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ON THE DISJOINT WEAK BANACH-SAKS OPERATORS

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Abstract We introduce and study a new class of operators that we call disjoint weak Banach-Saks operators. We establish some characterizations of this class of operators by different types of convergence (norm convergence, unbounded order convergence, unbounded norm convergence and unbounded absolute weak convergence) as well as by the positive weakly null sequences. Consequently, we give a new characterization of the disjoint weak Banach-Saks property by the positive disjoint weakly null sequences. Furthermore, we study the relationship between this class and other classes of operators.

1 Introduction

In [7], Y. Deng et al. introduced the disjoint weak Banach-Saks property (abb. DWBSP) in the Banach lattice. Recall that a Banach lattice E has the disjoint weak Banach-Saks property, if every disjoint weakly null sequence in E has a subsequence whose Cesàro sequence is norm convergent in E, as examples of such Banach lattices we have ℓ_1 and $L_p(c_0)$ (see [7]).

In this paper, we introduce the so-called disjoint weak Banach-Saks operator from a Banach lattice E into a Banach space X. Our definition is based on the disjoint weak Banach-Saks property. Mainly, in the Proposition 3.10 we establish some characterizations of this class of operators by different types of convergence and in the Proposition 3.12 we give a characterization of this class of operators by positive weakly null sequences. Consequently, we give a new characterization of the disjoint weak Banach-Saks property. Also, we give a generalization of Proposition 6.9 [9] and of Proposition 6.15 [9]. Furthermore, we study the relationship between this class of operators and that of weak Banach-Saks operators (resp. almost Banach-Saks operators, order weakly compact operators and weakly compact operators).

2 Preliminaries and Notations

To state our results, we need to fix some notations and recall some definitions. A Banach lattice is a Banach space $(E, ||.||)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm, is also a Banach lattice. A Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, the sequence (x_α) converges to 0 for the norm $\|.\|$, where the notation $x_\alpha \downarrow 0$ means that the sequence (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. In a vector lattice E, two elements x and y are said to be disjoint (in symbols $x \perp y$) whenever $|x| \wedge |y| = 0$ holds. For each $x, y \in E$ with $x \leq y$, the set $[x, y] = \{z \in E : x \leq z \leq y\}$ is called an order interval. A subset of E is said to be order bounded if it is included in some order interval. Recall that a nonzero element x of a vector lattice E is discrete if the order ideal generated by x equals the subspace generated by x. The vector lattice E is discrete, if it admits a complete disjoint system of discrete elements. In a Banach lattice every order bounded disjoint sequence converges weakly to zero (see [1]).

An order continuous Banach lattice E is said to have the subsequence splitting property if for any norm bounded sequence (x_n) there exist a subsequence (x_{n_k}) of (x_n) and two sequences (y_k) and (z_k) such that $x_{n_k} = y_k + z_k$, (y_k) is almost order bounded, (z_k) is pairwise disjoint and $y_k \perp z_k$ for all k (see [9]). A vector lattice E is said to be $\sigma-$laterally complete, if the supremum of every disjoint sequence of E^+ exists in E. A Banach lattice is said to have weakly sequentially continuous lattice operations whenever $x_n \xrightarrow{w} 0$ implies $|x_n| \xrightarrow{w} 0$. A Banach space is said to have the Schur property, whenever every weakly convergent sequence is norm convergent. A Banach space X has the weak Banach-Saks property if every weakly null sequence (x_n) in X has a subsequence (x_{n_k}) whose Cesàro sequence $(\frac{1}{m}\sum_{k=1}^{m} x_{n_k})$ is norm convergent to zero.

A subset A of a Banach lattice E is said to be almost order bounded if for any $\varepsilon > 0$ there exists $u \in E^+$ such that $A \subset [-u, u] + \varepsilon B_E$. We know from [10] that $A \subset [-u, u] + \varepsilon B_E$ if and only if $\sup_{x\in A} \|(|x| - u)^+\| \leq \varepsilon$ if and only if $\sup_{x\in A} \||x| - |u| \wedge |x|\| \leq \varepsilon$. Note that every norm convergent sequence is almost order bounded.

A net (x_α) of a vector lattice E is said to be uo-converge (abb. uo-converge) to x if $(|x_\alpha - x| \wedge u)$ converges in order to zero for every $u \in E^+$; we write $x_\alpha \xrightarrow{uo} x$. We mention that order convergence implies uo-convergence and they coincide for order bounded nets. We note that every disjoint net is uo-null (see [9]).

A net (x_α) of a Banach lattice E is said to be unbounded norm convergent (abb. un-convergent) to x if $(|x_\alpha - x| \wedge u)$ converges to zero for every $u \in E^+$; we write $x_\alpha \xrightarrow{un} x$. The example 2.6 of [7] shows that a disjoint sequence need not to be un-null.

A net (x_α) of a Banach lattice E is said to be unbounded absolute weakly convergent (abb. uaw-convergent) to x if $(|x_\alpha - x| \wedge u)$ converges weakly to zero for every $u \in E^+$; we write $x_\alpha \xrightarrow{uaw} x$. Every absolute weakly convergent net is uaw-convergent. But the converse is not true in general. The absolute weakly convergence coincides with uaw-convergence for the order bounded nets (see [15]). We note that every disjoint net is uaw-null (see [15, Lemma 2]).

