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INDUCED RICCI TYPE TENSORS OF GCR-LIGHTLIKE SUBMANIFOLDS OF INDEFINITE SASAKIAN MANIFOLDS

Varun Jain, Satvinder Singh Bhatia and Rakesh Kumar

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Abstract We derive the expressions for induced Ricci type tensors of a GCR-lightlike submanifold of an indefinite Sasakian space form and obtain some characterization theorems for it.

1 Introduction

In 1978, Bejancu [1] introduced the notion of CR-submanifolds of Kaehler manifolds as a generalization of invariant and anti-invariant submanifolds. Later, contact CR-submanifolds of Sasakian manifolds were introduced by Bejancu et al. [2] in 1981. Since contact geometry has vital roles in the theory of differential equations, optics and phase spaces of a dynamical system therefore contact geometry with definite and indefinite metric becomes the topic of main discussion. The CR structures on real hypersurfaces of complex manifolds have interesting applications in the general theory of relativity. For example, Penrose correspondence [18] is a correspondence between points of a Minkowski space and projective lines of a certain real hypersurfaces in a complex projective space, which is an interesting means of passing from the geometry of a Minkowski space to the geometry of a CR manifold. Duggal [4, 5] studied the geometry of CR submanifolds with Lorentzian metric and obtained their interaction with relativity. Theory of contact CR-lightlike and contact SCR-lightlike submanifolds of indefinite Sasakian manifolds was introduced by Duggal and Sahin [7]. But there was no inclusion relation between invariant and screen real submanifolds, so a new class of submanifolds called, Generalized Cauchy-Riemann (GCR)-lightlike submanifolds of indefinite Sasakian manifolds (which is an umbrella of invariant, screen real, contact CR-lightlike submanifolds) was derived by Duggal and Sahin [8] and further developed in [10]–[15], [17]. Since contact geometry and Ricci tensor have remarkable applications in mathematical physic and geometry, therefore we derive the expression of induced Ricci type tensor of a GCR-lightlike submanifold of an indefinite Sasakian space form and obtain some characterization theorems for it.

2 Lightlike Submanifolds

Let $(\overline{M}, \overline{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\overline{M}$ and g be the induced metric of $\overline{g}$ on M. If $\overline{g}$ is degenerate on the tangent bundle TM of M then M is called a lightlike submanifold of $\overline{M}$. For a degenerate metric g on M, $TM^{\perp}$ is a degenerate n-dimensional subspace of $T_x\overline{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $Rad(T_xM) = T_xM \cap T_xM^{\perp}$ which is known as radical (null) subspace. If the mapping $Rad(TM) : x \in M \longrightarrow Rad(T_xM)$, defines a smooth distribution on M of rank $r > 0$ then the submanifold M of $\overline{M}$ is called an r-lightlike submanifold and $Rad(TM)$ is called the radical distribution on M, (for detail see [6]).

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, that is,

$$TM = Rad(TM) \perp S(TM), \tag{2.1}$$

and $S(TM^\perp)$ is a complementary vector subbundle to $Rad(TM)$ in $TM^\perp$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\overline{M}\mid_M$ and to $Rad(TM)$ in $S(TM^\perp)^\perp$, respectively. Then

$$tr(TM) = ltr(TM) \perp S(TM^\perp), \tag{2.2}$$

and

$$T\overline{M}\mid_M = TM \oplus tr(TM) = \big(Rad(TM) \oplus ltr(TM)\big) \perp S(TM) \perp S(TM^\perp). \tag{2.3}$$

Let $\mathcal{U}$ be a local coordinate neighbourhood of M and consider a local quasi-orthonormal field of frames of $\overline{M}$ along M on $\mathcal{U}$ as $\{\xi_1, ..., \xi_r, W_{r+1}, ..., W_n, N_1, ..., N_r, X_{r+1}, ..., X_m\}$, where $\{\xi_1, ..., \xi_r\}, \{N_1, ..., N_r\}$ are local lightlike bases of $\Gamma(Rad(TM)\mid_{\mathcal{U}})$, $\Gamma(ltr(TM)\mid_{\mathcal{U}})$, respectively and $\{W_{r+1}, ..., W_n\}, \{X_{r+1}, ..., X_m\}$ are local orthonormal bases of $\Gamma(S(TM^\perp)\mid_{\mathcal{U}})$ and $\Gamma(S(TM)\mid_{\mathcal{U}})$, respectively. Then for this quasi-orthonormal fields of frames, we have the following important observation.

