Palestine Journal of Mathematics

Vol. 12(2)(2023) , 361–365

Spacetime classification by Ricci soliton and Gradient Ricci soliton

Ajit Barman

Communicated by Zafar Ahsan

MSC 2020 Classifications: 53C15, 53C25.

Keywords and phrases: Ricci soliton, Gradient Ricci soliton, Perfect fluid, concurrent vector field, parallel vector field, sectional curvature, Einstein's field equation.

Abstract
In the present paper, we have discussed Ricci soliton and gradient Ricci soliton in spacetime. Through these discussions, some new valuable things were found which will further enrich the Ricci soliton and gradient Ricci soliton in spacetime.

1 Introduction

In relativity and cosmology applied physics, semi-Riemannian manifolds (M) have many uses that are part of various modern geometries. Self-identifying solutions and translation solutions, often referred to as soliton solutions, have emerged as important subject matter in recent years as geometric flows as they present singularity models.There has been a lot of progress recently in the study of Ricci flow (i.e., Ricci soliton) and the mean curvature flow.

A Ricci soliton [8] is a natural generalisation of the Einstein metric. The Ricci tensor is a constant multiple of the semi-Riemannian metric g. A Ricci soliton is defined on a semi-Riemannian manifold as

$$L_V^* g(X,Y) + S_{ric}(X,Y) + \mu^* g(X,Y) = 0, \tag{1.1}$$

where L^* denotes the Lie derivative of semi-Riemannian metric g along a vector field V and μ^* is a constant.

A Ricci soliton is said to be shrinking, steady, or expanding depending on whether μ^* is negative, zero, or positive.

A gradient steady Ricci soliton is a semi-Riemannian manifold [5]. If a smooth function F^* exists such that the Ricci tensor of the metric is given by the Hessian of F^* :

$$S_{ric}(X,Y) = \nabla_X^* \nabla_X^* F^*, \tag{1.2}$$

where ∇^* be the Levi-Civita connection on manifolds. Gradient steady solitons play an important role in Hemilton's Ricci flow because they are compatible with translation solutions.

The result of the tireless work of many authors is Ricci Soliton, they are Hamilton [4], Mondal and Shaikh [8], Shaikh and Mondal [12], Shaikh, Mandal and Mondal [13] and many others.

The General relativity given by Einstein's equation [9] is the lower form of flow.

$$S_{ric}(X,Y) - \frac{1}{2} r^* g(X,Y) + \lambda^* g(X,Y) = \kappa^* T_{energy}(X,Y), \tag{1.3}$$

where r^* is the scalar curvature, $T_{energy}(X,Y)$ is the energy-momentum tensor of type $(0,2)$, λ^* is the cosmological constant and κ^* is the gravitational constant. Einstein's gave the equation

0

without the cosmological constant as follows [10]:

$$S_{ric}(X,Y) - \frac{1}{2}r^* g(X,Y) = \kappa^* T_{energy}(X,Y). \tag{1.4}$$

The energy-momentum tensor describes a Perfect fluid [9] if

$$T_{energy}(X,Y) = (\sigma^* + p^*)A(X)A(Y) + \rho^* g(X,Y), \tag{1.5}$$

where σ^* is the energy density and p^* is the fluid's isotropic pressure and 1-form A is defined by $A(X) = g(X, \xi^*)$, $\xi^* \in M$.

A memory carries many authors on the energy momentum tensors of a semi-Riemannian manifold (M^n, g) have studied by Barman [1], Ozen [11], Guler and Altay [3], Mallick, Suh and De ([6], [7]) and many others.

A concurrent vector field W [14] on a manifold is defined as

$$\nabla^*_X W = X, \tag{1.6}$$

for arbitrary vector field X on manifolds.

The present paper is organised as follows: After introduction, in Section 2, we show that if the Ricci solution is combined with the perfect fluid of Einstein's field equation, then we get the scalar curvature, provided the vector field is parallel to other vector fields. In the same section, we investigate whether the manifold of the Ricci soliton, which is a concurrent vector field of Einstein's field equation is flat if and only if the energy momentum tensor is divergence-free. Finally, we study the sectional curvature function, which is identically zero of the Ricci soliton, which is a concurrent vector field of Einstein's field equation and the manifold is equivalent:

(1) M is locally symmetric,

(2) If $L^* : T^*_p(M) \to T^*_q(M)$ is a curvature-preserving local isometry, then there exists an isometry ϕ^* of normal neighbourhoods of p and q such that $d\phi^*_p = L^*$,

(3) The local geodesic symmetry ζ^* is an isometry at each point p of M,

if the energy momentum tensor is divergence-free.

2 Ricci soliton on Spacetimes

Theorem 2.1: If the Ricci solution is combined with the perfect fluid of Einstein's field equation and the vector field is parallel to other vector fields, then the scalar curvature will be $r^* = -2[2A(\xi^*)\kappa^*(\sigma^* + p^*) + \kappa^*\rho^* + \mu^*]$.