We will use the term operator $T : E \longrightarrow F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. The operator T is regular if $T = T_1 - T_2$ where T_1 and T_2 are positive operators from E into F. Note that each positive linear mapping on a Banach lattice is continuous. If an operator $T : E \longrightarrow F$ between two Banach lattices is positive, then its adjoint $T' : F' \longrightarrow E'$ is likewise positive, where T' is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$.

An operator $T : E \longrightarrow F$ between two Riesz spaces is said to preserve disjointness whenever $x \perp y$ in E implies $Tx \perp Ty$ in F (see [1]).

A positive linear mapping S is disjointness preserving if and only if S is a Riesz homomorphism (see [6]).

An operator $T : E \longrightarrow Y$ is called M-weakly compact, if $(T(x_n))$ is norm-null for every bounded disjoint sequence (x_n) in E (see [1]).

An operator T from a Banach lattice E into a Banach space X is said to be almost disjoint Banach-Saks, if for every bounded disjoint sequence (x_n) of E, $(T(x_n))$ has a Cesàro convergent subsequence in X (see [11]).

An operator $T : E \longrightarrow Y$ from a Banach lattice to a Banach space is said to be order weakly compact if for every order bounded disjoint sequence (x_n) of E, we have $\|T(x_n)\| \longrightarrow 0$ ([1, Theorem 5.57 (Dodds)]).

An operator $T : X \longrightarrow Y$ between two Banach spaces is said to be weakly compact, if for every norm bounded sequence (x_n) of X the sequence $(T(x_n))$ has a weakly convergent subsequence in Y (see [1]).

In the rest of this paper X, Y will denote Banach spaces, and E, F will denote Banach lattices.

3 Main results

We start by the following definition.

Definition 3.1. An operator $T : E \longrightarrow Y$ is said to be disjoint weak Banach-Saks (abb; DWBS), if for each disjoint weakly null sequence (x_n) of E, $(T(x_n))$ has a subsequence whose Cesàro sequence is norm convergent in Y.

It is clear that E has the disjoint weak Banach-Saks property if, and only if, the identity operator Id_E of E is disjoint weak Banach-Saks. Also, we observe that this class of operators contains M-weakly compact operators.

We have the following characterisation of the positive disjoint weak Banach-Saks operator.

Theorem 3.2. *Let $T : E \longrightarrow F$ be a positive operator. The following assertions are equivalent:*

(1) T is a disjoint weak Banach-Saks operator.

(2) For each disjoint weakly null sequence $(x_n) \subset E^+$, $(T(x_n))$ has a subsequence whose Cesàro sequence is norm convergent in F.

Proof. $(1) \Longrightarrow (2)$ Is evident.
$(2) \Longrightarrow (1)$ Let (x_n) be a disjoint weakly null sequence of E. It follows from [14, Remark 1] that $(|x_n|)$ is a disjoint sequence of E^+ such that $|x_n| \longrightarrow 0$ for the topology $\sigma(E, E')$. From the assertion (2), there exists $(|x_{n_k}|)$ which is a subsequence of $(|x_n|)$ such that $(T(|x_{n_k}|))$ is Cesàro norm convergent. Now, by the inequality:

$$|T(x_n)| \leq T(|x_n|) \text{ for each n,}$$

we obtain

$$|\frac{1}{m}\sum_{k=1}^{m} T(x_{n_k})| \leq \frac{1}{m}\sum_{k=1}^{m} |T(x_{n_k})| \leq \frac{1}{m}\sum_{k=1}^{m} T(|x_{n_k}|)$$

and since $\|\frac{1}{m}\sum_{k=1}^{m} T(|x_{n_k}|)\| \longrightarrow 0$, then $\|\frac{1}{m}\sum_{k=1}^{m} T(x_{n_k})\| \longrightarrow 0$. Thus, $(T(x_n))$ has a Cesàro norm convergent subsequence in F, and hence T is a positive disjoint weak Banach-Saks operator. □

As a consequence of the last Theorem, we obtain the following characterization of the disjoint weak Banach-Saks property.

Corollary 3.3. *The following assertions are equivalent:*

(1) E has the disjoint weak Banach-Saks property.

(2) Every disjoint weakly null sequence $(x_n) \subset E^+$ has a subsequence whose Cesàro sequence is norm convergent.

If E is order continuous, we obtain the following characterizations of positive disjoint weak Banach-Saks operator.

Proposition 3.4. *Let $T : E \longrightarrow F$ be a positive operator such that E is order continuous. The following assertions are equivalent:*

(1) T is a disjoint weak Banach-Saks operator.

(2) For each weakly null and uo-null sequence $(x_n) \subset E^+$, $(T(x_n))$ has a subsequence whose Cesàro sequence is norm convergent in F.

(3) For each weakly null sequence $(x_n) \subset E^+$, $(T(x_n))$ has a subsequence whose Cesàro sequence is norm convergent in F.

Proof. $(1) \Longrightarrow (2)$ Let (x_n) be a weakly null and uo-null sequence of E^+. Since E is order continuous, then by [9, Corollary 3.6] there exist a subsequence (x_{n_k}) of (x_n) and a disjoint sequence (d_k) of E such that $\|x_{n_k} - d_k\| \longrightarrow 0$. As (x_n) is a weakly null sequence of E, then it follows from the proof of [7, Theorem 3.2] that (d_k) is a weakly null sequence of E. Thus, every subsequence of (d_k) is disjoint and weakly null. Hence, the assertion (1) yields that $(T(d_k))$ has a subsequence whose Cesàro sequence is norm convergent in F.