Theorem 2.1. *([6]). Let $(M, g, S(TM), S(TM^\perp))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then, there exists a complementary vector bundle $ltr(TM)$ of $Rad(TM)$ in $S(TM^\perp)^\perp$ and a basis of $\Gamma(ltr(TM)\mid_{\mathcal{U}})$ consisting of smooth section $\{N_i\}$ of $S(TM^\perp)^\perp\mid_{\mathcal{U}}$, where $\mathcal{U}$ is a coordinate neighborhood of M, such that*

$$\overline{g}(N_i, \xi_j) = \delta_{ij}, \quad \overline{g}(N_i, N_j) = 0, \quad \textit{for any } i, j \in \{1, 2, .., r\},$$

where $\{\xi_1, ..., \xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$.

Let $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$. Then according to the decomposition (2.3), the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \overline{\nabla}_X U = -A_U X + \nabla^\perp_X U,$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^\perp_X U\}$ belongs to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$, which is called the second fundamental form and A_U is a linear operator on M, known as the shape operator.

Considering the projection morphisms $\mathcal{L}$ and $\mathcal{S}$ of $tr(TM)$ on $ltr(TM)$ and $S(TM^\perp)$, respectively then Gauss and Weingarten formulae become

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad \overline{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{2.4}$$

where $h^l(X, Y) = \mathcal{L}(h(X, Y))$, $h^s(X, Y) = \mathcal{S}(h(X, Y))$, $D^l_X U = \mathcal{L}(\nabla^\perp_X U)$, $D^s_X U = \mathcal{S}(\nabla^\perp_X U)$.

As h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^\perp))$-valued respectively, therefore they are called as the lightlike second fundamental form and the screen second fundamental form on M. In particular

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \tag{2.5}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{2.6}$$

where $X \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^\perp))$. Using (2.2), (2.3) and (2.4)-(2.6), we obtain

$$\overline{g}(h^s(X, Y), W) + \overline{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{2.7}$$

$$\overline{g}(h^l(X, Y), \xi) + \overline{g}(Y, h^l(X, \xi)) + g(Y, \nabla_X \xi) = 0, \tag{2.8}$$

$$\overline{g}(D^s(X, N), W) = \overline{g}(N, A_W X), \tag{2.9}$$

$$\overline{g}(A_N X, N') + \overline{g}(N, A_{N'} X) = 0, \tag{2.10}$$

for any $\xi \in \Gamma(Rad(TM))$, $W \in \Gamma(S(TM^{\perp}))$ and $N, N' \in \Gamma(ltr(TM))$.

Let $\overline{P}$ be the projection morphism of TM on $S(TM)$. Then using (2.1), we can induce some new geometric objects on the screen distribution $S(TM)$ of M as

$$\nabla_X \overline{P}Y = \nabla^*_X \overline{P}Y + h^*(X,Y), \quad \nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi,$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla^*_X \overline{P}Y, A^*_\xi X\}$ and $\{h^*(X,Y), \nabla^{*t}_X \xi\}$ belongs to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$, respectively. ∇^* and ∇^{*t} are linear connections on complementary distributions $S(TM)$ and $Rad(TM)$, respectively. h^* and A^* are $\Gamma(Rad(TM))$-valued and $\Gamma(S(TM))$-valued bilinear forms and called as the second fundamental forms of distributions $S(TM)$ and $Rad(TM)$, respectively. By using above equation, we obtain

$$\overline{g}(h^l(X, \overline{P}Y), \xi) = g(A^*_\xi X, \overline{P}Y), \quad \overline{g}(h^*(X, \overline{P}Y), N) = g(A_N X, \overline{P}Y). \tag{2.11}$$

Denote by $\overline{R}$ and R the curvature tensors of $\overline{\nabla}$ and ∇, respectively then by straightforward calculations ([6]), we have

$$\begin{aligned} R(X,Y)Z \;=\; & \overline{R}(X,Y)Z - A_{h^l(X,Z)}Y + A_{h^l(Y,Z)}X - A_{h^s(X,Z)}Y + A_{h^s(Y,Z)}X \\ & -(\nabla_X h^l)(Y,Z) + (\nabla_Y h^l)(X,Z) - D^l(X, h^s(Y,Z)) + D^l(Y, h^s(X,Z)) \\ & -(\nabla_X h^s)(Y,Z) + (\nabla_Y h^s)(X,Z) - D^s(X, h^l(Y,Z)) \\ & +D^s(Y, h^l(X,Z)). \end{aligned} \tag{2.12}$$

