J. Weiss, M.Tabor, G. Carnevale,"The Painlevéproperty of partial differential equations", J. Phys. **24**(1983), 522-526.

[50] W. H. Steeb, N. Euler,"Nonlinear Evolution Equation and Painlevé Test", World Scientific, (1988).

[51] J. Hu, M. Yan,"Analytical Aspects of the Painlevé Test", (2000).

[52] M. J. Ablowitz, A. S. Fokas,"Complex Variables, Introduction and Applications", second edition, Cambridge University Press, London, (2003).

Author information

Basema Asaad AlTayyan, Department of Mathematics, Islamic University of Gaza, Palestine.
E-mail: basma-20111@live.com

Received: 2022-01-09
Accepted: 2022-05-28