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ON SOME NEW IDENTITIES OF LUCAS NUMBERS AND GENERALIZATION OF FIBONACCI TRACE SEQUENCES

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Abstract In this paper, we obtain some new identities of Lucas sequences and establish relationships with Fibonacci and Tribonacci sequences. We propose the generalized Fibonacci trace sequences which give the generalized Lucas sequences $\{l_{k,n}\}_{k\geq 2}$ of order k as defined by the recurrence relation $l_{k,n+k} = l_{k,n} + l_{k,n+1} + ... + l_{k,n+(k-1)}$ with initial conditions as solution of Hanoi Tower sequences. Further, the associated recursive matrices has been developed and studied.

1 Introduction

The recursive sequences along with associated matrices and their properties always attract the attention of people working in the area of combinatorics and linear algebra.

In spite of many articles, literature reviews, and books that have been published on Fibonacci and Lucas numbers [6], some special second-order recursive sequences such as Pell sequence, Pell-Lucas Sequence, etc. of integers are still interesting areas for investigation. Also for third-order integer sequences such as Tribonacci, Padovon, Perrin numbers, etc. have interesting properties which attract the researchers.

Tianxiao et al. [11] presented the matrix representation from recursive sequences of order three as $a_{k+3} = p.a_k + q.a_{k+1} + r.a_{k+2}$ with arbitrary initial conditions, and discussed some special third-order recurrence relations such as Padavon numbers and Perrin numbers. Yazlik and Taskara [13], defined generalized k-Horadam sequence $\{H_{k,n}\}_{n\in\mathbb{N}}$ and studied their properties, further they obtained a generating function for the generalized k-Horadam sequence. Here Horadam sequence is a special second-order sequence alike Fibonacci, Pell, Lucas, etc. Koken and Bozkurt [5] obtained many identities related to Fibonacci and Lucas of order two and established some well-known equalities by matrix method. Kumari et al. [7] have discussed the commutativity of multinacci matrices and their application in cryptography. In [2], Younseok Choo derived general identities for generalized Fibonacci sequences and established relations with other known and unknown identities. Kumari et al. [8] have proposed some new families of identities of k-Mersenne and generalized k-Gaussian Mersenne numbers and their polynomials. For the generalizations of Lucas and Fibonacci like sequences, properties, their applications, and some of its variants, see [1, 9, 10, 11, 12].

In this paper, Section 1 is devoted to the preliminaries on Lucas sequences and matrices. In Section 2, some new identities on Lucas sequences and matrices of orders 2 and 3 have been obtained. Finally in Section 3, we proposed a generalized Fibonacci trace sequences as well as the associated recursive matrices and obtained formula for the initial values of Fibonacci trace sequences as solution of Hanoi Tower sequences.

1.1 Lucas Sequence and Lucas Matrices

The second order Fibonacci sequence$\{f_n\}$ and Lucas sequence $\{l_n\}$ [3, 4, 6] are the sequences of integers defined as,

$$f_{n+2} = f_n + f_{n+1}, \ n \geq 0, \quad \text{with } f_0 = 0, f_1 = 1 \quad [A000045], \tag{1.1}$$
$$l_{n+2} = l_n + l_{n+1}, \ n \geq 0, \quad \text{with } l_0 = 2, \ l_1 = 1 \quad [A000032], \tag{1.2}$$

where both the Fibonacci and Lucas sequences can be extended in negative direction too with the identity $f_{-n} = (-1)^{n+1} f_n$ and $l_{-n} = (-1)^n l_n$ for $n \in \mathbb{N} \cup \{0\}$, respectively.

Let us assume that $f_{k,n}$ and $l_{k,n}$ denotes the nth term of Fibonacci and Lucas sequences of order k throughout the paper.

Rewriting Eqn. (1.2) with new notation, we get

$$l_{2,n} = l_{2,n+2} - l_{2,n+1}, \ n < 0, \ \text{with } l_{2,0} = 2, l_{2,1} = 1$$
$$or, \ l_{2,-n} = (-1)^n l_{2,n} \quad \text{for } n \in \mathbb{N} \cup \{0\}.$$

The first few values of above sequences are:

n	...	-4	-3	-2	-1	**0**	**1**	2	3	4	5	6	7	8	9	...
$f_{2,n}$	...	-3	2	-1	1	**0**	**1**	1	2	3	5	8	13	21	34	...
$l_{2,n}$	...	7	-4	3	-1	**2**	**1**	3	4	7	11	18	29	47	76	...

The recursive matrices $Q_2^{(n)}$ and $L_2^{(n)}$ of Fibonacci and Lucas numbers, respectively, are given by

$$Q_2^{(n)} = \begin{bmatrix} f_{2,n+1} & f_{2,n} \\ f_{2,n} & f_{2,n-1} \end{bmatrix} \text{ with } Q_2^{(1)} = Q_2 = \begin{bmatrix} f_{2,2} & f_{2,1} \\ f_{2,1} & f_{2,0} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$
$$L_2^{(n)} = \begin{bmatrix} l_{2,n+1} & l_{2,n} \\ l_{2,n} & l_{2,n-1} \end{bmatrix} \text{ with } L_2^{(0)} = \begin{bmatrix} l_{2,1} & l_{2,0} \\ l_{2,0} & l_{2,-1} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}.$$

It can be noted that $Q_2^{(n)} = Q_2^n$, the n times multiplication of Q_2 [10].

