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On (m,n)-class Q and (m,n)-class Q* operators

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Abstract For a positive real number m and an integer $n \geq 1$, an operator $T \in \mathcal{B}(\mathcal{H})$ is (m,n)-class $\mathcal{Q}$ operator if $\|Tx\|^2 \leq \dfrac{m^{\frac{2}{n+1}}}{n+1}\left(\|T^{n+1}x\|^2 + n\|x\|^2\right)$ for every $x \in \mathcal{H}$ and (m,n)-class $\mathcal{Q}^*$ operator if $\|T^*x\|^2 \leq \dfrac{m^{\frac{2}{n+1}}}{n+1}\left(\|T^{n+1}x\|^2 + n\|x\|^2\right)$ for every $x \in \mathcal{H}$. In this paper we study some properties of (m,n)-class $\mathcal{Q}$ and (m,n)-class $\mathcal{Q}^*$ operators. Also we characterize (m,n)-class $\mathcal{Q}$ and (m,n)-class $\mathcal{Q}^*$ composition operators on L^2 space.

1 Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space and $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *hyponormal* if $TT^* \leq T^*T$, and *paranomal* if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for all $x \in \mathcal{H}$ [4]. Hyponormal operators are paranormal [4]. $T \in \mathcal{B}(\mathcal{H})$ is said to be *Class* $\mathcal{Q}$ if $T^{*2}T^2 - 2T^*T + I \geq 0$ [2]. *Class* $\mathcal{Q}$ operators properly includes the paranormal operators [2]. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *Class* $\mathcal{Q}^*$ if $T^{*2}T^2 - 2TT^* + I \geq 0$ [12]. For a positive real number m and an integer $n \geq 1$, $T \in \mathcal{B}(\mathcal{H})$ is (m,n)-*paranomal* if $\|Tx\|^{n+1} \leq m\|T^{n+1}x\|\|x\|^n$ for all $x \in \mathcal{H}$ and $(m,n)^*$-*paranomal* if $\|T^*x\|^{n+1} \leq m\|T^{n+1}x\|\|x\|^n$ for all $x \in \mathcal{H}$ [3]. An operator $T \in$ is said to be a normaloid if $\|T\| = \sup\{|\lambda| : \lambda \in \sigma(T)\}$, where $\sigma(T)$ denotes the spectrum of T. It is well-known that every class $\mathcal{Q}$ operators are not necessarily normaloid.

Let $(X, \mathcal{A}, \mu)$ be a σ- finite measure space. A transformation T is said to be measurable if $T^{-1}(B) \in \mathcal{A}$, for every $B \in \mathcal{A}$. If T is a nonsingular measurable transformation on $(X, \mathcal{A}, \mu)$ and the Randon- Nikodym derivative $\dfrac{d\mu T^{-1}}{d\mu}$ denoted by h is essentially bounded, then the composition operator C_T on $L^2(\mu)$ is defined by $C_T f = f \circ T, f \in L^2(\mu)$ [11]. Let $L^\infty(\mu)$ denote the space of all essentially bounded complex valued measurable functions on X. For $\pi \in L^\infty(\mu)$, the multiplication operator M_π on $L^2(\mu)$ is given by $M_\pi f = \pi f,\ f \in L^2(\mu)$. The weighted composition operator W on $L^2(X, \mathcal{A}, \mu)$ induced by T and a complex valued measurable function π is given by

$$W = \pi(f \circ T),$$

for $f \in L^2(\mu)$. Let π_k denote $\pi(\pi \circ T)(\pi \circ T^2) \cdots (\pi \circ T^{k-1})$. Then, $W^k(f) = \pi_k(f \circ T)^k$ [9]. More details on general properties of (measure based) composition operators can be found in [8, 11]. The conditional expectation operator $E(.|_{T^{-1}(\mathcal{A})}) = E(f)$ is defined for each non-negative function $f \in L^p(\mu)$, $1 \leq p < \infty$ and is uniquely determined by the conditions

(i) $E(f)$ is $T^{-1}(\mathcal{A})$ measurable

(ii) If B is any $T^{-1}(\mathcal{A})$ measurable set for which $\int_B f d\mu$ converges then $\int_B f d\mu = \int_B E(f) d\mu$.