An odd dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ is called an ϵ-contact metric manifold, if there exist a $(1,1)$ tensor field ϕ, a vector field V, called the characteristic vector field, and a 1-form η such that (see [16])

$$\overline{g}(\phi X, \phi Y) = \overline{g}(X,Y) - \epsilon\eta(X)\eta(Y), \quad \overline{g}(V,V) = \epsilon,$$

$$\phi^2(X) = -X + \eta(X)V, \quad \overline{g}(X,V) = \epsilon\eta(X), \quad d\eta(X,Y) = \overline{g}(X, \phi Y),$$

for any $X, Y \in \Gamma(TM)$,where $\epsilon = \pm 1$, then it follows that

$$\phi V = 0, \quad \eta o \phi = 0, \quad \eta(V) = \epsilon,$$

and $(\phi, V, \eta, \overline{g})$ is called an ϵ-contact metric structure of $\overline{M}$. We say that $\overline{M}$ has a normal contact structure if $N_\phi + d\eta \otimes V = 0$, where N_ϕ is Nijenhuis tensor field of ϕ. A normal ϵ-contact metric manifold is called an ϵ-Sasakian manifold and for an ϵ-Sasakian manifold, we have

$$\overline{\nabla}_X V = -\phi X, \quad (\overline{\nabla}_X \phi)Y = -\overline{g}(X,Y)V + \epsilon\eta(Y)X.$$

Let $\overline{M}$ be an indefinite Sasakian space form $\overline{M}(c)$ of constant ϕ-holomorphic sectional curvature c then the curvature tensor $\overline{R}$ of $\overline{M}(c)$ is given by

$$\begin{aligned} \overline{g}(\overline{R}(X,Y)Z, W) \;=\; & \frac{(c+3)}{4}\{\overline{g}(Y,Z)\overline{g}(X,W) - \overline{g}(X,Z)\overline{g}(Y,W)\} \\ & + \frac{(c-1)}{4}\{\epsilon\eta(X)\eta(Z)\overline{g}(Y,W) - \epsilon\eta(Y)\eta(Z)\overline{g}(X,W) \\ & +\epsilon\eta(Y)\eta(W)\overline{g}(X,Z) - \epsilon\eta(X)\eta(W)\overline{g}(Y,Z) + \overline{g}(\phi Y, Z)\overline{g}(\phi X, W) \\ & +\overline{g}(\phi Z, X)\overline{g}(\phi Y, W) - 2\overline{g}(\phi X, Y)\overline{g}(\phi Z, W)\}, \end{aligned} \tag{2.13}$$

now using (2.12), we get

$$\begin{aligned}
\overline{g}(R(X,Y)Z,W) &= \frac{(c+3)}{4}\{\overline{g}(Y,Z)\overline{g}(X,W)-\overline{g}(X,Z)\overline{g}(Y,W)\}\\
&+\frac{(c-1)}{4}\{\epsilon\eta(X)\eta(Z)\overline{g}(Y,W)-\epsilon\eta(Y)\eta(Z)\overline{g}(X,W)\\
&+\epsilon\eta(Y)\eta(W)\overline{g}(X,Z)-\epsilon\eta(X)\eta(W)\overline{g}(Y,Z)+\overline{g}(\phi Y,Z)\overline{g}(\phi X,W)\\
&+\overline{g}(\phi Z,X)\overline{g}(\phi Y,W)-2\overline{g}(\phi X,Y)\overline{g}(\phi Z,W)\}-\overline{g}(A_{h^l(X,Z)}Y,W)\\
&+\overline{g}(A_{h^l(Y,Z)}X,W)-\overline{g}(A_{h^s(X,Z)}Y,W)+\overline{g}(A_{h^s(Y,Z)}X,W)\\
&-\overline{g}((\nabla_X h^l)(Y,Z),W)+\overline{g}((\nabla_Y h^l)(X,Z),W)\\
&-\overline{g}(D^l(X,h^s(Y,Z)),W)+\overline{g}(D^l(Y,h^s(X,Z)),W)\\
&-\overline{g}((\nabla_X h^s)(Y,Z),W)+\overline{g}((\nabla_Y h^s)(X,Z),W)\\
&-\overline{g}(D^s(X,h^l(Y,Z)),W)+\overline{g}(D^s(Y,h^l(X,Z)),W). \qquad (2.14)
\end{aligned}$$

3 Generalized Cauchy-Riemann Lightlike Submanifold

Calin [3] proved that if the characteristic vector field V is tangent to M then it belongs to $S(TM)$. So, we assume characteristic vector V is tangent to M throughout this paper.