Lemma 1.1. *[6] Let $\{l_{2,n}\}$ be the Lucas sequence and Q_2^n be the Fibonacci matrix then*

$$l_{2,n} = trace(Q_2^n) = f_{2,n-1} + f_{2,n+1}, \qquad for \ n \in \mathbb{Z}. \tag{1.3}$$

Some properties of generalized Fibonacci matrices obtained in [10] are given in the following theorem, that we use in this paper to establish some new identities and results.

Theorem 1.2. *[10] Let Q_k^n be the generalized Fibonacci matrix of order k (multinacci matrix), then it has following properties:*

(i) $Q_k^0 = I_k$, *where I_k is identity matrix of order k.*

(ii) $(Q_k^1)^n = Q_k^n$ *and* $(Q_k^{-1})^n = Q_k^{-n}$.

(iii) $Q_k^n Q_k^l = Q_k^{n+l}$.

(iv) $\det(Q_k^n) = [(-1)^{k-1}]^n = (-1)^{(k-1)n}$.

(v) $Q_k^n Q_k^{-n} = I_k$, *where I_n is identity matrix of order n.*

2 Main work

Theorem 2.1 (Matrix formation). *Let $n \in \mathbb{Z}$ and $L_2^{(n)}$ be the nth Lucas matrix, then we have*

$$L_2^{(n)} = Q_2^n L_2^{(0)} = L_2^{(0)} Q_2^n, \qquad \forall \, n \in \mathbb{Z}. \tag{2.1}$$

Proof. To prove it, we start with the RHS of Eqn. (2.1)

$$\begin{aligned} Q_2^n L_2^{(0)} &= \begin{bmatrix} f_{2,n+1} & f_{2,n} \\ f_{2,n} & f_{2,n-1} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} f_{2,n+1} + 2f_{2,n} & 2f_{2,n+1} - f_{2,n} \\ f_{2,n} + 2f_{2,n-1} & 2f_{2,n} - f_{2,n-1} \end{bmatrix} \\ &= \begin{bmatrix} f_{2,n+2} + f_{2,n} & f_{2,n+1} + f_{2,n-1} \\ f_{2,n+1} + f_{2,n-1} & f_{2,n} + f_{2,n-2} \end{bmatrix} \\ &= \begin{bmatrix} l_{2,n+1} & l_{2,n} \\ l_{2,n} & l_{2,n-1} \end{bmatrix} = L_2^{(n)} \qquad \text{(using Lemma 1.1).} \end{aligned}$$

By a similar argument, the relation $L_2^{(n)} = L_2^{(0)} Q_2^n$ holds. □

It may be noted that for Lucas matrix $L_2^{(n)}$, we have

$$L_2^{(n)} L_2^{(-n)} = L_2^{(0)} Q_2^n Q_2^{-n} L_2^{(0)} = L_2^{(0)} L_2^{(0)} = 5I_2, \tag{2.2}$$

where Q_2^n is Fibonacci Matrix.

Proposition 2.2. Let $L_2^{(n)}$ be the n-th Lucas matrix of order two, then the determinant of $L_2^{(n)}$ is

$$\det(L_2^{(n)}) = (-1)^{n+1} 5 \qquad \forall\, n \in \mathbb{Z}.$$

Proof. From Eqn. (2.1), we have $L_2^{(n)} = Q_2^n L_2^{(0)}$. Thus

$$\begin{aligned} \det(L_2^{(n)}) &= \det(Q_2^n L_2^{(0)}) \\ &= \det(Q_2^n) \det(L_2^{(0)}) \\ &= (-1)^{n+1} 5 \qquad \text{(using (iv) of Theorem 1.2)} \end{aligned}$$

as required. □

Remark 2.3. For the Lucas sequence $\{l_{2,n}\}$, we have $l_{2,n+1} l_{2,n-1} - l_{2,n}^2 = (-1)^{n-1} 5 \quad \forall\, n \in \mathbb{Z}$.

In order to obtain the inverse of nth Lucas Matrix $L_2^{(n)}$, we have the following theorem.

Theorem 2.4. *Let L_2^* be the inverse of the nth Lucas matrix $L_2^{(n)}$ then, we have $L_2^* = \frac{1}{5} L_2^{(-n)}$ where $L_2^{(-n)} = \begin{bmatrix} l_{2,-n+1} & l_{2,-n} \\ l_{2,-n} & l_{2,-n-1} \end{bmatrix}$, $n > 0$.*

Proof. Here,

$$\begin{aligned} L_2^{(n)} L_2^* &= L_2^{(n)} (\frac{1}{5} L_2^{(-n)}) = \frac{1}{5} (L_2^{(n)} L_2^{(-n)}) \\ &= \frac{1}{5} 5 I_2 = I_2 \qquad \text{(using Eqn. (2.2)).} \end{aligned} \tag{2.3}$$

As required. □

Corollary 2.5. *Let $\{l_{2,n}\}$ and $\{f_{2,n}\}$ be the Lucas and Fibonacci sequences, respectively. Then the following relations hold good for every $n \in \mathbb{Z}$.*

(i) $l_{2,n} = f_{2,n} + 2f_{2,n-1}$.

(ii) $l_{2,n} = 2f_{2,n+1} - f_{2,n}$.

(iii) $l_{2,m+1} l_{2,n+1} + l_{2,m} l_{2,n} = 5 f_{2,m+n+1}$.

(iv) $l_{2,m+1} l_{2,n} + l_{2,m} l_{2,n-1} = 5 f_{2,m+n} = l_{2,m} l_{2,n+1} + l_{2,m-1} l_{2,n}$.