$(2) \Longrightarrow (1)$ It follows from the fact that every disjoint sequence is uo-null ([9, Corollary 3.6]).

$(2) \Longrightarrow (3)$ Let (x_n) be a weakly null sequence of E^+. Since E is order continuous, then it follows from [9, Proposition 4.5] and [9, Proposition 4.7] that $x_{n_k} \xrightarrow{uo} 0$ for some subsequence (x_{n_k}). Hence, the assertion (2) yields that $(T(x_{n_k}))$ has a subsequence whose Cesàro sequence is norm convergent in F.

$(3) \Longrightarrow (2)$ It is Obvious. □

Proposition 3.5. *We have the following assertions:*

(1) If $T : E \longrightarrow X$ is a disjoint weak Banach-Saks operator, then for each operator $S : X \longrightarrow Y$ the composed operator $S \circ T$ is disjoint weak Banach-Saks.

(2) If $T : F \longrightarrow Y$ is a disjoint weak Banach-Saks operator and $S : E \longrightarrow F$ is a disjointness preserving operator, then the composed operator $T \circ S$ is disjoint weak Banach-Saks.

Proof. (1) Let (x_n) be a disjoint weakly null sequence of E. Since T is a disjoint weakly Banach-Saks operator, then there exists (x_{n_k}) which is a subsequence of (x_n) such that the Cesàro sequence of $(T(x_{n_k}))$ converges to 0. As

$$\|\frac{1}{m}\sum_{k=1}^{m} S \circ T(x_{n_k})\| \leq \|S\|\|\frac{1}{m}\sum_{k=1}^{m} T(x_{n_k})\|,$$

then the Cesàro sequence of $(S \circ T(x_{n_k}))$ converges to 0. Thus $S \circ T$ is a disjoint weak Banach-Saks operator.

(2) Let (x_n) be a disjoint weakly null sequence of E. Since S is a disjointness preserving operator, then $(S(x_n))$ is a disjoint sequence, and hence $(S(x_n))$ is a disjoint weakly null sequence in F. As T is a disjoint weak Banach-Saks operator, then $(T \circ S(x_n))$ is a Cesàro norm convergent subsequence in Y. Therefore, $T \circ S$ is a disjoint weak Banach-Saks operator.

□

As consequences of the above proposition, we obtain the following results:

Corollary 3.6. *The following statements are equivalent:*

(1) For each Y, every operator $T : E \longrightarrow Y$ is disjoint weak Banach-Saks.

(2) E has the disjoint weak Banach-Saks property.

Corollary 3.7. *The following statements are equivalent:*

(1) For each E, every disjointness preserving operator $T : E \longrightarrow F$ is disjoint weak Banach-Saks.

(2) F has the disjoint weak Banach-Saks property.

In the following result, we prove that the class of disjoint weak Banach-Saks operators satisfies the domination problem.

Theorem 3.8. *Let $S, T : E \longrightarrow F$ be two operators with $0 \leq S \leq T$. If T is a disjoint weak Banach-Saks operator, then S is also a disjoint weak Banach-Saks operator.*

Proof. Let $S, T : E \longrightarrow F$ be two operators such that $0 \leq S \leq T : E \longrightarrow F$. We suppose that T is a disjoint weak Banach-Saks operator. Let (x_n) be a disjoint weakly null sequence of E, it follows from [14, Remark 1] that $(|x_n|)$ is a disjoint weakly null sequence. Since T is disjoint weak Banach-Saks, then there exists $(|x_{n_k}|)$ a subsequence of $(|x_n|)$ such that $(T(|x_{n_k}|))$ is Cesàro norm convergent. On the other hand, we have $0 \leq S \leq T$ implies that

$$|S(x_n)| \leq S(|x_n|) \leq T(|x_n|),$$

this shows that $$\|\frac{1}{m}\sum_{k=1}^{m} S(x_{n_k})\| \leq \|\frac{1}{m}\sum_{k=1}^{m} |S(x_{n_k})|\| \leq \|\frac{1}{m}\sum_{k=1}^{m} T(|x_{n_k}|)\|,$$

and hence $(S(x_{n_k}))$ is Cesàro norm convergent. Therefore, S is a disjoint weak Banach-Saks operator. □

We note that there exist Banach lattices E and F and an operator $T : E \longrightarrow F$ which is disjoint weak Banach-Saks and such that its modulus $|T|$ does not exist. We consider from [3] the operator $T : L_1[0, 1] \longrightarrow c_0$ defined by:

$$\begin{array}{rlcl} T : & L_1[0,1] & \longrightarrow & c_0 \\ & f & \longmapsto & (\int_0^1 f(x) r_1 dx, \int_0^1 f(x) r_2 dx, ...) \end{array}$$

where (r_n) is the sequence of Rademacher functions on $[0,1]$. It is clear that T is disjoint weak Banach-Saks $\big(L_1[0,1]$ has the disjoint weak Banach-Saks property$\big)$, but from the remark page 361 of [3] the modulus of T does not exist.