We refer the reader to [1, 6, 7, 10] for more details on the properties of conditional expectation.

In this paper we initiate the study of (m,n)-class $\mathcal{Q}$ and (m,n)- class $\mathcal{Q}^*$ operators. The classes (m,n)-class $\mathcal{Q}$ and (m,n)- class $\mathcal{Q}^*$ are extension of class Q and class Q* operators, respectively. We study some properties and give examples of these classes of operators. Moreover, (m,n)-class $\mathcal{Q}$ and (m,n)- class $\mathcal{Q}^*$ composition operators on L^2-space are characterized.

2 (m,n)-class $\mathcal{Q}$ and (m,n)-class $\mathcal{Q}^*$ operators

Throughout this paper, let m be a positive real number and $n \geq 1$ be an integer. In this section we give some basic properties of (m,n)-class $\mathcal{Q}$ and (m,n)-class $\mathcal{Q}^*$ operators.

An operator $T \in \mathcal{B}(\mathcal{H})$ is (m,n)-paranomal if and only if

$$m^{\frac{2}{n+1}} T^{*n+1}T^{n+1} - (n+1)a^n T^*T + m^{\frac{2}{n+1}} n\, a^{n+1}\ I \geq 0,$$

for each $a > 0$ [3].

For $T \in \mathcal{B}(\mathcal{H})$, we define

$$\mathcal{Q}_{(m,n)} = m^{\frac{2}{n+1}} T^{*n+1}T^{n+1} - (n+1)T^*T + m^{\frac{2}{n+1}}\ n\ I.$$

Definition 2.1. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *(m,n)*-class $\mathcal{Q}$ if $\mathcal{Q}_{(m,n)} \geq 0$. That is, T is a *(m,n)*-class $\mathcal{Q}$ if

$$\|Tx\|^2 \leq \frac{m^{\frac{2}{n+1}}}{n+1}\left(\|T^{n+1}x\|^2 + n\|x\|^2\right),$$

for every $x \in \mathcal{H}$.

If $m = n = 1$, then *(m,n)*-class $\mathcal{Q}$ and class $\mathcal{Q}$ coincides.

Example 2.2. Let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots).$$

Then, $T^*(x_1, x_2, x_3, \cdots) = (x_2, x_3, x_4 \cdots)$.
Also,

$$T^*T(x_1, x_2, x_3, \cdots) = (x_1, x_2, x_3, \cdots)$$

and

$$T^{*n+1}T^{n+1}(x_1, x_2, x_3, \cdots) = (x_1, x_2, x_3, \cdots).$$

Hence,

$$\begin{aligned}\mathcal{Q}_{(m,n)} &= m^{\frac{2}{n+1}} T^{*n+1}T^{n+1} - (n+1)T^*T + m^{\frac{2}{n+1}}\ n\ I\\ &= (m^{\frac{2}{n+1}} - 1)(1+n)I \geq 0, \text{ for all } m \geq 1.\end{aligned}$$

If $m \geq 1$, then T is (m,n)-class $\mathcal{Q}$.

Theorem 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$. T is (m,n)-paranormal if and only if λT is (m,n)-class $\mathcal{Q}$ operator, for each $\lambda > 0$.*

Proof. By definition, λT (m,n)-class $\mathcal{Q}$

$$\Leftrightarrow m^{\frac{2}{n+1}} |\lambda|^{2(n+1)} T^{*n+1}T^{n+1} - (n+1)\, |\lambda|^2\, T^*T + m^{\frac{2}{n+1}}\ n\ I \geq 0,\ \forall\, \lambda > 0.$$

$$\Leftrightarrow m^{\frac{2}{n+1}} T^{*n+1}T^{n+1} - (n+1)\left(\frac{1}{\lambda^2}\right)^n T^*T + m^{\frac{2}{n+1}}\ n\left(\frac{1}{\lambda^2}\right)^{n+1} I \geq 0, \forall\, \lambda > 0$$

$$\Leftrightarrow m^{\frac{2}{n+1}} T^{*n+1}T^{n+1} - (n+1)\ a^n\ T^*T + m^{\frac{2}{n+1}}\ n\ a^{n+1}\ I \geq 0,\ a > 0.$$

$\Leftrightarrow T$ is (m, n)-paranormal.