2.1 Third Order Lucas Sequence

Definition 2.6. For $n \in \mathbb{Z}$, the tribonacci sequences $\{f_{3,n}\}$ is given by recurrence relation

$$f_{3,n+3} := f_{3,n} + f_{3,n+1} + f_{3,n+2} \quad \text{with } f_{3,0} = f_{3,1} = 0,\ f_{3,2} = 1 \quad [A000073].$$

The third order Lucas sequence is a sequence of integers analogous to tribonacci sequences given by

$$l_{3,n+3} = l_{3,n} + l_{3,n+1} + l_{3,n+2}, \ \ n \geq 0, \tag{2.4}$$

with initial values

$$l_{3,0} = 3, l_{3,1} = 1, l_{3,2} = 3 \ \ [A001644] \tag{2.5}$$
$$l_{3,0} = 0, l_{3,1} = 1, l_{3,2} = 3 \ \ [A007486]. \tag{2.6}$$

For above sequence (2.4) with (2.5), the term "trucas" is used by Johnson in [4].
Sequence (2.4) can also be extended in negative direction, given by rearranging (2.4) as

$$l_{3,n} = l_{3,n+3} - (l_{3,n+1} + l_{3,n+2}), \ \ n \leq 0.$$

The first few values of above two ended sequence are as follows:

OEIS	n	...	-6	-5	-4	-3	-2	-1	**0**	**1**	**2**	3	4	5	6	...
[A001644]	$l_{3,n}$	...	11	-1	-5	5	-1	-1	**3**	**1**	**3**	7	11	21	39	...
[A007486]	$l_{3,n}$	...	-1	-4	4	-1	-1	2	**0**	**1**	**3**	4	8	15	27	...

The matrix representation for third order Lucas sequence given by the square matrix $L_3^{(n)}$ called the Lucas matrix of order 3 is defined as

$$L_3^{(n)} = \begin{bmatrix} l_{3,n+2} & l_{3,n+1} + l_{3,n} & l_{3,n+1} \\ l_{3,n+1} & l_{3,n} + l_{3,n-1} & l_{3,n} \\ l_{3,n} & l_{3,n-1} + l_{3,n-2} & l_{3,n-1} \end{bmatrix} \quad \text{with} \quad L_3^{(0)} = \begin{bmatrix} l_{3,2} & l_{3,1} + l_{3,0} & l_{3,1} \\ l_{3,1} & l_{3,0} + l_{3,-1} & l_{3,0} \\ l_{3,0} & l_{3,-1} + l_{3,-2} & l_{3,-1} \end{bmatrix}.$$

Now we aim to obtain some new identities and relations on Lucas matrices for both the initial values.

Case-I: For Initial Condition (2.5)

Theorem 2.7. *Let Q_3^n be the tribonacci matrix. If $3rd$ order Lucas sequence $\{l_{3,n}\}$ satisfies the initial value given in Eqn. (2.5), then we write*

$$L_3^{(n)} = Q_3^n L_3^{(0)}, \quad \textit{where } n \in \mathbb{Z} \textit{ and } L_3^{(0)} = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 3 \\ 3 & -2 & -1 \end{bmatrix}.$$

Proof. First we shall prove this theorem by mathematical induction on n in positive direction. For $n = 0$, statement holds obviously. Consider for $n = 1$,

$$Q_3^1 L_3^{(0)} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 3 \\ 3 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 4 & 3 \\ 3 & 4 & 1 \\ 1 & 2 & 3 \end{bmatrix} = L_3^{(1)}.$$

Assume that result is true for $n = k$, (i.e $L_3^{(k)} = Q_3^k L_3^{(0)}$).

Now proving the result for $n = k+1$, we have

$$
\begin{aligned}
Q_3^{k+1}L_3^{(0)} &= \begin{bmatrix} f_{3,k+3} & f_{3,k+2}+f_{3,k+1} & f_{3,k+2} \\ f_{3,k+2} & f_{3,k+1}+f_{3,k} & f_{3,k+1} \\ f_{3,k+1} & f_{3,k}+f_{3,k-1} & f_{3,k} \end{bmatrix} \begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 3 \\ 3 & -2 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 3f_{3,k+3}+4f_{3,k+2}+f_{3,k+1} & 4f_{3,k+3}+2f_{3,k+1} & f_{3,k+3}+2f_{3,k+2}+3f_{3,k+1} \\ 3f_{3,k+2}+4f_{3,k+1}+f_{3,k} & 4f_{3,k+2}+2f_{3,k} & f_{3,k+2}+2f_{3,k+1}+3f_{3,k} \\ 3f_{3,k+1}+4f_{3,k}+f_{3,k-1} & 4f_{3,k+1}+2f_{3,k-1} & f_{3,k+1}+2f_{3,k}+3f_{3,k-1} \end{bmatrix} \\
&= \begin{bmatrix} l_{3,k+3} & l_{3,k+2}+l_{3,k+1} & l_{3,k+2} \\ l_{3,k+2} & l_{3,k+1}+l_{3,k} & l_{3,k+1} \\ l_{3,k+1} & l_{3,k}+l_{3,k-1} & l_{3,k} \end{bmatrix} \qquad (\text{using } L_3^{(k)} = Q_3^k L_3^{(0)}) \\
&= L_3^{(k+1)}.
\end{aligned}
$$

Thus, the result is true for $n = k+1$.

For negative direction, consider $n = -k$, where $k \geq 0$. Then by applying induction on k it can be proved for negative direction too. Combining both the cases, result holds for all $n \in \mathbb{Z}$. □

On conclusion, following corollary is consequence of the Theorem 2.7.