Definition 3.1. Let $(M, g, S(TM))$ be a real lightlike submanifold of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$. Then, M is called a generalized Cauchy-Riemann (GCR)-lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:

(A) there exist two subbundles D_1 and D_2 of $Rad(TM)$ such that

$$Rad(TM) = D_1 \oplus D_2, \quad \phi(D_1) = D_1, \quad \phi(D_2) \subset S(TM),$$

(B) there exist two subbundles D_0 and $\overline{D}$ of $S(TM)$ such that

$$S(TM) = \{\phi D_2 \oplus \overline{D}\} \perp D_0 \perp V, \quad \phi(\overline{D}) = L \perp S,$$

where D_0 is an invariant non-degenerate distribution on M, $\{V\}$ is one dimensional distribution spanned by V, L and S are vector subbundles of $ltr(TM)$ and $S(TM)^\perp$, respectively.

Let Q, P_1, P_2 be the projection morphisms on D, ϕS, ϕL respectively, then

$$X = QX + \eta(X)V + P_1X + P_2X,$$

for any $X \in \Gamma(TM)$. Applying ϕ to the above equation, we obtain

$$\phi X = fX + \omega P_1 X + \omega P_2 X, \qquad (3.1)$$

where $fX \in \Gamma(D)$, $\omega P_1 X \in \Gamma(S)$ and $\omega P_2 X \in \Gamma(L)$, or we can write (3.1), as

$$\phi X = fX + \omega X,$$

where fX and ωX are the tangential and the transversal components of ϕX, respectively. Similarly

$$\phi U = BU + CU, \quad U \in \Gamma(tr(TM)), \qquad (3.2)$$

where BU and CU are the sections of TM and $tr(TM)$, respectively. Differentiating (3.1) and using (2.5)-(2.8) and (3.2), we have

$$D^l(X, \omega P_1 Y) = -\nabla^l_X \omega P_2 Y + \omega P_2 \nabla_X Y - h^l(X, fY) + Ch^l(X,Y),$$

$$D^s(X, \omega P_2 Y) = -\nabla^s_X \omega P_1 Y + \omega P_1 \nabla_X Y - h^s(X, fY) + Ch^s(X,Y),$$

for all $X, Y \in \Gamma(TM)$.

Definition 3.2. A GCR-lightlike submanifold of an indefinite Sasakian manifold is called:

(i) a totally geodesic GCR-lightlike submanifold if its second fundamental form h vanishes, that is, $h(X,Y) = 0$, for any $X, Y \in \Gamma(TM)$.

(ii) a D-geodesic GCR-lightlike submanifold if $h(X,Y) = 0$, for any $X, Y \in \Gamma(D)$.

(iii) a $\overline{D}$-geodesic GCR-lightlike submanifold $h(X,Y) = 0$, for any $X, Y \in \Gamma(\overline{D})$.

(iv) a mixed-geodesic GCR-lightlike submanifold if $h(X,Y) = 0$, for any $X \in \Gamma(D)$ and $Y \in \Gamma(\overline{D})$.

4 Induced Ricci Type Tensors of GCR-Lightlike Submanifold

Let $\{E_1, E_2, ..., E_m\}$ be a local orthonormal frame field on M such that $\{E_1, E_2, ..., E_p, E_{p+1} = \phi E_1, E_{p+2} = \phi E_2, ..., E_{2p} = \phi E_p\}$, $\{\xi_1, \xi_2, ..., \xi_s, \xi_{s+1} = \phi\xi_1, \xi_{s+2} = \phi\xi_2, ..., \xi_{2s} = \phi\xi_s\}$, $\{E_0\}$, $\{\xi_{2s+1}, \xi_{2s+2}, ..., \xi_r\}$ and $\{\phi\xi_{2s+1}, \phi\xi_{2s+2}, ..., \phi\xi_r\}$ be local frame fields on D_0, D_1, V, D_2 and ϕD_2 respectively and $\{F_1, F_2, ..., F_q\}$ be a local frame field on $\overline{D}$, then by direct computation, we have