□

Now we prove that the part of (m,n)-class $\mathcal{Q}$ is again (m,n)-class $\mathcal{Q}$.

Theorem 2.4. *Let $T \in \mathcal{B}(\mathcal{H})$ be a (m,n)-class $\mathcal{Q}$ operator and $\mathcal{M}$ be a closed subspace of $\mathcal{H}$ which is invariant under T. Then $T|_{\mathcal{M}}$ is a (m,n)-class $\mathcal{Q}$ operator.*

Proof. Let $x \in \mathcal{M}$. Then

$$\begin{aligned}\|T|_{\mathcal{M}}x\|^2 &= \|Tx\|^2\\ &\leq \frac{m^{\frac{2}{n+1}}}{n+1}\left(\|T^{n+1}x\|^2 + n\|x\|^2\right)\\ &= \frac{m^{\frac{2}{n+1}}}{n+1}\left(\|(T|_{\mathcal{M}})^{n+1}x\|^2 + n\|x\|^2\right).\end{aligned}$$

Thus, $T|_{\mathcal{M}}$ is a (m,n)-class $\mathcal{Q}$ operator. □

Theorem 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ and $c = \dfrac{n+1}{m^{\frac{2}{n+1}}\, n}$. If $\sqrt{c}\,T$ is a contraction, then T is a (m,n)-class $\mathcal{Q}$ operator.*

Proof. Since $\sqrt{c}\,T$ is a contraction, we have $-c\,T^*T + I \geq 0$. Hence

$$\frac{1}{n}T^{*n+1}T^{n+1} - c\,T^*T + I \geq 0.$$

Thus, T is a (m,n)-class $\mathcal{Q}$ operator. □

Theorem 2.6. *Let $T \in \mathcal{B}(\mathcal{H})$ be a weighted shift operator with non zero weights $\{\alpha_k\}, k = 1, 2, \cdots$. Then T is a (m,n)-class $\mathcal{Q}$ operator if and only if*

$$\frac{n+1}{m^{\frac{2}{n+1}}}\left(|\alpha_k|^2\right) \leq |\alpha_k|^2|\,\alpha_{k+1}|^2 \cdots |\alpha_{k+n}|^2 + n, \forall\, k \in \mathbb{N}.$$

Proof. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$. Since $Te_k = \alpha_k\, e_{k+1}$, we have $T^{n+1}e_k = \alpha_k\, \alpha_{k+1} \cdots \alpha_{k+n}\, e_{k+n+1}$.
Now,

$$\begin{aligned}T \text{ is } (m,n)\text{-class } \mathcal{Q} &\Leftrightarrow \|Tx\|^2 \leq \frac{m^{\frac{2}{n+1}}}{n+1}\left(\|T^{n+1}x\|^2 + n\|x\|^2\right), \ \forall\, x \in \mathcal{H}.\\ &\Leftrightarrow \|Te_k\|^2 \leq \frac{m^{\frac{2}{n+1}}}{n+1}\left(\|T^{n+1}e_k\|^2 + n\|e_k\|^2\right), \ \forall\, k \in \mathbb{N}.\\ &\Leftrightarrow \frac{n+1}{m^{\frac{2}{n+1}}}\, |\alpha_k|^2 \leq |\alpha_k|^2\, |\alpha_{k+1}|^2 \ldots\ldots |\alpha_{k+n}|^2 + n, \ \forall\, k \in \mathbb{N}.\end{aligned}$$

□

Let $T \in \mathcal{B}(\mathcal{H})$ be a weighted shift operator with non zero weights $\{\alpha_k\}$, $k = 1, 2, \cdots$. It is evident that T is (m,n) -paranormal if and only if

$$|\alpha_k|^{n+1} \leq m\, |\alpha_k|\, |\alpha_{k+1}| \cdots |\alpha_{k+n}|, \ \forall\, k \in \mathbb{N} \tag{2.1}$$

From Theorem 2.3, every (m,n)-paranormal operator is (m,n)-class $\mathcal{Q}$ operator. The following example show that the reverse inclusion may not hold.

Example 2.7. Let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (0, \frac{1}{2}x_1, \frac{1}{4}x_2, \frac{1}{4}x_3, \cdots).$$

By Theorem 2.6, T is a $(\frac{1}{3}, 3)$-class $\mathcal{Q}$ operator. If $k = 4$, then equation (2.1) does not holds. Hence, T is not $(\frac{1}{3}, 3)$-paranormal operator.