Observation 1. Lucas sequence of order three can be also defined as sequence of traces of tribonacci matrices i.e $l_{3,n} = trace(Q_3^n) = f_{3,n+2} + f_{3,n} + 2f_{3,n-1}$.

Corollary 2.8. *Let the third order Lucas sequence $\{l_{3,n}\}$ is satisfying I.C (2.5) and $\{f_{3,n}\}$ be the tribonacci sequence then the following relation holds for every $n \in \mathbb{Z}$:*

(i) $l_{3,n} = f_{3,n+2} + f_{3,n} + 2f_{3,n-1}$.

(ii) $l_{3,n} = f_{3,n+1} + 2f_{3,n} + 3f_{3,n-1}$.

(iii) $l_{3,n} = 3f_{3,n} + 4f_{3,n-1} + f_{3,n-2}$.

(iv) $l_{3,n+1} + l_{3,n} = 4f_{3,n+2} + 2f_{3,n}$.

(v) $l_{3,n+1} + l_{3,n} = 2f_{3,n+4} - 4f_{3,n+1}$.

Proof. It can be easily proved using mathematical induction on n and Theorem 2.7. □

Case-II: For Initial Condition (2.6)

Theorem 2.9. *Let Q_3^n be the tribonacci matrix. If the Lucas sequence $\{l_{3,n}\}$ satisfies the initial value given in Eqn. (2.6), then we write*

$$
L_3^{(n)} = Q_3^n L_3^{(0)} \qquad \text{where } n \in \mathbb{Z} \text{ and } L_3^{(0)} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}.
$$

Proof. For $n = 0$, statement holds obviously. Now we shall prove this theorem by mathematical induction on n in positive direction. For $n = 1$, we have

$$
Q_3^1 L_3^{(0)} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 3 \\ 3 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} = L_3^{(1)}.
$$

Assume that the result is true for $n = k$, i.e $L_3^{(k)} = Q_3^k L_3^{(0)}$. Now for $n = k+1$, using $L_3^{(0)} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ a similar calculation to Theorem 2.7 yields $Q_3^{k+1} L_3^{(0)} = L_3^{k+1}$.

For negative direction, consider $n = -k$, where $k \geq 0$. Then by applying induction on k, it can be proved for negative direction too. Combining all above cases, it gives the result holds for all $n \in \mathbb{Z}$. □

Thus, we have $Q_3^n L_3^{(0)} = L_3^{(n)}$ and on conclusion following corollary is consequence of the Theorem 2.9.

Corollary 2.10. *Let $\{f_{3,n}\}$ be the tribonacci sequence and the Lucas sequence $\{l_{3,n}\}$ satisfy the initial values of Eqn. (2.6) then the following relations holds good for every $n \in \mathbb{Z}$.*

(i) $l_{3,n} = 3f_{3,n} + f_{3,n-1} + f_{3,n-2}$.

(ii) $l_{3,n} = f_{3,n+1} + 2f_{3,n}$.

(iii) $l_{3,n+1} + l_{3,n} = 4f_{3,n+2} + 3f_{3,n+1} + f_{3,n}$.

(iv) $l_{3,n+1} + l_{3,n} = f_{3,n+4} + 2f_{3,n+2} + f_{3,n+1}$.

(v) $l_{3,n+1} + l_{3,n} = f_{3,n+4} - f_{3,n-1}$.

Proof. Proof of corollary followed from induction on n and Theorem 2.9. □

Remark 2.11. For Case-II, $l_{3,n} \neq trace(Q_3^n)$.

The following results hold for both the initial conditions (2.6) and (2.5).

Proposition 2.12. For $n \in \mathbb{Z}$, we have $Q_3^n L_3^{(0)} = L_3^{(0)} Q_3^n$, where $L_3^{(0)}$ is the initial Lucas matrix.

Proof. For initial condition (2.6), we have $L_3^{(0)} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

Now,

$$\begin{aligned} Q_3^n L_3^{(0)} &= \begin{bmatrix} f_{3,n+2} & f_{3,n+1} + f_{3,n} & f_{3,n+1} \\ f_{3,n+1} & f_{3,n} + f_{3,n-1} & f_{3,n} \\ f_{3,n} & f_{3,n-1} + f_{3,n-2} & f_{3,n-1} \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 3f_{3,n+2} + f_{3,n+1} + f_{3,n} & f_{3,n+2} + 3f_{3,n+1} + 2f_{3,n} & f_{3,n+2} + 2f_{3,n+1} \\ 3f_{3,n+1} + f_{3,n} + f_{3,n-1} & f_{3,n+1} + 3f_{3,n} + 2f_{3,n-1} & f_{3,n+1} + 2f_{3,n} \\ 3f_{3,n} + f_{3,n-1} + f_{3,n-2} & f_{3,n} + 3f_{3,n-1} + 2f_{3,n-2} & f_{3,n} + 2f_{3,n-1} \end{bmatrix} \\ &= \begin{bmatrix} f_{3,n+3} + 2f_{3,n+2} & f_{3,n+4} - f_{3,n-1} & f_{3,n+2} + 2f_{3,n+1} \\ f_{3,n+2} + 2f_{3,n+1} & f_{3,n+3} - f_{3,n-2} & f_{3,n+1} + 2f_{3,n} \\ f_{3,n+1} + 2f_{3,n} & f_{3,n+2} - f_{3,n-3} & f_{3,n} + 2f_{3,n-1} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} L_3^{(0)} Q_3^n &= \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} f_{3,n+2} & f_{3,n+1} + f_{3,n} & f_{3,n+1} \\ f_{3,n+1} & f_{3,n} + f_{3,n-1} & f_{3,n} \\ f_{3,n} & f_{3,n-1} + f_{3,n-2} & f_{3,n-1} \end{bmatrix} \\ &= \begin{bmatrix} 3f_{3,n+2} + f_{3,n+1} + f_{3,n} & 3f_{3,n+1} + 4f_{3,n} + 2f_{3,n-1} + f_{3,n-2} & 3f_{3,n+1} + f_{3,n} + f_{3,n-1} \\ 3f_{3,n+1} + f_{3,n} + f_{3,n-1} & 3f_{3,n} + 4f_{3,n-1} + 2f_{3,n-2} + f_{3,n-3} & 3f_{3,n} + f_{3,n-1} + f_{3,n-2} \\ 3f_{3,n} + f_{3,n-1} + f_{3,n-2} & 3f_{3,n-1} + 4f_{3,n-2} + 2f_{3,n-3} + f_{3,n-4} & 3f_{3,n-1} + f_{3,n-2} + f_{3,n-3} \end{bmatrix} \\ &= \begin{bmatrix} f_{3,n+3} + 2f_{3,n+2} & f_{3,n+4} - f_{3,n-1} & f_{3,n+2} + 2f_{3,n+1} \\ f_{3,n+2} + 2f_{3,n+1} & f_{3,n+3} - f_{3,n-2} & f_{3,n+1} + 2f_{3,n} \\ f_{3,n+1} + 2f_{3,n} & f_{3,n+2} - f_{3,n-3} & f_{3,n} + 2f_{3,n-1} \end{bmatrix} \quad \text{(using tribonacci sequence (2.6))}. \end{aligned}$$