Proposition 3.9. *Let $T : E \longrightarrow F$ be an order bounded disjointness preserving operator such that F is Dedekind-complete. Then, T is disjoint weak Banach-Saks if, and only if, $|T|$ is disjoint weak Banach-Saks.*

Proof. Let $T : E \longrightarrow F$ be an order bounded disjointness preserving operator. By [5, Theorem 2.2], we have

$$|T|(|x|) = |T(x)| = ||T|(x)| \ \ \forall x \in E$$

and hence

$$\|\frac{1}{m}\sum_{k=1}^{m}|T|(x_{n_k})\| = \||T|(\frac{1}{m}\sum_{k=1}^{m}x_{n_k})\| = \||T(\frac{1}{m}\sum_{k=1}^{m}x_{n_k})|\| = \|\frac{1}{m}\sum_{k=1}^{m}T(x_{n_k})\|, \text{ for each } (x_n) \text{ of } E;$$

as desired. □

Proposition 3.10. *If E is order continuous, then for every operator $T : E \longrightarrow Y$ the following assertions are equivalent:*

(1) T is disjoint weak Banach-Saks.

(2) For each weakly null sequence (x_n) of E such that $x_n \xrightarrow{un} 0$, $(T(x_n))$ has a subsequence whose Cesàro sequence is norm convergent in Y.

(3) For each weakly null sequence (x_n) of E such that $x_n \xrightarrow{uaw} 0$, $(T(x_n))$ has a subsequence whose Cesàro sequence is norm convergent in Y.

(4) For each weakly null sequence (x_n) of E such that $x_n \xrightarrow{uo} 0$, $(T(x_n))$ has a subsequence whose Cesàro sequence is norm convergent in Y.

Proof. $(1) \Longrightarrow (2)$ Let (x_n) be a weakly null sequence of E such that $x_n \xrightarrow{un} 0$. It follows from [7, Theorem 3.2] that there exist (x_{n_k}) a subsequence of (x_n) and a disjoint sequence (d_k) of E such that $\|x_{n_k} - d_k\| \longrightarrow 0$. So, $\|T(x_{n_k}) - T(d_k)\| \longrightarrow 0$ in Y. Since (x_n) is a weakly null sequence of E, then it follows from the proof of [7, Theorem 3.2] that (d_k) is a weakly null sequence of E. As T is a disjoint weak Banach-Saks operator, the assumption (1) yields that $(T(d_k))$ has a subsequence whose Cesàro sequence is norm convergent in Y. Hence, $(T(x_{n_k}))$ has a subsequence whose Cesàro sequence is norm convergent in Y.

$(2) \Longrightarrow (3)$ Let (x_n) be a weakly null sequence of E such that $x_n \xrightarrow{uaw} 0$. Since E is order continuous, then it follows from [15, Theorem 4] that $x_n \xrightarrow{un} 0$. Therefore, the assertion (2) yields that $(T(x_n))$ has a subsequence whose Cesàro sequence is norm convergent in Y.

$(3) \Longrightarrow (4)$ Let (x_n) be a weakly null sequence of E such that $x_n \xrightarrow{uo} 0$. Since E is order continuous, then it follows from [7, Propostion 2.5] and [15, Theorem 4] that $x_n \xrightarrow{uaw} 0$. Hence, by the assertion (3), we infer that $(T(x_n))$ has a subsequence whose Cesàro sequence is norm convergent in Y.

$(4) \Longrightarrow (1)$ Let (x_n) be a disjoint weakly null sequence of E. By [9, Corollary 3.6], we have that $x_n \xrightarrow{uo} 0$ and hence it follows from the assertion (4) that $(T(x_n))$ has a subsequence whose Cesàro sequence is norm convergent in Y. Thus, T is a disjoint weak Banach-Saks operator. □

Proposition 3.11. *If F is order continuous, then for every operator $T : E \longrightarrow F$ the following assertions are equivalent:*

(1) T is disjoint weak Banach-Saks.

(2) For each disjoint weakly null sequence (x_n) of E, $(T(x_n))$ has a subsequence whose Cesàro sequence is almost order bounded.

(3) For each disjoint weakly null sequence (x_n) of E^+, $(T(x_n))$ has a subsequence whose Cesàro sequence is almost order bounded.

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) Are obvious.

(3) $\Longrightarrow$ (2) Let (x_n) be a disjoint weakly null sequence of E. Then, it follows from Wnuk [14, Remark 1] that $(|x_n|)$ is a disjoint sequence of E^+ such that $|x_n| \xrightarrow{w} 0$. We have $x_n = x_n^+ - x_n^-$ and $|x_n| = x_n^+ + x_n^-$. It is easy from the inequality $0 \leq x_n^+ \leq |x_n|$ (resp, $0 \leq x_n^- \leq |x_n|$), to prove that (x_n^+) (resp, (x_n^-)) is a disjoint weakly null sequence of E^+. So, every subsequence of (x_n^+) is disjoint weakly null in E^+. The assumption (3) yields that every subsequence of $(T(x_n^+))$ has a further subsequence whose Cesàro sequence is almost order bounded in F. Since F is order continuous, then by [9, Lemma 6.3] there exist a subsequence $(T(x^+_{\phi(n)}))$ of $(T(x_n^+))$ and a vector $y \in F$ such that the Cesàro sequence of any subsequence of $(T(x^+_{\phi(n)}))$ converges uo and in norm to $y = 0$. As a result, the Cesàro sequence of any subsequence of $(T(x^+_{\phi(n)}))$ is almost order bounded in F. On the other hand, $(x^-_{\phi(n)})$ is a disjoint weakly null sequence of E^+. Then, it follows from the assertion (3) that the sequence $(T(x^-_{\phi(n)}))$ has a subsequence $(T(x^-_{\psi(\phi(n))}))$ whose Cesàro sequence is almost order bounded in F. As it was shown above, the Cesàro sequence of any subsequence of $(T(x^+_{\phi(n)}))$ is almost order bounded in F; hence the Cesàro sequence of $(T(x^+_{\psi(\phi(n))}))$ is almost order bounded. Therefore, the Cesàro sequence of the subsequence $T(x_{\psi(\phi(n))}) = T(x^+_{\psi(\phi(n))}) - T(x^-_{\psi(\phi(n))})$ of the sequence $(T(x_n))$ is almost order bounded in F, where ϕ and ψ are increasing mappings from $\mathbb{N}$ into $\mathbb{N}$.