It is well known that every paranormal operators are normaloid and the set of all (m,n)-class $\mathcal{Q}$ operators includes paranormal operators. But every (m,n)-class $\mathcal{Q}$ operators need not be a normaloid. For example, let $T = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}$. If $|\lambda|^2 \leq \dfrac{m}{2}$, then T is $(m,1)$-class $\mathcal{Q}$ but not normaloid.

Theorem 2.8. *Let $T \in \mathcal{B}(\mathcal{H})$ be a (m,n)-class $\mathcal{Q}$ operator and an isometric operator $A \in \mathcal{B}(\mathcal{H})$ be such that $AT = TA$. Then TA is a (m,n)-class $\mathcal{Q}$ operator.*

Proof. Let

$$S_{m,n} = m^{\frac{2}{n+1}}(TA)^{*n+1}(TA)^{n+1} - (n+1)(TA)^*(TA) + m^{\frac{2}{n+1}}\ n\ I$$

Since $AT = TA$ and $A^*A = I$, we have

$$S_{m,n} = m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)T^*T + m^{\frac{2}{n+1}}\ n\ I.$$

Since T is (m,n)-class $\mathcal{Q}$, we have $S_{m,n} \geq 0$. That is, TA is (m,n)-class $\mathcal{Q}$. □

Theorem 2.9. *Let $T \in \mathcal{B}(\mathcal{H})$ be a (m,n)-class $\mathcal{Q}$ operator and T is unitarily equivalent to a operator B. Then B is a (m,n)-class $\mathcal{Q}$ operator.*

Proof. Since T is unitarily equivalent to B, there exist a unitary operator U such that $B = U^*TU$. Now, $m^{\frac{2}{n+1}}B^{*n+1}B^{n+1} - (n+1)B^*B + m^{\frac{2}{n+1}}\ n\ I$

$$\begin{aligned} &= m^{\frac{2}{n+1}}U^*(T^*)^{n+1}(T)^{n+1}U - (n+1)U^*T^*TU + m^{\frac{2}{n+1}}\ nI \\ &= U^*\left(m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)T^*T + m^{\frac{2}{n+1}}\ n\ I\right)U \geq 0. \end{aligned}$$

That is, B is a (m,n)-class $\mathcal{Q}$ operator. □

An operator $T \in \mathcal{B}(\mathcal{H})$ is $(m,n)^*$- paranomal if and only if

$$m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)a^nTT^* + m^{\frac{2}{n+1}}n\ a^{n+1}\ I \geq 0,$$

for each $a > 0$ [3].

For $T \in \mathcal{B}(\mathcal{H})$, we define

$$\mathcal{Q}_{(m,n)^*} = m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)TT^* + m^{\frac{2}{n+1}}\ n\ I.$$

Definition 2.10. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be (m,n)-class $\mathcal{Q}^*$ if $\mathcal{Q}_{(m,n)^*} > 0$. Equivalently, T is (m,n)-class $\mathcal{Q}^*$ if

$$\|T^*x\|^2 \leq \frac{m^{\frac{2}{n+1}}}{n+1}\left(\|T^{n+1}x\|^2 + n\|x\|^2\right),$$

for every $x \in \mathcal{H}$.

Theorem 2.11. *Let $T \in \mathcal{B}(\mathcal{H})$ be a weighted shift operator with non zero weights $\{\alpha_k\}$, $k = 1, 2, \cdots$ T is a (m,n)-class $\mathcal{Q}^*$ operator if and only if*

$$\frac{n+1}{m^{\frac{2}{n+1}}}\left(|\alpha_k|^2\right) \leq |\alpha_{k+1}|^2\ |\alpha_{k+2}|^2 \cdots |\alpha_{k+n+1}|^2 + n, \forall\ k \in \mathbb{N}.$$