Therefore, $Q_3^n L_3^{(0)} = L_3^{(0)} Q_3^n$.
By a similarly argument, we can prove the result for the initial condition (2.5). □

Theorem 2.13. *Let $n \in \mathbb{Z}$, then for both the initial values given in Eqn. (2.5) and (2.6), we have*

$$L_3^{(n)} L_3^{(-n)} = (L_3^{(0)})^2 \qquad \forall \ n \in \mathbb{Z}.$$

Proof.

$$\begin{aligned} L_3^{(n)} L_3^{(-n)} &= (Q_3^n L_3^{(0)})(Q_3^{-n} L_3^{(0)}) && \forall \ n \in \mathbb{Z} \\ &= (Q_3^n L_3^{(0)} Q_3^{-n} L_3^{(0)}) \\ &= (Q_3^n Q_3^{-n} L_3^{(0)} L_3^{(0)}) && \text{(using Proposition 2.12)} \\ &= I_3 (L_3^{(0)})^2 && \text{(using Theorem 1.2 (v))} \\ &= (L_3^{(0)})^2. \end{aligned}$$

As required. □

Proposition 2.14. For any $n \in \mathbb{Z}$, the determinant of $L_3^{(n)}$ matrices are given by

$$\det(L_3^{(n)}) := \begin{cases} 44 & : \text{for the case-I (Eqn. (2.5))}, \\ 11 & : \text{for the case-II (Eqn. (2.6))}. \end{cases} \tag{2.7}$$

Proof. Using Theorem 2.7 and 2.9, we have $L_3^{(n)} = Q_3^n L_3^{(0)}$. Thus

$$\begin{aligned} \det(L_3^{(n)}) &= \det(Q_3^n L_3^{(0)}) \\ &= \det(Q_3^n) \det(L_3^{(0)}) \\ &= \begin{cases} (-1)^{2n} 44 & : \text{for the case-I} \\ (-1)^{2n} 11 & : \text{for the case-II} \end{cases} && \text{(using Theorem 1.2 (iv))} \\ &= \begin{cases} 44 & : \text{for the case-I} \\ 11 & : \text{for the case-II} \end{cases}. \end{aligned}$$

This completes the proof. □

Thus by Proposition 2.14, $\det(L_3^{(0)}) \neq 0$ implies the existence of inverse of Lucas matrices of order 3 for all n $\in \mathbb{Z}$. Hence, the following theorem.

Theorem 2.15. *For all $n \in \mathbb{Z}$, the inverse of Lucas matrices $L_3^{(n)}$ is given by*

$$Inv(L_3^{(n)}) = H^{-1} L_3^{(-n)} \quad \textit{where} \ \ H = (L_3^{(0)})^2.$$

Proof. Let $A = L_3^{(n)}$ and $B = [H^{-1} L_3^{(-n)}]$, We claim: $AB = I_n$.
To prove this theorem, first we need to show the following result

$$L_3^{(n)} H^{-1} = H^{-1} L_3^{(n)}.$$

Proof of the result.

$$\begin{aligned} L_3^{(n)} H^{-1} &= \{Q_3^n L_3^{(0)}\}\{(L_3^{(0)})^{-2}\} \\ &= Q_3^n (L_3^{(0)})^{-1} \\ &= (L_3^{(0)})^{-1} Q_3^n && \text{(using Proposition 2.12)} \\ &= (L_3^{(0)})^{-2} (L_3^{(0)}) Q_3^n = H^{-1}(L_3^{(n)}) \end{aligned}$$

Now,

$$\begin{aligned} AB &= L_3^{(n)} [H^{-1} L_3^{(-n)}] \\ &= (H^{-1})[L_3^{(n)} L_3^{(-n)}] \\ &= (L_3^{(0)})^{-2} (L_3^{(0)})^2 = I_3. \end{aligned}$$

This completes the proof. □

Thus the following remark obtained as summary of the results on third order Lucas matrix.