(2) $\Longrightarrow$ (1) Let (x_n) be a disjoint weakly null sequence of E. Since any subsequence (x_{n_k}) of (x_n) is also a disjoint weakly null sequence of E, the assertion (2) yields that $(T(x_{n_k}))$ has a subsequence whose Cesàro sequence is almost order bounded. In particular, any subsequence of $(T(x_n))$ has a further subsequence whose Cesàro sequence is almost order bounded. As F is order continuous, then it follows from [9, Lemma 6.3] that there exist a subsequence $(T(x_{n_k}))$ of $(T(x_n))$ and a vector $y \in F$ such that the Cesàro sequence of any subsequence of $(T(x_{n_k}))$ converges uo- and in norm to $y = 0$.

□

Whenever E and F are order continuous, we obtain the following characterizations.

Proposition 3.12. *Let E and F be order continuous. For an operator $T : E \longrightarrow F$, the following assertions are equivalent:*

(1) T is disjoint weak Banach-Saks.

(2) For each weakly and uo-null sequence (x_n) of E^+, $(T(x_n))$ has a subsequence whose Cesàro sequence is almost order bounded in F.

(3) For each weakly null sequence (x_n) of E^+, $(T(x_n))$ has a subsequence whose Cesàro sequence is almost order bounded in F.

Proof. (1) $\Longrightarrow$ (2) Let (x_n) be a weakly null and uo-null sequence of E^+. Since E is order continuous, then by [9, Lemma 6.7] there exist a subsequence (x_{n_k}) of (x_n) and a disjoint sequence (d_k) of E such that $\|x_{n_k} - d_k\| \longrightarrow 0$. Since (x_n) is a weakly null sequence of E, then it follows from the proof [7, Theorem 3.2] that (d_k) is a weakly null sequence of E. Hence, every subsequence of (d_k) is disjoint and weakly null. As T is disjoint weak Banach-Saks, then it follows from the Proposition 3.11 that every subsequence of $(T(d_k))$ has a further subsequence whose Cesàro sequence is almost order bounded in F. Since F is order continuous, then it follows from [9, Lemma 6.3] that $(T(d_k))$ has a subsequence whose Cesàro sequence is norm convergent in F. Therefore, $(T(x_{n_k}))$ has a subsequence whose Cesàro sequence is almost order bounded in F.

(2) $\Longrightarrow$ (1) Let (x_n) be a disjoint weakly null sequence of E^+. By [9, Corollary 3.6], we have that $x_n \xrightarrow{uo} 0$ and hence it follows from the assertion (2) that $(T(x_n))$ has a subsequence whose Cesàro sequence is almost order bounded in F. As F is order continuous, then it follows from the Proposition 3.11 that T is disjoint weak Banach-Saks.

(2) $\Longrightarrow$ (3) Let (x_n) be a weakly null sequence of E^+. Since E is order continuous, then it follows from [9, Proposition 4.5] and [9, Proposition 4.7] that there exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \xrightarrow{uo} 0$. Therefore, by assertion (2) we conclude that $(T(x_{n_k}))$ has a subsequence whose Cesàro sequence is almost order bounded in F.

(3) $\Longrightarrow$ (2) It is obvious. □

As a consequence of the Proposition 3.4, the Proposition 3.11 and the Proposition 3.12; we obtain Proposition 6.9 [9].

Corollary 3.13. *([9, Proposition 6.9])*
Let E be order continuous. Then, the following assertions are equivalent:

(1) E has the disjoint weak Banach-Saks property.

(2) Every disjoint weakly null sequence (x_n) of E has a subsequence whose Cesàro sequence is almost order bounded.

(3) Every disjoint weakly null sequence (x_n) of E^+ has a subsequence whose Cesàro sequence is almost order bounded.

(4) Every weakly and uo-null sequence (x_n) of E has a subsequence whose Cesàro sequence is norm convergent.

(5) Every weakly and uo-null sequence (x_n) of E^+ has a subsequence whose Cesàro sequence is almost order bounded.

(6) Every weakly null sequence (x_n) of E^+ has a subsequence whose Cesàro sequence is norm convergent.

(7) Every weakly null sequence (x_n) of E^+ has a subsequence whose Cesàro sequence is almost order bounded.

We denote by $L_{DWBS}(E, Y)$ the space of all disjoint weak Banach-Saks operators from E into Y.