Proof. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$. Since $Te_k = \alpha_k\ e_{k+1}$, we have $T^{n+1}e_k = \alpha_k\ \alpha_{k+1} \cdots \alpha_{k+n}\ e_{k+n+1}$ and $T^*e_k = \overline{\alpha_{k-1}}\ e_{k-1}$.
Now, T is (m,n)-class $\mathcal{Q}^*$

$$\begin{aligned} &\Leftrightarrow \|T^*x\|^2 \leq \frac{m^{\frac{2}{n+1}}}{n+1}\left(\|T^{n+1}x\|^2 + n\|x\|^2\right), \forall\ x \in \mathcal{H}. \\ &\Leftrightarrow \|T^*e_k\|^2 \leq \frac{m^{\frac{2}{n+1}}}{n+1}\left(\|T^{n+1}e_k\|^2 + n\|e_k\|^2\right),\ \forall\ k \in \mathbb{N}. \\ &\Leftrightarrow \frac{n+1}{m^{\frac{2}{n+1}}}\ |\alpha_k|^2 \leq\ |\alpha_{k+1}|^2\ |\alpha_{k+2}|^2.....|\alpha_{k+n+1}|^2 + n,\ \forall\ k \in \mathbb{N}. \end{aligned}$$

□

Theorem 2.12. *Let $T \in \mathcal{B}(\mathcal{H})$ and let $c = \dfrac{n+1}{m^{\frac{2}{n+1}}\, n}$. If $\sqrt{c}\, T^*$ is a contraction, then T is a (m,n)-class $\mathcal{Q}^*$ operator.*

Proof. Since $\sqrt{c}\, T^*$ is a contraction, we have $-c\, TT^* + I \geq 0$. Hence

$$\frac{1}{n} T^{*n+1} T^{n+1} - c\, TT^* + I \geq 0.$$

That is, T is a (m,n)-class $\mathcal{Q}^*$ operator. □

Theorem 2.13. *Let $T \in \mathcal{B}(\mathcal{H})$. T is $(m,n)^*$-paranormal if and only if λT is (m,n)-class $\mathcal{Q}^*$, for each $\lambda > 0$.*

Proof. The reult follows by a similar argument as in the proof of Theorem 2.3. □

By a similar argument as in the case of (m,n)-class $\mathcal{Q}$, the following results hold:

(i) If $T \in \mathcal{B}(\mathcal{H})$ is a (m,n)-class $\mathcal{Q}^*$ operator, then the restriction $T|_{\mathcal{M}}$ is (m,n)-class $\mathcal{Q}^*$, where $\mathcal{M}$ is a closed subspace of $\mathcal{H}$ which is invariant under T.

(ii) If $T \in \mathcal{B}(\mathcal{H})$ is a (m,n)-class $\mathcal{Q}^*$ operator and $A \in \mathcal{B}(\mathcal{H})$ is an isometric operator such that $AT = TA$, then TA is a (m,n)-class $\mathcal{Q}^*$ operator.

(iii) If $T \in \mathcal{B}(\mathcal{H})$ is a (m,n)-class $\mathcal{Q}^*$ operator and T is unitarily equivalent to an operator B, then B is a (m,n)-class $\mathcal{Q}^*$ operator.

3 (m,n)-class $\mathcal{Q}$ and (m,n)-class $\mathcal{Q}^*$ composition operators

In this section we characterize (m,n)-class $\mathcal{Q}$ and (m,n)-class $\mathcal{Q}^*$ composition operators on L^2 space. Throughout this section, T is a nonsingular measurable transformation on a σ- finite measure space $(X, \mathcal{A}, \mu)$.

Proposition 3.1. *[1, 5]*

Let C_T be the composition operator on $L^2(\mu)$ and P be the projection from $L^2(X, \mathcal{A}, \mu)$ onto $\overline{R(C_T)}$. Then for every $f \in L^2(\mu)$, the following holds

(i) $C_T^* f = h.E(f) \circ T^{-1}$.

(ii) $C_T C_T^* f = (h \circ T) P f$, $C_T^* C_T = hf$.

(iii) $(C_T C_T^*)^k f = (h \circ T)^k P f$, $k \in \mathbb{N}$.

(iv) $(C_T^* C_T)^k f = h^k f$, $k \in \mathbb{N}$.

Let h_k denotes the Radon-Nikodym derivative of the measure $\mu(T^k)^{-1}$ with respect to μ.

