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A note on the isoperimetric inequality in the plane

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Abstract It is well known that among all closed bounded curves in the plane with the given perimeter, the circle encloses the maximum area. There are many proofs in the literature. In this article, we have provided a new proof using some ideas of Demar [5].

1 Introduction

The isoperimetric inequality is a very well-known geometric problem. The problem is finding a simple closed curve that encloses the maximum area than any other simple closed curves with the same length. From ancient times it has been known that the solution is the circle. There is a famous story about this problem. The story dates back the problem to Virgil's epic Latin poem The *Aeneid* [11], written in the period 29 BC to 19 BC. This poem is about the foundation of the city of Carthage. The Greeks [10] were the first who considered this problem and tried to solve it.

Many mathematicians from ancient times tried to solve this problem. It was believed that Zenodours [4] in the second century first gave a complete proof. Later Steiner, Euler, Weierstrass, and many mathematicians proved it using different techniques. For complete history and some proofs see Blåsjö [2], Osserman [9].

In 1975, Demar[5] proved the isoperimetric inequality in the plane using some geometric arguments. He proved that if a closed continuous curve solves the isoperimetric inequality in the plane, then any triangle formed by the intersection of two tangents at any two points and the chord joining them is isosceles. And he then showed that every point on that curve at which a tangent can be drawn lies at the same distance from a fixed point inside the curve. In this article, we only use the first idea of Demar[5] that all such triangles are isosceles, and then we shall prove that only for the region having maximum area all such triangles with equal base are congruent.

2 Isoperimetric Inequality in the Plane.

Our main target is to prove the following theorem.

Theorem 2.1 (Isoperimetric Inequality). *If γ be a simple closed curve in the plane with length L and bounds a region of area A, then*

$$L^2 \geq 4\pi A,$$

where the equality holds if and only if γ is a circle.

It is well known that the solution is a convex region in the plane [2, 5]. So only the convex regions will be focused on here. The existence of the solution to the above inequality can be proved easily by using compactness.

Let $\mathbb{B}$ be the set of all nonempty compact subsets in $\mathbb{R}^2$. The Hausdorff distance [6] in $\mathbb{B}$ is defined by

$$HD(A,B) = \max\left\{\sup_{a\in A} \text{dist}(a,B),\ \sup_{b\in B} \text{dist}(A,b)\right\}$$
$$= \sup_{x\in\mathbb{R}^2} |\text{dist}(x,A) - \text{dist}(x,B)|.$$

Then $(\mathbb{B}, HD)$ becomes a metric space [6]. The following compactness theorem will solve the existence problem.

Theorem 2.2. [8, Theorem 4.1] *The space of all compact subsets of a ball in $\mathbb{R}^2$ is compact under the Hausdorff metric.*

Proof of the Existence of Solution of Theorem 2.1. Let γ be a simple closed curve of length L. Then γ, by translation, can be put in a planar disk D of radius $2L$. Take $\mathbb{B}$ as the set of all simple closed curves of length L. Then by the Theorem 2.2 $(\mathbb{B}, HD)$ is compact. Now the function $f : \mathbb{B} \to \mathbb{R}$, defined by

$$f(\gamma) = \text{Area enclosed by } \gamma$$

is continuous. We note that the function f is not onto since $\mathbb{B}$ does not contain any curve of area more than $4\pi L^2$. Since $\mathbb{B}$ is compact, hence $max(f)$ exists. So there is a simple closed curve with length L having maximum area. □

Let γ be a simple closed curve bounding a convex region R. Now standing at any point A on the curve γ and facing the region R, there is a right-hand and left-hand direction along γ from A. A right-hand tangent at A is a ray from A tangent to γ in the right-hand direction, and similarly, a left-hand tangent can be defined [7]. Now the interior angle between these two tangents, i.e., the angle between right hand and left-hand tangent at any point, measured in the sector containing R, is at most π for a convex region [5].

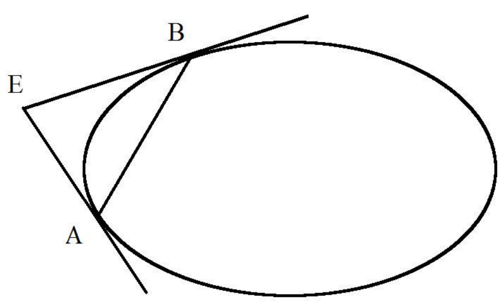

Figure 1.

Consider two points A and B on γ such that tangents at A and B meets each other at E. If the length AB is l then we call the triangle $\triangle ABE$, the tangent triangle on the curve $\widehat{AB}$ of base length l and is denoted it by $\triangle_l \widehat{AB}$, see Figure 1. It is not always possible that for any base length, we get a tangent triangle, e.g., there is no tangent triangle of a circle with a base length the same as the diameter. But for a sufficiently small length, it is possible to construct a tangent triangle at any two points on the curve by taking this length as the base of the triangle. Now for the general convex curve, two tangent triangles with equal base length may not be congruent, but in a circle, all the tangent triangles with equal bases are congruent.

Proof of Theorem 2.1. We prove this by contradiction. Let γ be a simple closed curve of length L, enclosing a convex region R. Let us assume that R is not circular and has the property that it has a maximum area than any other region with the same boundary length L. Since the set of points where the convex curve is not differentiable is at most countable [3, pp. 154] hence there is a dense set of points of γ where tangents can be drawn. Demar [5] proved that any tangent triangle of R is isosceles, see Figure 2. We will show that all the tangent triangles having the same base length are congruent.