$$\sum_{i=1}^{m} g(U, E_i)g(E_i, W) = g(U, W), \tag{4.1}$$

$$\sum_{i=r+1}^{m} g(U, E_i)g(E_i, W) = g(\overline{P}U, \overline{P}W), \tag{4.2}$$

$$\sum_{i=1}^{m-q} g(U, E_i)g(E_i, W) = g(QU, QW), \tag{4.3}$$

$$\sum_{i=1}^{q} g(U, E_i)g(E_i, W) = g(PU, PW), \tag{4.4}$$

for any $U, W \in \Gamma(TM)$ and where P is the projection morphism on $\overline{D}$. Let $\overline{Ric}$ be the Ricci curvature tensor of $\overline{M}$ and $R^{(0,2)}$ be the induced Ricci type tensor of M and given respectively by

$$\overline{Ric}(X,Y) = trace(Z \to \overline{R}(Z,X)Y), \quad \forall\, X, Y \in \Gamma(T\overline{M}).$$
$$R^{(0,2)}(X,Y) = trace(Z \to R(Z,X)Y), \quad \forall\, X, Y \in \Gamma(TM).$$

Using the decomposition in (2.1), the induced Ricci type tensor is given by

$$R^{(0,2)}(U,W) = \sum_{a=1}^{r} \overline{g}(R(U,\xi_a)W, N_a) + \sum_{b=r+1}^{m} g(R(U,U_b)W, U_b). \tag{4.5}$$

Using (2.14), we obtain

$$\begin{aligned}
\sum_{a=1}^{r} \overline{g}(R(U,\xi_a)W, N_a) \;=\;& -\frac{(c+3)r}{4} g(U,W) + \frac{(c-1)r}{4} \epsilon\eta(U)\eta(W) \\
& -\frac{(c-1)}{2} \sum_{a=1}^{r} \overline{g}(\phi U, \xi_a)\overline{g}(\phi W, N_a) \\
& -\frac{(c-1)}{4} \sum_{a=1}^{r} \overline{g}(\phi W, \xi_a)\overline{g}(\phi U, N_a) \\
& +\frac{(c-1)}{4} \sum_{a=1}^{r} \overline{g}(\phi W, U)\overline{g}(\phi\xi_a, N_a) + \sum_{a=1}^{r} \overline{g}(A_{h^l(\xi_a, W)}U, N_a) \\
& -\sum_{a=1}^{r} \overline{g}(A_{h^l(U,W)}\xi_a, N_a) + \sum_{a=1}^{r} \overline{g}(A_{h^s(\xi_a, W)}U, N_a) \\
& -\sum_{a=1}^{r} \overline{g}(A_{h^s(U,W)}\xi_a, N_a).
\end{aligned} \tag{4.6}$$

Now, using equation (2.30) of [6] at page 158, for any $U \in \Gamma(T(M))$ define a differential 1-form as $\nu_a(U) = \overline{g}(U, N_a)$, for all $a \in \{1, 2, ..., r\}$, then any vector field U on M is expressed as $U = \overline{P}U + \sum_{a=1}^{r} \nu_a(U)\xi_a + \eta(U)V$, where $\overline{P}$ is the projection morphism of TM on $S(TM)$, therefore we have $\sum_{a=1}^{r} \overline{g}(U, N_a)g(\xi_a, W) = g(U, W) - g(\overline{P}U, W) - \epsilon\eta(U)\eta(W)$. Hence, using (2.9) and (2.10)in (4.6), we obtain

$$\begin{aligned} \sum_{a=1}^{r} \overline{g}(R(U, \xi_a)W, N_a) &= -\frac{(c+3)r + 3(c-1)}{4} g(U, W) + \frac{3(c-1)}{4} g(fU, fW) \\ &+ \frac{(c-1)(r+3)}{4} \epsilon\eta(U)\eta(W) - \sum_{a=1}^{r} \overline{g}(A_{N_a}U, h^l(\xi_a, W)) \\ &+ \frac{(c-1)}{4} \sum_{a=1}^{r} g(\phi W, U)\overline{g}(\phi\xi_a, N_a) + \sum_{a=1}^{r} \overline{g}(A_{N_a}\xi_a, h^l(U, W)) \\ &+ \sum_{a=1}^{r} \overline{g}(D^s(U, N_a), h^s(\xi_a, W)) \\ &- \sum_{a=1}^{r} \overline{g}(D^s(\xi_a, N_a), h^s(U, W)). \end{aligned} \tag{4.7}$$