Theorem 3.2. *C_T is (m,n)-class $\mathcal{Q}$ if and only if $m^{\frac{2}{n+1}}(h_{n+1} + n) \geq (n+1)h$.*

Proof. By definition, C_T is a (m,n)-class $\mathcal{Q}$ operator if and only if

$$m^{\frac{2}{n+1}} C_T^{*n+1} C_T^{n+1} - (n+1) C_T^* C_T + m^{\frac{2}{n+1}}\, n\, I \geq 0.$$

By Proposition 3.1, $(n+1) C_T^* C_T f = (n+1) h f$ and

$$\begin{aligned} m^{\frac{2}{n+1}} C_T^{*n+1} C_T^{n+1} f &= m^{\frac{2}{n+1}} C_T^{*n+1} (f \circ T^{n+1}) \\ &= m^{\frac{2}{n+1}} h_{n+1} E(f \circ T^{n+1}) \circ T^{-(n+1)} \\ &= m^{\frac{2}{n+1}} h_{n+1} E(1) f \\ &= m^{\frac{2}{n+1}} h_{n+1} f. \end{aligned}$$

Hence, C_T is of (m,n)-class $\mathcal{Q}$ operator if and only if $m^{\frac{2}{n+1}} h_{n+1} - (n+1)h + m^{\frac{2}{n+1}} n I \geq 0$. That is, C_T is of (m,n)-class $\mathcal{Q}$ operator if and only if $m^{\frac{2}{n+1}}(h_{n+1} + n) \geq (n+1)h$. □

Theorem 3.3. *C_T^* is a (m,n)-class $\mathcal{Q}$ operator if and only if $m^{\frac{2}{n+1}}(h_{n+1}\circ T^{n+1}+n)\geq (n+1)h\circ T$.*

Proof. By definition, C_T^* is a (m,n)-class $\mathcal{Q}$ operator if and only if

$$m^{\frac{2}{n+1}}C_T^{n+1}C_T^{*n+1}-(n+1)C_TC_T^*+m^{\frac{2}{n+1}}\ n\,I\geq 0.$$

By Proposition 3.1, $(n+1)C_TC_T^*f=(n+1)(h\circ T)Pf=(n+1)(h\circ T)f$ and

$$\begin{aligned} m^{\frac{2}{n+1}}C_T^{n+1}C_T^{*n+1}f &= m^{\frac{2}{n+1}}C_T^{n+1}(h_{n+1}.E(f)\circ T^{-(n+1)}\\ &= m^{\frac{2}{n+1}}(h_{n+1}.E(f)\circ T^{-(n+1)})\circ T^{n+1}\\ &= m^{\frac{2}{n+1}}h_{n+1}\circ T^{n+1}E(f)\\ &= m^{\frac{2}{n+1}}h_{n+1}\circ T^{n+1}f. \end{aligned}$$

Thus,
C_T^* is a (m,n)-class $\mathcal{Q}$ operator if and only if $m^{\frac{2}{n+1}}h_{n+1}\circ T^{n+1}-(n+1)(h\circ T)+m^{\frac{2}{n+1}}nI\geq 0$.
That is, C_T^* is a (m,n)-class $\mathcal{Q}$ operator if and only if $m^{\frac{2}{n+1}}(h_{n+1}\circ T^{n+1}+n)\geq (n+1)h\circ T$. □

Example 3.4. Let $X=\mathbb{N}\cup\{0\}$, $\mathcal{A}=P(X)$ and μ be the measure defined by

$$\mu(A)=\sum_{k\in A}m_k,$$

where

$$m_k=\begin{cases}1 & \text{if } k=0\\ \dfrac{1}{4^{k-1}} & \text{if } k\geq 1.\end{cases}$$

Let $T:X\to X$ defined by

$$T(k)=\begin{cases}0, & k=0,1\\ k-1, & k\geq 2.\end{cases}$$

For $q>1$, we have

$$T^q(k)=\begin{cases}0, & k=0,1,2,\dots,q\\ k-q, & k\geq q+1.\end{cases}$$

Therefore, $h(k)=\frac{\mu T^{-1}(\{k\})}{\mu\{k\}}=\begin{cases}2, & k=0\\ \frac{1}{4}, & k\geq 1.\end{cases}$

For $q>1$,

$$h_q(k)=\begin{cases}2+\frac{1}{4}+\frac{1}{4^2}+\ldots+\frac{1}{4^{q-1}} & k=0.\\ \frac{1}{4^q}, & k\geq 1.\end{cases}$$

If $m\geq 2$ and $n=1$, then $m(h_2+1)\geq 2h$. Hence C_T is (m,n)-class $\mathcal{Q}$ operator.