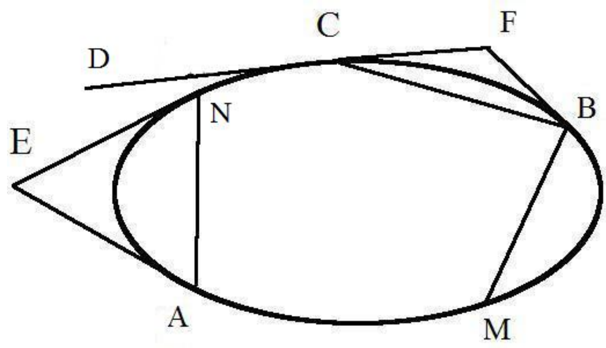

Figure 2.

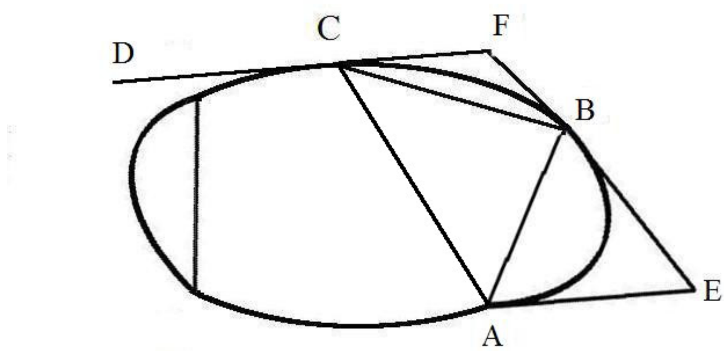

Figure 3.

Let the two tangent triangles $\triangle_r\widehat{BC}$ and $\triangle_r\widehat{AN}$ are not congruent, see Figure 2. Let M be a point on γ such that $BM = r$. Now interchanging the positions of the curves $\widehat{AN}$ and $\widehat{BM}$ we get the curve γ_1 enclosing the region R', see Figure 3. Obviously γ_1 has same length L and occupies the same area as R.

Now we have two adjacent tangent triangles $\triangle_r\widehat{AB}$ and $\triangle_r\widehat{BC}$ which are not congruent. Since their base are equal hence $\angle EAB \neq \angle FCB$. Assume that

$$\angle EAB > \angle FCB.$$

Now the triangle $\triangle ABC$ is isosceles. So we get

$$\angle EAC > \angle FCA.$$

Now reflect the whole area $ACFBE$ with respect to the perpendicular bisector of AC. Then we get a new curve γ' by replacing $\widehat{AC}$ by its image $\widehat{A'C'}$ under the reflection. Hence we get a new region R'', see Figure 4, and since reflection does not change area and perimeter hence both R' and R'' have same area and perimeter.

Now the image of the triangles $\triangle ABE$, $\triangle CFB$ and $\triangle ABC$ under the reflection are $\triangle A'BE'$, $\triangle C'F'B$ and $\triangle BA'C'$ respectively.

Hence we get

$$\angle FCB = \angle F'C'B < \angle E'A'B.$$

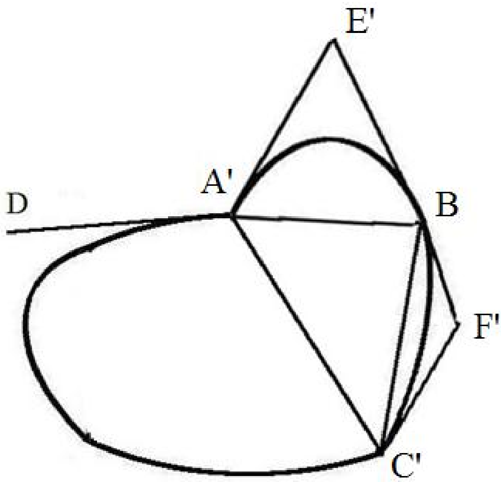

Figure 4.

This implies that

$$\angle FCA = \angle F'C'A' < \angle E'A'C' = \angle EAC,$$

since $\triangle BA'C'$ is isosceles. So combining above two inequalities we get

$$\pi = \angle FCA + \angle ACD < \angle E'A'C' + \angle C'A'D.$$

But $A'E'$ and $A'D$ are right hand and left hand tangents of γ', therefore R'' is not convex. It is a contradiction that R is the region of maximum area among all regions with the same perimeter. Hence all the tangent triangles having equal base are congruent and since all the tangent triangles are isosceles, so γ must be a circle. □

References

[1] R. V. Benson, *Euclidean Geometry and Convexity*, McGraw-Hill, New York (1966).
[2] V. Blåsjö, The Isoperimetric Problem, *Amer. Math. Monthly*, **112(6)**, 526–566 (2005).
[3] H. Busemann, *The Geometry of Geodesics*, Academic Press, New York (1955).
[4] J. L. Coolidge, *A History of Geometrical Methods*, Clarendon Press, Oxford (1940).
[5] R. F. Demar, A Simple Approach to Isoperimetric Problems in the Plane, *Math. Mag.* **48**, 1–12 (1975).
[6] S. G. Krantz, H. R. Parks, *Geometric Integration Theory*, Springer, Boston (2008).
[7] M. A. Krasnosel'skiĭ, Ya. B. Rutickiĭ, *Convex Functions and Orlicz Spaces*, Noordhoff Ltd. Groningen (1961).
[8] F. Morgan, Compactness, *Pro. Mathematica*, **22(43-44)**, 121–131 (2008).
[9] R. Osserman, The Isoperimetric Inequality, *Bull. Amer. Math. Soc.* **86(6)**, 1182–1238 (1978).
[10] T. I. Porter, *A History of the Classical Isoperimetric Problem*, Chicago Univ. Press, Chicago (1933).
[11] D. Witt, N. Wentworth, *The Dido Episode in the Aeneid of Virgil*, Chicago Univ. Press, Chicago (1907).

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