Using (2.7), (2.11), (2.14) and (4.2), we obtain

$$\begin{aligned} \sum_{b=r+1}^{m} g(R(U, U_b)W, U_b) &= \frac{(c+3)}{4} g(\overline{P}U, \overline{P}W) + \frac{(c-1)\epsilon - (c+3)(m-r)}{4} g(U, W) \\ &- \frac{3(c-1)}{4} g(fU, fW) + \frac{(m-r-2)(c-1)}{4} \eta(U)\eta(W) \\ &+ \sum_{b=r+1}^{m} \overline{g}(h^l(U_b, W), h^*(U, U_b)) - \sum_{b=r+1}^{m} \overline{g}(h^l(U, W), h^*(U_b, U_b)) \\ &+ \sum_{b=r+1}^{m} \overline{g}(h^s(U_b, W), h^s(U, U_b)) \\ &- \sum_{b=r+1}^{m} \overline{g}(h^s(U, W), h^s(U_b, U_b)). \end{aligned} \tag{4.8}$$

Thus substituting (4.7) and (4.8) in (4.5), we obtain the expression of induced Ricci type tensor of a GCR-lightlike submanifolds as

$$\begin{aligned} R^{(0,2)}(U, W) &= -\frac{((c+3)m + (3-\epsilon)(c-1))}{4} g(U, W) + \frac{(c+3)}{4} g(\overline{P}U, \overline{P}W) \\ &+ \frac{(c-1)(m+1)}{4} \epsilon\eta(U)\eta(W) + \frac{(c-1)}{4} \sum_{a=1}^{r} g(\phi W, U)\overline{g}(\phi\xi_a, N_a) \\ &- \sum_{a=1}^{r} \overline{g}(A_{N_a}U, h^l(\xi_a, W)) + \sum_{a=1}^{r} \overline{g}(A_{N_a}\xi_a, h^l(U, W)) \\ &+ \sum_{a=1}^{r} \overline{g}(D^s(U, N_a), h^s(\xi_a, W)) - \sum_{a=1}^{r} \overline{g}(D^s(\xi_a, N_a), h^s(U, W)) \\ &+ \sum_{b=r+1}^{m} \overline{g}(h^l(U_b, W), h^*(U, U_b)) - \sum_{b=r+1}^{m} \overline{g}(h^l(U, W), h^*(U_b, U_b)) \\ &+ \sum_{b=r+1}^{m} \overline{g}(h^s(U_b, W), h^s(U, U_b)) - \sum_{b=r+1}^{m} \overline{g}(h^s(U, W), h^s(U_b, U_b)). \end{aligned}$$

Clearly, the induced Ricci type tensor $R^{(0,2)}$ is not symmetric because the induced connection ∇ is not a metric connection on M. The induced Ricci type tensor $R^{(0,2)}$ is called induced Ricci tensor and denoted by Ric of M if it is symmetric, (see [9]). Thus, the symmetry of the induced Ricci type tensor is very important. From the above expression, it is clear that if $c = 1$ and M is a totally geodesic lightlike submanifold then the induced Ricci type tensor is symmetric.

Next, using orthonormal frame fields on $\overline{D}$, $D_0 \oplus \{V\}$, ϕD_2 and $Rad(TM)$, we can also find expressions of induced Ricci type tensor as

$$\begin{aligned} R^{(0,2)}(U,W) &= \sum_{i=1}^{q} g(R(U,F_i)W,F_i) + \sum_{k=0}^{2p} g(R(U,E_k)W,E_k) + \sum_{l=2s+1}^{r} g(R(U,\phi\xi_l)W,\phi N_l) \\ &+ \sum_{a=1}^{r} \overline{g}(R(U,\xi_a)W,N_a). \end{aligned}$$