Proof. When we sort equations (1.4) and (1.5), we get to write

$$S_{ric}(X,Y) = \kappa^*(\sigma^* + p^*)A(X)A(Y) + (\kappa^*\rho^* + \frac{1}{2}r^*)g(X,Y). \tag{2.1}$$

Again when we do equations (1.1) and (2.1) together, we see that

$$L^*_V g(X,Y) + 2\kappa^*(\sigma^* + p^*)A(X)A(Y) + (\kappa^*\rho^* + \frac{1}{2}r^* + \mu^*)g(X,Y) = 0. \tag{2.2}$$

If we use the definition of Lie derivative for equation (2.2), the new equation will be

$$\begin{aligned} g(\nabla^*_X V, Y) + g(X, \nabla^*_Y V) + 2\kappa^*(\sigma^* + p^*)A(X)A(Y) \\ +(\kappa^*\rho^* + \frac{1}{2}r^* + \mu^*)g(X,Y) = 0. \end{aligned} \tag{2.3}$$

Since if V is parallel to the vector fields of X and Y, i.e.,

$$\nabla^*_X V = \nabla^*_Y V = 0. \tag{2.4}$$

By combining equations (2.3) and (2.4), we can obtain that

$$2\kappa^*(\sigma^* + p^*)A(X)A(Y) + (\kappa^*\rho^* + \frac{1}{2}r^* + \mu^*)g(X,Y) = 0. \tag{2.5}$$

Putting $Y = \xi^*$ in equation (2.5), it follows that

$$[2\kappa^*(\sigma^* + p^*)A(\xi^*) + (\kappa^*\rho^* + \frac{1}{2}r^* + \mu^*)]A(X) = 0. \tag{2.6}$$

From equation (2.6), it can be said that $A(X) \neq 0$ but $r^* = -2[2A(\xi^*)\kappa^*(\sigma^* + p^*) + \kappa^*\rho^* + \mu^*]$. Then the theorem is proved.

Theorem 2.2: The Ricci soliton's manifold M of Einstein's field equation would be flat $(R^*(X,W)V = 0)$ if and only if the energy momentum tensor is divergence-free $[(\nabla^*_W T_{energy})(X,Y) = 0]$, then this vector field must be a concurrent vector field $(\nabla^*_W X = W)$.

Proof. From equation (1.1) and equation (1.4), it implies that

$$L^*_V g(X,Y) + 2\kappa^* T_{energy}(X,Y) + (r^* + \mu^*)g(X,Y) = 0. \tag{2.7}$$

From the definition of Lie derivative for equation (2.7), the equation will be changed as follows:

$$g(\nabla^*_X V, Y) + g(X, \nabla^*_Y V) + 2\kappa^* T_{energy}(X,Y) + (r^* + \mu^*)g(X,Y) = 0. \tag{2.8}$$

Taking the covariant derivative of the equation (2.8), we obtain that

$$\begin{aligned} \nabla^*_W g(\nabla^*_X V, Y) - g(\nabla^*_W \nabla^*_X V, Y) - g(\nabla^*_X V, \nabla^*_W Y) \\ +\nabla^*_W g(X, \nabla^*_Y V) - g(\nabla^*_W X, \nabla^*_Y V) - g(X, \nabla^*_W \nabla^*_Y V) \\ +2\kappa^*(\nabla^*_W T_{energy})(X,Y) = 0. \end{aligned} \tag{2.9}$$

Interchange X and W by W and X respectively, in the equation (2.9), we get that

$$\begin{aligned} \nabla^*_X g(\nabla^*_W V, Y) - g(\nabla^*_X \nabla^*_W V, Y) - g(\nabla^*_W V, \nabla^*_X Y) \\ +\nabla^*_X g(W, \nabla^*_Y V) - g(\nabla^*_X W, \nabla^*_Y V) - g(W, \nabla^*_X \nabla^*_Y V) \\ +2\kappa^*(\nabla^*_X T_{energy})(W,Y) = 0. \end{aligned} \tag{2.10}$$

Adding Equation (2.9) and Equation (2.10), we can write the new equation

$$\begin{aligned} R^*(X,W,V,Y) = 2\kappa^*[(\nabla^*_W T_{energy})(X,Y) + (\nabla^*_X T_{energy})(W,Y)] \\ -g(\nabla^*_{[X,W]} V, Y) + \nabla^*_W g(\nabla^*_X V, Y) + g(\nabla^*_X V, \nabla^*_W Y) \\ -\nabla^*_W g(X, \nabla^*_Y V) + g(\nabla^*_W X, \nabla^*_Y V) + g(X, \nabla^*_W \nabla^*_Y V) \\ +\nabla^*_X g(\nabla^*_W V, Y) - g(\nabla^*_W V, \nabla^*_X Y) + \nabla^*_X g(W, \nabla^*_Y V) \\ -g(\nabla^*_X W, \nabla^*_Y V) - g(W, \nabla^*_X \nabla^*_Y V), \end{aligned} \tag{2.11}$$

where $R^*(X,W,V,Y) = g(R^*(X,W)V,Y)$ and $R^*(X,W)V$ is the manifold's curvature tensor. Using equation (1.6) and equation (2.11), we can conclude that

$$R^*(X,W,V,Y) = 2\kappa^*[(\nabla^*_W T_{energy})(X,Y) + (\nabla^*_X T_{energy})(W,Y)].$$

Since the energy momentum tensor is divergence-free [2], i.e., $(\nabla^*_W T_{energy})(X,Y) = (\nabla^*_X T_{energy})(W,Y) = 0$. The proof of the theorem is completed.