Now we characterize for (m,n)-class $\mathcal{Q}$ weighted composition operators.

Proposition 3.5. *[1]*
If W is a weighted composition operator, then the following holds for every $f\in L^2(\mu)$ and $\pi>0$.

(i) $W^*W(f)=hE(\pi^2)\circ T^{-1}(f)$.

(ii) $WW^*(f)=\pi(h\circ T)E(\pi f)$.

Theorem 3.6. *W is a (m,n)-class $\mathcal{Q}$ operator if and only if*

$$m^{\frac{2}{n+1}}((h_{n+1}\ E_{n+1}(\pi_{n+1}^2)\circ T^{-(n+1)}+n)\geq (n+1)hE(\pi^2)\circ T^{-1}.$$

Proof. W is a (m,n)-class $\mathcal{Q}$ operator if and only if

$$m^{\frac{2}{n+1}} W^{*n+1} W^{n+1} - (n+1) W^* W + m^{\frac{2}{n+1}}\; n\, I \geq 0.$$

By Proposition 3.5, $(n+1)W^*Wf = (n+1)hE(\pi^2) \circ T^{-1} f$.
Since $W^{*k}W^k(f) = h_k E_k(\pi_k^2) \circ T^{-k}(f)$, we have

$$m^{\frac{2}{n+1}} W^{*n+1} W^{n+1} f = m^{\frac{2}{n+1}}\; h_{n+1}\; E_{n+1}(\pi_{n+1}^2) \circ T^{-(n+1)} f$$

Hence, W is a (m,n)-class $\mathcal{Q}$ operator if and only if

$$m^{\frac{2}{n+1}} h_{n+1}\; E_{n+1}(\pi_{n+1}^2) \circ T^{-(n+1)} - (n+1) hE(\pi^2) \circ T^{-1} + m^{\frac{2}{n+1}} nI \geq 0.$$

That is, W is a (m,n)-class $\mathcal{Q}$ operator if and only if

$$m^{\frac{2}{n+1}} ((h_{n+1}\; E_{n+1}(\pi_{n+1}^2) \circ T^{-(n+1)} + n) \geq (n+1) hE(\pi^2) \circ T^{-1}.$$

□

The following results hold by similar arguments:

(i) C_T is a (m,n)-class $\mathcal{Q}^*$ operator if and only if $m^{\frac{2}{n+1}}(h_{n+1} + n) \geq (n+1) h \circ T$.

(ii) C_T^* is a (m,n)-class $\mathcal{Q}^*$ operator if and only if $m^{\frac{2}{n+1}}(h_{n+1} \circ T^{n+1} + n) \geq (n+1)h$.

Example 3.7. In example 3.4, if we choose $m \geq 4$ and $n = 1$ then C_T is a (m,n)-class $\mathcal{Q}^*$ operator.

Theorem 3.8. *W is a (m,n)-class $\mathcal{Q}^*$ operator if and only if*

$$\langle m^{\frac{2}{n+1}} (h_{n+1}\; E_{n+1}(\pi_{n+1}^2) \circ T^{-(n+1)} + n) f - (n+1)\pi(h \circ T) E(\pi f), f \rangle \geq 0.$$

Proof. W is a (m,n)-class $\mathcal{Q}^*$ operator if and only if

$$m^{\frac{2}{n+1}} W^{*n+1} W^{n+1} - (n+1) WW^* + m^{\frac{2}{n+1}}\; n\, I \geq 0.$$

By Proposition 3.5, $WW^*(f) = \pi(h \circ T)E(\pi f)$. Since $W^{*k}W^k(f) = h_k E_k(\pi_k^2) \circ T^{-k}(f)$, it follows that $\langle m^{\frac{2}{n+1}} (h_{n+1}\; E_{n+1}(\pi_{n+1}^2) \circ T^{-(n+1)} + n) f - (n+1)\pi(h \circ T) E(\pi f), f \rangle \geq 0$. □

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