Denote $D = D_0 \oplus \{V\}$ then we have

$$\begin{aligned} R^{(0,2)}_D(U,W) &= \sum_{k=0}^{2p} g(R(U,E_k)W,E_k) + \sum_{l=2s+1}^{r} g(R(U,\phi\xi_l)W,\phi N_l) \\ &+ \sum_{a=1}^{r} \overline{g}(R(U,\xi_a)W,N_a), \end{aligned} \tag{4.9}$$

$$R^{(0,2)}_{\overline{D}}(U,W) = \sum_{i=1}^{q} g(R(U,F_i)W,F_i).$$

Using (2.9), (2.11), (2.14) and (4.3), we obtain

$$\begin{aligned} \sum_{k=0}^{2p} g(R(U,E_k)W,E_k) &= -\frac{(c-3)}{4} g(QU,QW) + \frac{(c-1)(2p-1)}{4}\epsilon\eta(U)\eta(W) \\ &- \left\{\frac{(2p+1)(c+3)}{4} - \frac{(c-1)}{4}\right\} g(U,W) \\ &+ \sum_{k=0}^{2p} \overline{g}(h^l(E_k,W),h^*(U,E_k)) - \sum_{k=0}^{2p} \overline{g}(h^l(U,W),h^*(E_k,E_k)) \\ &+ \sum_{k=0}^{2p} \overline{g}(h^s(E_k,W),h^s(U,E_k)) \\ &- \sum_{k=0}^{2p} \overline{g}(h^s(U,W),h^s(E_k,E_k)), \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} \sum_{l=2s+1}^{r} g(R(U,\phi\xi_l)W,\phi N_l) &= -\frac{(c+3)(r-2s)}{4} g(U,W) + \frac{(c-1)(r-2s)}{4}\epsilon\eta(U)\eta(W) \\ &+ \frac{(c+3)}{4} \sum_{l=2s+1}^{r} g(\phi\xi_l,W) g(U,\phi N_l) \\ &- \sum_{l=2s+1}^{r} g(A_{h^l(U,W)}\phi\xi_l,\phi N_l) + \sum_{l=2s+1}^{r} g(A_{h^l(\phi\xi_l,W)}U,\phi N_l) \\ &- \sum_{l=2s+1}^{r} g(A_{h^s(U,W)}\phi\xi_l,\phi N_l) \\ &+ \sum_{l=2s+1}^{r} g(A_{h^s(\phi\xi_l,W)}U,\phi N_l). \end{aligned} \tag{4.11}$$

Using (4.7), (4.10) and (4.11) in (4.9), we obtain

$$\begin{aligned}
R_D^{(0,2)}(U,W) &= -\{\frac{(c+3)(m-q)}{4}+\frac{(c-1}{2}\}g(U,W)-\frac{(c-3)}{4}g(QU,QW)\\
&+\frac{(c-1)(m-q+1)}{4}\epsilon\eta(U)\eta(W)+\frac{3(c-1)}{4}g(fU,fW)\\
&+\frac{(c-1)}{4}\sum_{a=1}^{r}g(\phi W,U)\overline{g}(\phi\xi_a,N_a)\\
&+\frac{(c+3)}{4}\sum_{l=2s+1}^{r}g(\phi\xi_l,W)g(U,\phi N_l)\\
&+\sum_{a=1}^{r}\overline{g}(A_{N_a}\xi_a,h^l(U,W))-\sum_{a=1}^{r}\overline{g}(A_{N_a}U,h^l(\xi_a,W))\\
&-\sum_{a=1}^{r}\overline{g}(D^s(\xi_a,N_a),h^s(U,W))+\sum_{a=1}^{r}\overline{g}(D^s(U,N_a),h^s(\xi_a,W))\\
&+\sum_{k=0}^{2p}\overline{g}(h^l(E_k,W),h^*(U,E_k))-\sum_{k=0}^{2p}\overline{g}(h^l(U,W),h^*(E_k,E_k))\\
&+\sum_{k=0}^{2p}\overline{g}(h^s(E_k,W),h^s(U,E_k))-\sum_{k=0}^{2p}\overline{g}(h^s(U,W),h^s(E_k,E_k))\\
&-\sum_{l=2s+1}^{r}g(A_{h^l(U,W)}\phi\xi_l,\phi N_l)+\sum_{l=2s+1}^{r}g(A_{h^l(\phi\xi_l,W)}U,\phi N_l)\\
&-\sum_{l=2s+1}^{r}g(A_{h^s(U,W)}\phi\xi_l,\phi N_l)+\sum_{l=2s+1}^{r}g(A_{h^s(\phi\xi_l,W)}U,\phi N_l).
\end{aligned}$$