Proposition 2.1: [9] A semi-Riemannian manifold M is flat if and only if the sectional curvature function K^* is identically zero.

From the Theorem 2.2 and Proposition 2.1, we have

Theorem 2.3: The Ricci soliton's sectional curvature function K^*, which is a concurrent vector field of Einstein's field equation, is identically zero if the energy momentum tensor is divergence-free $[(\nabla^*_W T_{energy})(X,Y) = 0]$.

Corollary 2.1: [9] A semi-Riemannian manifold of constant sectional curvature is locally symmetric.

Using Theorem 2.3 and Corollary 2.1, we get a new theorem.

Theorem 2.4: The Ricci soliton's manifold M, a concurrent vector field of Einstein's field equations, is locally symmetric $(\nabla^* R^* = 0)$ but the energy momentum tensor is divergence-free.

Corollary 2.2: [9] A semi-Riemannian manifold M are equivalent:
(1) M is locally symmetric,
(2) If $L^* : T^*_p(M) \to T^*_q(M)$ is a local isometry that preserves curvature, then there is an isometry ϕ^* of normal neighbourhoods of p and q such that $d\phi^*_p = L^*$,
(3) At each point p of M the local geodesic symmetry ζ^* is an isometry.

From Theorem 2.4 and Corollary 2.2, we can write

Theorem 2.5 The manifold M of the Ricci soliton, which is a concurrent vector field of Einstein's field equation, is equivalent to:
(1) M is locally symmetric,
(2) If $L^* : T^*_p(M) \to T^*_q(M)$ is a curvature-preserving local isometry, then there exists an isometry ϕ^* of normal neighbourhoods of p and q such that $d\phi^*_p = L^*$,
(3) The local geodesic symmetry ζ^* is an isometry at each point p of M,
if the energy momentum tensor is divergence-free.

Acknowledgement. The author is grateful to the referee for his/ her valuable comments on the improvement of my paper.

References

[1] A. Barman, Some properties of a semiconformal curvature tensor on a Riemannian manifold, *Mathematics Student,* **91**(2022), 201-208.

[2] K. L. Duggal and R. Sharma, Symmetries of spacetimes and Riemannian manifolds, *Springer-Science Business media,* 1999, 61-63.

[3] S. Guler and S. A. Altay, On Ricci symmetric generalized quasi-Einstein space-times, *Miskolc Math. notes,* **16**(2015), 853-868.

[4] R. S. Hamilton, The Ricci Flow on Surfaces, *Contemporary Mathematics,* **71**(1988), 237-261.

[5] R. S. Hamilton, The formation of singularities in the Ricci flow, *Surveys in Differential Geometry, International Press, Cambridge,* **2**(1995), 7-13.

[6] S. Mallick, Y. J. Suh and U. C. De, A spacetime with pseudo-projective curvature tensor, *Journal of Mathematical Physics,* **57**(2016), 1-10.

[7] S. Mallick, Y. J. Suh and U. C. De, Spacetimes with different forms of energy momentum tensor, *Journal of Geometry and Physics,* **151**(2020), 1-8.

[8] C. K. Mondal and A. A. Shaikh, On Ricci solitons whose potential is convex, *Proc. Indian Acad.Sci. (Math. Sci.),* **130(55)**(2020), 1-7.

[9] B. O'Neill, Semi-Riemannian geometry with applications to relativity, *Academic Press,* p-79, 221, 223.

[10] A. Z. Petrov, Einstein spaces, *Pergamon Press, Oxford,* 1949.

[11] F. Z. Ozen, m-projecttively flat spacetimes, *Math. Reports,* **14**(2012), 363-370.

[12] A. A. Shaikh and C. K. Mondal, Isometry theorem of gradient Shrinking Ricci solitons, *J. Geom. Phys.,* **163**(2021), 104-110.

[13] A. A. Shaikh, P. Mandal and C. K. Mondal, Existence of finite global norm of potential vector field in a Ricci soliton, arXiv:2102.08188.

[14] S. I. Tachibana, On Finsler spaces which admit a concurrent vector field, *Tensor (N.S.),* **1**(1950), 1–5.

Author information

Ajit Barman, Department of Mathematics, Ramthakur College, Arundhuti Nagar- 799003, West Tripura, Tripura, India.
E-mail: ajitbarmanaw@yahoo.in

Received: 2022-05-14
Accepted: 2022-07-13