Proposition 3.14. *$L_{DWBS}(E, Y)$ is a closed subset of the space of all operators from E into Y.*

Proof. Let (T_m) be a sequence of disjoint weak Banach-Saks operators which is norm convergent to the operator T. We will show that T is also a disjoint weak Banach-Saks operator. For this, let (x_n) be a disjoint weakly null sequence of E, then it follows that $(T_m(x_n))$ has a subsequence $(T_m(x_{n_k}))$ whose Cesàro sequence is norm convergent in Y. Thus, we have $\lim\limits_{n\longrightarrow\infty} \frac{1}{n}\sum_{k=1}^{n} T_m(x_{n_k}) = 0$. On the other hand, (T_m) is norm convergent to the operator T. So, given any $\varepsilon > 0$, there is an $m_0 \in \mathbb{N}$ such that $\|T_m - T\| \leq \frac{\varepsilon}{2M}$ for each $m > m_0$, where $M = \sup_n \|x_n\|$. Let $m > m_0$, for sufficiently large n we have $\|\frac{1}{n}\sum_{k=1}^{n} T_m(x_{n_k})\| \leq \frac{\varepsilon}{2}$. Therefore, by the inequality

$$\begin{aligned} \|T(x_n)\| &\leq \|(T - T_m)(x_n)\| + \|T_m(x_n)\| \\ &\leq M.\frac{\varepsilon}{2M} + \|T_m(x_n)\| \end{aligned}$$

we obtain that

$$\begin{aligned} \|\frac{1}{n}\sum_{k=1}^{n} T(x_{n_k})\| &\leq \frac{\varepsilon}{2} + \|\frac{1}{n}\sum_{k=1}^{n} T_m(x_{n_k})\| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence we find that $\lim\limits_{n\longrightarrow\infty} \frac{1}{n}\sum_{k=1}^{n} T(x_{n_k}) = 0$. Therefore, $T \in L_{DWBS}(E, Y)$. $\square$

Proposition 3.15. *If F is order continuous, then $L_{DWBS}(E, F)$ is a closed vector subspace of the space of all operators from E into F.*

Proof. Let $T, S \in L_{DWBS}(E, F)$. Is $T+S \in L_{DWBS}(E, F)$? Let (x_n) be a disjoint weakly null sequence of E^+, then every subsequence of (x_n) is disjoint weakly null of E^+. Since T is disjoint weak Banach-Saks operator, then it follows from the Proposition 3.11 that every subsequence of $(T(x_n))$ has a further subsequence whose Cesàro sequence is almost order bounded in F. As F is order continuous, then from [9, Lemma 6.3] there exists a subsequence $(T(x_{\phi(n)}))$ of $(T(x_n))$ and a vector $y \in F$ such that the Cesàro sequence of any subsequence of $(T(x_{\phi(n)}))$ converges uo-and in norm to $y = 0$, and hence is almost order bounded in F. On the other hand, S is disjoint weak Banach-Saks operator. It follows from the Proposition 3.11 that the sequence $(S(x_{\phi(n)}))$ has a subsequence $(S(x_{\psi(\phi(n))}))$ whose Cesàro sequence is almost order bounded in F. As, it was mentioned above, the Cesàro sequence of any subsequence of $(T(x_{\phi(n)}))$ is almost order bounded in F; hence the Cesàro sequence of $(T(x_{\psi(\phi(n))}))$ is almost order bounded. Therefore, the Cesàro sequence of the subsequence $(T+S)(x_{\psi(\phi(n))}) = T(x_{\psi(\phi(n))}) + S(x_{\psi(\phi(n))})$ of the sequence $(T+S)(x_n)$ is almost order bounded in F. Where ϕ and ψ are increasing mappings from $\mathbb{N}$ to $\mathbb{N}$ □

As a consequence of the Proposition 3.15, we obtain the following result.

Corollary 3.16. *Let $T : E \longrightarrow F$ be an operator such that F is order continuous. If the modulus of T exists and is disjoint weak Banach-Saks, then T is disjoint weak Banach-Saks.*

Proof. The proof is the same as that of the second part of [3, Proposition 2.1]. □

Lemma 3.17. *Let (x_α) be a net in E such that $x_\alpha \xrightarrow{uaw} 0$. Then, there exist an increasing sequence of indices (α_k) and a disjoint sequence (d_k) such that $x_{\alpha_k} - d_k \xrightarrow{w} 0$.*

Proof. The proof of this lemma is the same as that of [7, Theorem 3.2]. It suffices to replace the norm in the inequality $\|x_{\alpha_k} \wedge x_{\alpha_i}\| \leq \frac{1}{2^{k+1}}$ by a positive linear functional in order to replace the norm convergence by the weak convergence. □

Proposition 3.18. *If Y has the Schur property, then for every operator $T : E \longrightarrow Y$ the following statements are equivalent:*

(1) T is disjoint weak Banach-Saks.

(2) For each weakly null sequence (x_n) of E such that $x_n \xrightarrow{uaw} 0$, $(T(x_n))$ has a subsequence whose Cesàro sequence is norm convergent.

Proof. (1) $\Longrightarrow$ (2) Let (x_n) be a weakly null sequence of E such that $x_n \xrightarrow{uaw} 0$. Then it follows from the Lemma 3.17 that there exist (x_{n_k}) a subsequence of (x_n) and a disjoint sequence (d_k) of E such that $x_{n_k} - d_k \xrightarrow{w} 0$. As (x_{n_k}) is weakly null in E, then (d_k) is weakly null in E. Since T is a disjoint weak Banach-Saks operator, then $(T(d_k))$ has a Cesàro norm convergent subsequence $(T(d_{\phi(k)}))$ which is also weakly convergent. Since $x_{n_{\phi(k)}} - d_{\phi(k)} \xrightarrow{w} 0$, then $(T(x_{n_k}))$ has a Cesàro weakly convergent subsequence $(T(x_{n_{\phi(k)}}))$ in Y. On the other hand, Y has the Schur property, implies that $(T(x_{n_k}))$ has a Cesàro convergent subsequence $(T(x_{n_{\phi(k)}}))$.