Remark 2.16. Let $n \in \mathbb{Z}$, Q_3^n be tribonacci matrix, $L_3^{(n)}$ be 3rd order Lucas matrix and $L_3^{(0)}$ be the initial matrix as defined above, then Lucas matrices hold following properties:

(i) $L_3^{(n)} = Q_3^n L_3^{(0)}$.

(ii) $L_3^{(n)} L_3^{(-n)} = (L_3^{(0)})^2$.

(iii) $L_3^{(m)} L_3^{(n)} = L_3^{(m+n)} L_3^{(0)} = L_3^{(n)} L_3^{(m)}$, for $m, n \in \mathbb{Z}$.

(iv) $det(L_3^{(n)}) = det(L_3^{(0)})$.

(v) $(L_3^{(n)})^{-1} = (L_3^{(0)})^{-2} L_3^{(-n)}$.

3 Generalization of Sequence of Traces

Let Q_k^n be generalized Fibonacci matrix (in short, GFM) of order k corresponding to generalized Fibonacci sequence $\{f_{k,n}\}$, which is given as follows [10]:

$$f_{k,k+n} = f_{k,k+n-1} + f_{k,k+n-2} + f_{k,k+n-3} + ... + f_{k,n+1} + f_{k,n}, \quad k \in \mathbb{Z}^+ \tag{3.1}$$

with initial values

$$f_{k,0} = f_{k,1} = f_{k,2} = ... = f_{k,k-2} = 0 \text{ and } f_{k,k-1} = 1 \tag{3.2}$$

and

$$Q_k^n = \begin{bmatrix} f_{k,k+n-1} & f_{k,k+n-2} + f_{k,k+n-3} + ... + f_{k,n} & f_{k,k+n-2} + ... + f_{k,n+1} & \cdots & f_{k,k+n-2} \\ f_{k,k+n-2} & f_{k,k+n-3} + f_{k,k+n-4} + ... + f_{k,n-1} & f_{k,k+n-3} + ... + f_{k,n} & \cdots & f_{k,k+n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{k,k+n-(k-1)} & f_{k,n} + f_{k,n-1} + ... + f_{k,-k+n+2} & f_{k,n} + ... + f_{k,-k+n+3} & \cdots & f_{k,k+n-k} \\ f_{k,k+n-k} & f_{k,n-1} + f_{k,n-2} + ... + f_{k,-k+n+1} & f_{k,n-1} + ... + f_{k,-k+n+2} & \cdots & f_{k,n-1} \end{bmatrix}$$

Now we define a new generalized sequence$\{l_{k,n}\}$ of order k analogous to generalized Fibonacci sequence, which we call generalized Lucas sequence (GLS(k)) obtained from the sequence of $traces$ of GFM(Q_k^n) i.e $\{l_{k,n}\} = trace(Q_k^n)$. The initial value for sequence $\{l_{k,n}\}$ can be obtained by Eqn. (3.2).

Theorem 3.1. *The sequence* $\{l_{k,n}\}$ *forms a* k^{th} *order recurrence relation as follows:*

$$l_{k,k+n} = l_{k,k+n-1} + l_{k,k+n-2} + l_{k,k+n-3} + ... + l_{k,n+2} + l_{k,n+1} + l_{k,n} \quad n \geq 0, k \in \mathbb{Z}^+, \tag{3.3}$$

with initial values $l_{k,0} = k$ *and* $l_{k,r} = 2^r - 1$, $1 \leq r < k$.

Proof. Since, sequence of traces(Q_k^n) is given by

$$\begin{aligned} trace(Q_k^n) &= f_{k,k+n-1} + \sum_{i=n-1}^{k+n-3} f_{k,i} + \sum_{i=n-1}^{k+n-4} f_{k,i} + ... + \sum_{i=n-1}^{n-2} f_{k,i} + \sum_{i=n-1}^{n-1} f_{k,i} \\ l_{k,n} &= f_{k,k+n-1} + 1f_{k,k+n-3} + 2f_{k,k+n-4} + 3f_{k,k+n-5} + ... + (k-3)f_{k,n+1} \\ &\quad +(k-2)f_{k,n} + (k-1)f_{k,n-1} \end{aligned}$$

$$l_{k,n} = \begin{bmatrix} 1 & 1 & 2 & 3 & ... & (k-3) & (k-2) & (k-1) \end{bmatrix} \begin{bmatrix} f_{k,k+n-1} \\ f_{k,k+n-3} \\ f_{k,k+n-4} \\ f_{k,k+n-5} \\ \vdots \\ f_{k,n} \\ f_{k,n-1} \end{bmatrix}_{k \times 1} \tag{3.4}$$

Now, using Eqn. (3.4) in the RHS of Eqn. (3.3), we get

$$= \begin{bmatrix}1 & 1 & 2 & 3 & \dots & (k-1)\end{bmatrix} \begin{bmatrix} f_{k,2k+n-2} + f_{k,2k+n-3} + \dots + f_{k,k+n} + f_{k,k+n-1} \\ f_{k,2k+n-4} + f_{k,2k+n-5} + \dots + f_{k,k+n-2} + f_{k,k+n-3} \\ f_{k,2k+n-5} + f_{k,2k+n-6} + \dots + f_{k,k+n-3} + f_{k,k+n-4} \\ f_{k,2k+n-6} + f_{k,2k+n-7} + \dots + f_{k,k+n-4} + f_{k,k+n-5} \\ \vdots \\ f_{k,k+n-1} + f_{k,k+n-2} + \dots + f_{k,n+1} + f_{k,n} \\ f_{k,k+n-2} + f_{k,k+n-3} + \dots + f_{k,n} + f_{k,n-1} \end{bmatrix}_{k\times 1}$$