(2) $\Longrightarrow$ (1) Let (x_n) be a disjoint weakly null sequence of E. Then it follows from [15, Lemma 2] that $x_n \xrightarrow{uaw} 0$. Therefore, $(T(x_n))$ has a Cesàro convergent subsequence in Y. □

As consequence of the Proposition 3.11 and the Proposition 3.18, we obtain the following results.

Corollary 3.19. *If the lattice operations of E are weakly sequentially continuous and F has the Schur property, then for every operator $T : E \longrightarrow F$ the following statements are equivalent:*

(1) T is disjoint weak Banach-Saks.

(2) For each weakly null sequence (x_n) of E such that $x_n \xrightarrow{uaw} 0$, $(T(x_n))$ has a subsequence whose Cesàro sequence is norm convergent.

(3) For each weakly null sequence (x_n) of E, $(T(x_n))$ has a subsequence whose Cesàro sequence is norm convergent.

(4) For each weakly null sequence (x_n) of E^+, $(T(x_n))$ has a subsequence whose Cesàro sequence is norm convergent.

Proof. (1) $\Longleftrightarrow$ (2) It follows from the Proposition 3.18.

(2) $\Longrightarrow$ (3) Let (x_n) be a weakly null sequence of E. Since the lattice operations of E are weak sequentially continuous, then it follows from [8, Theorem 2.1] that $x_n \xrightarrow{uaw} 0$. Hence, by (2) we conclude that $(T(x_n))$ has a subsequence whose Cesàro sequence is norm convergent in Y.

(3) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (4) Are obvious.

(4) $\Longrightarrow$ (3) The proof is the same as that of the implication ((3) $\Longrightarrow$ (2)) of the Proposition 3.11. □

Corollary 3.20. *Let $T : E \longrightarrow F$ be a positive operator such that F has the Schur property. The following assertions are equivalent:*

(1) T is a disjoint weak Banach-Saks operator.

(2) For each weakly null sequence $(x_n) \subset E^+$, $(T(x_n))$ has a subsequence whose Cesàro sequence is norm convergent in F.

Proof. (1) $\Longrightarrow$ (2) Let (x_n) be a weakly null sequence of E^+, then $|x_n| \xrightarrow{w} 0$ and hence it follows from [15] that $x_n \xrightarrow{uaw} 0$. Therefore, by Proposition 3.18 $(T(x_n))$ has a subsequence whose Cesàro sequence is norm convergent in F.

(2) $\Longrightarrow$ (1) It follows from the Theorem 3.2. □

Note that a weak Banach-Saks operator is disjoint weak Banach-Saks. But the converse is not true in general. In fact, the Banach lattice $E = L_p(c_0) = L_p([0,1]; c_0)$, where $0 < p < 1$, has the disjoint weak Banach-Saks property ([9, example 6.10]) but fails to have the weak Banach-Saks property. Thus, Id_E the identity operator of E is a disjoint weak Banach-Saks operator which is not weak Banach-Saks.

In the following result, we give sufficient conditions on E and Y under which each disjoint weak Banach-Saks operator from E into Y is weak Banach-Saks.

Theorem 3.21. *Each disjoint weak Banach-Saks operator $T : E \longrightarrow Y$ is weak Banach-Saks, if one of the following assertions is valid:*

(1) E has the subsequence splitting property.

(2) E is order continuous and atomic.

(3) E and E' are order continuous.

(4) The lattice operations of E are weakly sequentially continuous and Y is a Banach lattice with the Schur property.

Proof. 1) Let (x_n) be a weakly null sequence of E. Since E has the subsequence splitting property, by passing to a subsequence we may assume that $x_n = y_n + z_n$, where (y_n) is almost order bounded and (z_n) is disjoint. Since E is order continuous, then by passing to a further subsequence, we may assume that every subsequence of (y_n) and therefore of $(T(y_n))$ is Cesàro norm convergent ([9, lemma 6.3]). As (x_n) is a weakly null sequence, then (z_n) is also a weakly null sequence. Since T is a disjoint weak Banach-Saks operator, then $(T(z_n))$ has a Cesàro norm convergent subsequence $(T(z_{n_k}))$. We put $T(x_{n_k}) = T(y_{n_k}) + T(z_{n_k}) = T(y_{n_k} + z_{n_k})$, where (y_{n_k}) is a subsequence of (y_n). We have $(T(x_{n_k}))$ is a Cesàro norm convergent subsequence of $(T(x_n))$. Therefore, T is a weak Banach-Saks operator.

2) It follows from [9, Lemma 6.7] and [9, Lemma 6.14].

3) Let (x_n) be a weakly null sequence of E. Since both E and E' are order continuous, then it follows from [13, Theorem 5] that (x_n) is uo-null sequence of E. As T is a disjoint weak Banach-Saks operator, then by the Proposition 3.10 we infer that $(T(x_n))$ has a Cesàro norm convergent subsequence in Y. Thus, T is weak Banach-Saks.

4) It follows from the Corollary 3.19. □

Note that an almost Banach-Saks operator is disjoint weak Banach-Saks. But the converse is not true in general. In fact, the identity operator $Id_{\ell^1} : \ell^1 \longrightarrow \ell^1$ is disjoint weak Banach-Saks (because ℓ^1 has the disjoint weak Banach-Saks property), but it follows from [11] that Id_{ℓ^1} is not almost Banach-Saks.