$$= \begin{bmatrix}1 & 1 & 2 & 3 & \dots & (k-1)\end{bmatrix} \begin{bmatrix} f_{k,2k+n-1} \\ f_{k,2k+n-3} \\ f_{k,2k+n-4} \\ f_{k,2k+n-5} \\ \vdots \\ f_{k,k+n} \\ f_{k,k+n-1} \end{bmatrix} \quad \text{(using Eqn. (3.1))}$$

$$= l_{k,k+n} \tag{3.5}$$

Thus, in particular for $n = 0$ to $k-1$, $\{l_{k,n}\}$ is

$$k, 1, 3, 7, 15, 31, 63, 127, 255, 511, \dots, 2^{k-1} - 1 \tag{3.6}$$

which is the initial value for GLS(k) obtained with the help of initial values of generalized Fibonacci sequence given in Eqn. (3.2). □

Thus, we have the following definition for the generalized Lucas sequences $\{l_{k,n}\}$.

Definition 3.2. For $k \in \mathbb{N}$, the generalized Lucas sequence $\{l_{k,n}\}$ of order k is given by

$$l_{k,k+n} = l_{k,k+n-1} + l_{k,k+n-2} + l_{k,k+n-3} + \dots + l_{k,n+2} + l_{k,n+1} + l_{k,n}; \quad n \geq 0, k(\geq 2) \in \mathbb{N},$$

where $l_{k,r} = trace(Q_k^r), 0 \leq r < k$.

The first few terms of two ended sequence $\{l_{k,n}\}$ are

$$\dots, -1, k, 1, 3, 7, 15, 31, 63, 127, 255, \dots, 2^{k-1} - 1, 2^k - 1, 2^{k+1} - 2 - k, 2^{k+2} - 3 - 2k, \dots.$$

The following table show the list of initial values for sequence of traces.

Order	$l_{k,0}$	$l_{k,1}$	$l_{k,2}$	$l_{k,3}$	$l_{k,4}$	$l_{k,5}$	$l_{k,6}$	$l_{k,7}$	$l_{k,8}$	$l_{k,9}$	$l_{k,10}$
$k=2$	2	1									
$k=3$	3	1	3								
$k=4$	4	1	3	7							
$k=5$	5	1	3	7	15						
$k=6$	6	1	3	7	15	31					
$k=7$	7	1	3	7	15	31	63				
$k=8$	8	1	3	7	15	31	63	127			
$k=9$	9	1	3	7	15	31	63	127	255		
$k=10$	10	1	3	7	15	31	63	127	255	511	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Table 1. List of initial values for sequence of traces

3.1 Hanoi Tower Sequence as Initial Values of GLS(k)

In this subsection, we are establishing the relationship between Hanoi Tower sequences and initial values for GLS(k) and achieved that initial values of GLS(k) follow the Hanoi Tower sequence.

Theorem 3.3. *Let $A_k = (a_{ij})_{k\times k}$ be a square matrix of order k and Q_k^1 be first Fibonacci matrix of same order then trace of $Q_k^1 A_k$ is given by*

$$trace(Q_k^1 A_k) = \sum_{i=1}^{k} a_{i1} + \sum superdiagonal(A_k). \tag{3.7}$$

Proof. Let us assume that $Q_k^1 A_k = B$, thus trace of B can be obtained by taking sum of diagonal elements of B. Here

$$B = Q_k^1 A_k = \begin{bmatrix} 1 & 1 & 1 & ... & 1 & 1 \\ 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & 0 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & ... & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1k-1} & a_{1k} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2k-1} & a_{2k} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3k-1} & a_{3k} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{k-11} & a_{k-12} & a_{k-13} & \cdots & a_{k-1k-1} & a_{k-1k} \\ a_{k1} & a_{k2} & a_{k3} & \cdots & a_{kk-1} & a_{kk} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{k} a_{i1} & \sum_{i=1}^{k} a_{i2} & \sum_{i=1}^{k} a_{i3} & \cdots & \sum_{i=1}^{k} a_{ik-1} & \sum_{i=1}^{k} a_{ik} \\ a_{11} & a_{12} & a_{13} & \cdots & a_{1k-1} & a_{1k} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2k-1} & a_{2k} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3k-1} & a_{3k} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{k-11} & a_{k-12} & a_{k-13} & \cdots & a_{k-1k-1} & a_{k-1k} \end{bmatrix}.$$

Here on multiplication with Q_k^1 to any matrix A it gives B where row-1 to row-(k-1) of A becomes row-2 to row-(k) of B and element b_{11} of B is the sum of first column of A. Thus

$$\begin{aligned} trace(B) &= \sum_{i=1}^{k} a_{i1} + a_{12} + a_{23} + a_{34} + ... + a_{k-1k} \\ &= \sum_{i=1}^{k} a_{i1} + \sum superdiagonal(A_k) \end{aligned}$$

as required. □

Corollary 3.4. *On multiplication with Q_k^1 to any matrix A of same size, row-2 to row-(k) of the resultant matrix B have row-1 to row-(k-1) of A and first row will be the sum of corresponding columns of A.*

Observation 2. Let Q_k^1 be first Fibonacci matrix, then

$$q_{i,i} = q_{i,i+1}; \qquad i = 1, 2, ..., k-1$$

Observation 3. For $1 \leq r < k$, we have $\sum superdiagonal(Q_k^r) = trace(Q_k^r)$.

Further, we are interested to establish sequence traces of Q_k^r matrix for which we are observing the pattern of Q_k^r matrix. We know that elements of a Fibonacci matrix Q_k^r for $2 \leq r < k$ is distributed as $\left[\begin{array}{c|c} M_{r\times k} & \\ \hline I_{k-r} & O \end{array}\right]_{k\times k}$ where I_{k-r} is identity matrix situated at left bottom corner, O is zero matrix and entries of M matrix is given by $m_{ij} = 2^{r-i};\ 1 \leq j \leq k-(r-i)$ with $i \leq r$. It has been noted that in matrix $M_{r\times k}$, $m_{ij} = m_{ii} \quad for\ 1 \leq j \leq i, i \leq r$ so we can say $\sum_{i=1}^{r} m_{ii} = trace(Q_k^{r-1}) + 1$ (because rest diagonal entries of Q_k^r are 0).

Theorem 3.5. *Let* $2 \leq r < k$ *and* Q_k^r *be Fibonacci matrix of order k, then*

$$trace(Q_k^r) = 2trace(Q_k^{r-1}) + 1.$$

Proof. Here Q_k^r means multiplication of Q_k^{r-1} with Q_k^1 and structure of Q_k^1 is that every entry of first row is 1 and other entries are 0 except 1 at sub-diagonal. i.e

$$Q_k^1 = (q_{i,j}), \quad \text{where } q_{i,j} = \begin{cases} 1 & : if\ i = 1 \text{ (first row)}, \\ 1 & : if\ j = i-1, i = 2, 3, ..., k, \\ 0 & : \text{otherwise.} \end{cases}$$

Since, form Observation 3, $m_{ij} = m_{ii} \quad for\ 1 \leq j \leq i, i \leq r$, so

$$\begin{aligned} (1,1) - entry\ of\ Q_k^r &= \sum fist\ column(Q_k^{r-1}) \\ &= \sum fist\ column\ of\ M\ +\ 1 \\ &= trace(Q_k^{r-1}) + 1. \end{aligned} \tag{3.8}$$

Now, since (1,1)-entry of $Q_k^r = trace(Q_k^{r-1}) + 1$. So from Theorem 3.3, we have

$$\begin{aligned} trace(Q_k^r) &= (trace(Q_k^{r-1}) + 1) + \sum Superdiagonal(Q_k^{r-1}) \\ &= trace(Q_k^{r-1}) + 1 + trace(Q_k^{r-1}) \\ &= 2trace(Q_k^{r-1}) + 1. \end{aligned}$$

This completes the proof. □

Observation 4. Initial values of GLS(k) are the solution of Tower of Hanoi problems, so for initial values of generalized Lucas sequence, we do not need to start from Fibonacci sequences. It can be directly obtained by any one of the following:

(i) $l_{k,r} = 2l_{k,r-1} + 1, r \geq 2;\ l_{k,1} = 1.$

(ii) $l_{k,r} = \sum_{i=0}^{r-1} 2^i.$

(iii) $l_{k,r} = 2^r - 1.$

In particular for $k = 2$, one can see book [6] that is discussed in Eqn. (1.3) and for $k = 3$ it has been obtained in Corollary 2.8(i) given as

$$\begin{aligned} l_{2,n} = trace(Q_2^n) &= f_{2,n+1} + f_{2,n-1}, \\ l_{3,n} = trace(Q_3^n) &= f_{3,n+2} + f_{3,n} + 2f_{3,n-1}. \end{aligned}$$

Thus proposed generalized recursive matrix becomes

$$L_k^{(n)} = \begin{bmatrix} l_{k+n-1} & l_{k+n-2} + l_{k+n-3} + ... + l_n & l_{k+n-2} + ... + l_{n+1} & \cdots & l_{k+n-2} \\ l_{k+n-2} & l_{k+n-3} + l_{k+n-4} + ... + l_{n-1} & l_{k+n-3} + ... + l_n & \cdots & l_{k+n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{k+n-(k-1)} & l_n + l_{n-1} + ... + l_{-k+n+2} & l_n + ... + l_{-k+n+3} & \cdots & l_{k+n-k} \\ l_{k+n-k} & l_{n-1} + l_{n-2} + ... + l_{-k+n+1} & l_{n-1} + ... + l_{-k+n+2} & \cdots & l_{n-1} \end{bmatrix}$$

where initial matrix $L_k^{(0)}$ is

$$\begin{bmatrix} 2^{k-1}-1 & 2^{k-1} & 2^{k-1}-k & \dots & 7.2^{k-4} & 3.2^{k-3} & 2^{k-2}-1 \\ 2^{k-2}-1 & 2^{k-2} & 2^{k-2}+1 & \dots & 7.2^{k-5} & 3.2^{k-4} & 2^{k-3}-1 \\ 2^{k-3}-1 & 2^{k-3} & 2^{k-3}+1 & \dots & 7.2^{k-6} & 3.2^{k-5} & 2^{k-4}-1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \\ 1 & 2 & 3 & \dots & k-2 & k-1 & k \\ k & 1-k & 2-k & \dots & -3 & -2 & -1 \end{bmatrix}.$$

4 Conclusion

In summary, we have proposed a generalized Fibonacci trace sequences ($l_{k,n} = trace(Q_k^n)$) as well as the associated recursive matrix. We have obtained some new identities for Lucas sequences of order two ($k = 2$) and three ($k = 3$) and established connection with Fibonacci and Tribonacci sequences. Moreover, we have obtained a formula for the initial values of generalized Fibonacci trace sequences as solution of Hanoi Tower sequence preserving the existing properties and observed that initial values can be evaluated independently.

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