In the following result, we give necessary and sufficient conditions on E and Y under which each disjoint weak Banach-Saks operator from E into Y is almost Banach-Saks.

Theorem 3.22. *The following assertions are equivalent:*

(1) Each disjoint weak Banach-Saks operator $T : E \longrightarrow Y$ is almost Banach-Saks.

(2) One of the following assertions is valid:

(a) E' is order continuous;

(b) Y has the Banach-Saks property.

Proof. $(1) \implies (2)$ Assume that neither the norm of E' is order continuous nor Y has the Banach-Saks property. Then, by [12, Theorem 2.4.14] and [12, Proposition 2.3.11] E contains a complemented copy of ℓ^1 and there exists a positive projection $P : E \longrightarrow \ell^1$. On the other hand, since Y does not have the Banach-Saks property then there exists (y_n) a norm bounded sequence of Y with no Cesàro convergent subsequence.

We consider the following operator:

$$\begin{array}{lccc} S: & \ell^1 & \longrightarrow Y & \\ & (\lambda_n) & \longmapsto & \sum_{n=1}^{\infty} \lambda_n y_n \end{array}$$

S is well defined.

Now, we consider the composed operator $T = S \circ P$. Since ℓ^1 has the weak Banach-Saks property, then both S and P are weak Banach-Saks operators. Thus, T is weak Banch-Saks and hence T is disjoint weak Banach-Saks. To finish the proof, we have to claim that T is not an almost Banach-Saks operator. Otherwise, since the injection $\imath : \ell^1 \longrightarrow E$ is a lattice homomorphism then it follows from [11, Propostion 3.2] that $T \circ \imath$ is an almost Banach-Saks operator. But by taking (e_n) the unit basis of ℓ^1 as a norm bounded disjoint sequence, we have $T \circ \imath(e_n) = y_n$ with no Cesàro convergent subsequence, which is a contradiction.

$(2)(a) \implies (1)$ Let $T : E \longrightarrow Y$ be a disjoint weak Banach-Saks operator and let (x_n) be a norm bounded disjoint sequence in E. Since E' is order continuous, then by [12, Theorem 2.4.14] the sequence (x_n) is also weakly null and so $(T(x_n))$ has a subsequence whose Cesàro sequence is norm convergent in Y. Hence, the operator T is almost Banach-Saks.

$(2)(b) \implies (1)$ It is obvious. □

As consequences of the Theorem 3.22, we have the following results.

Corollary 3.23. *The following assertions are equivalent:*

(1) Each disjoint weak Banach-Saks operator $T : E \longrightarrow E$ is almost Banach-Saks.

(2) E' is order continuous.

Corollary 3.24. *([9, Proposition 6.15]) A Banach lattice E with the DWBSP has the DBSP if, and only if, it contains no lattice copy of ℓ^1.*

Recall from [1, Theorem 5.57] that an operator $T : E \longrightarrow Y$ is order weakly compact if and only if for every order bounded disjoint sequence (x_n) of E, we have $\|T(x_n)\| \longrightarrow 0$.

Theorem 3.25. *If either E is σ-laterally complete or AM-space with unit, then every o-weakly compact operator from E into Y is disjoint weak Banach-Saks.*

Proof. Let $T : E \longrightarrow Y$ be an order weakly compact operator and let (x_n) be a disjoint weakly null sequence in E. Since E is σ-laterally complete (resp, E is AM-space with unit), then (x_n) is order bounded and hence $(T(x_n))$ is norm-null. Therefore, $(T(x_n))$ has a Cesàro convergent subsequence. □

We note that a disjoint weak Banach-Saks operator is not necessarily o-weakly compact. In fact, the identity operator $Id_{\ell^\infty} : \ell^\infty \longrightarrow \ell^\infty$ is disjoint weak Banach-Saks $\big($because ℓ^∞ has the disjoint weak Banach-Saks property$\big)$ but it is not o-weakly compact(because ℓ^∞ is not order continuous).

By the same proof of [2, Theorem 2.2], we can investegate the following result.

Theorem 3.26. *If E has the disjoint weak Banach-Saks property, then the following assertions are equivalent:*

(1) Each order bounded operator from E into F is order weakly compact.

(2) Each order bounded disjoint weak Banach-Saks operator $T : E \longrightarrow F$ is order weakly compact.

(3) One of the following assertions is valid:

(a) E is order continuous;

(b) F is order continuous.

We note that the identity operator Id_{c_0} of the Banach lattice c_0 is a disjoint weak Banach-Saks operator which is not weakly compact. Conversely, weakly compact operators are not in general disjoint weak Banach-Saks. In fact, the identity operator of the Baerstein space (see [4]) is weakly compact but fails to be disjoint weak Banach-Saks.

Theorem 3.27. *If either E' has the positive Schur property or Y has the Schur property, then every weakly compact operator from E into Y is disjoint weak Banach-Saks.*

Proof.

- Let $T : E \longrightarrow Y$ be a weakly compact operator. If E' has the positive Schur property, then it follows from [11, Proposition 3.18] that T is almost Banach-Saks and hence T is disjoint weak Banach-Saks.
- Let (x_n) be a weakly disjoint sequence in E and $T : E \longrightarrow Y$ be a weakly compact operator. It follows from [1] that $(T(x_n))$ has a weakly null convergent subsequence in Y. Since Y has the Schur property, then $(T(x_n))$ has a norm null convergent subsequence.

□

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