Also using (2.7), (2.11), (2.14) and (4.4), we obtain

$$\begin{aligned}
R_{\overline{D}}^{(0,2)}(U,W) &= \frac{(c+3)}{4}g(PU,PW)-\frac{q(c+3)}{4}g(U,W)+\frac{q(c-1)\epsilon}{4}\eta(U)\eta(W)\\
&-\sum_{i=1}^{q}\overline{g}(h^l(U,W),h^*(F_i,F_i))+\sum_{i=1}^{q}\overline{g}(h^l(F_i,W),h^*(U,F_i))\\
&-\sum_{i=1}^{q}\overline{g}(h^s(F_i,F_i),h^s(U,W))+\sum_{i=1}^{q}\overline{g}(h^s(F_i,W),h^s(U,F_i)).
\end{aligned}$$

Let $X,Y\in\Gamma(D\oplus\{V\})$ and $Z,W\in\Gamma(\overline{D})$, then particularly, we have

$$\begin{aligned}
R_{\overline{D}}^{(0,2)}(X,Y) &= -\frac{q(c+3)}{4}g(X,Y)+\frac{q(c-1)\epsilon}{4}-\sum_{i=1}^{q}\overline{g}(h^l(X,Y),h^*(F_i,F_i))\\
&+\sum_{i=1}^{q}\overline{g}(h^l(F_i,Y),h^*(X,F_i))-\sum_{i=1}^{q}\overline{g}(h^s(F_i,F_i),h^s(X,Y))\\
&+\sum_{i=1}^{q}\overline{g}(h^s(F_i,Y),h^s(X,F_i)). \qquad (4.12)
\end{aligned}$$

$$\begin{aligned} R_D^{(0,2)}(X,Y) &= -\frac{(c+3)(m-q)+(3c-5)}{4}g(X,Y) - \frac{(c-1)(m-q+1)}{4}\epsilon \\ &+\frac{3(c-1)}{4}g(fX,fY) + \frac{(c-1)}{4}\sum_{a=1}^{r} g(\phi Y,X)\overline{g}(\phi\xi_a,N_a) \\ &+\sum_{a=1}^{r}\overline{g}(A_{N_a}\xi_a,h^l(X,Y)) - \sum_{a=1}^{r}\overline{g}(A_{N_a}X,h^l(\xi_a,Y)) \\ &-\sum_{a=1}^{r}\overline{g}(D^s(\xi_a,N_a),h^s(X,Y)) + \sum_{a=1}^{r}\overline{g}(D^s(X,N_a),h^s(\xi_a,Y)) \\ &+\sum_{k=0}^{2p}\overline{g}(h^l(E_k,Y),h^*(X,E_k)) - \sum_{k=0}^{2p}\overline{g}(h^l(X,Y),h^*(E_k,E_k)) \\ &+\sum_{k=0}^{2p}\overline{g}(h^s(E_k,Y),h^s(X,E_k)) - \sum_{k=0}^{2p}\overline{g}(h^s(X,Y),h^s(E_k,E_k)) \\ &-\sum_{l=2s+1}^{r} g(A_{h^l(X,Y)}\phi\xi_l,\phi N_l) + \sum_{l=2s+1}^{r} g(A_{h^l(\phi\xi_l,Y)}X,\phi N_l) \\ &-\sum_{l=2s+1}^{r} g(A_{h^s(X,Y)}\phi\xi_l,\phi N_l) + \sum_{l=2s+1}^{r} g(A_{h^s(\phi\xi_l,Y)}X,\phi N_l). \end{aligned} \tag{4.13}$$

$$\begin{aligned} R_{\overline{D}}^{(0,2)}(X,Z) &= -\sum_{i=1}^{q}\overline{g}(h^l(X,Z),h^*(F_i,F_i)) + \sum_{i=1}^{q}\overline{g}(h^l(F_i,Z),h^*(X,F_i)) \\ &-\sum_{i=1}^{q}\overline{g}(h^s(F_i,F_i),h^s(X,Z)) + \sum_{i=1}^{q}\overline{g}(h^s(F_i,Z),h^s(X,F_i)), \end{aligned} \tag{4.14}$$

$$\begin{aligned} R_D^{(0,2)}(X,Z) &= \frac{(c-1)}{4}\sum_{a=1}^{r} g(\phi Z,X)\overline{g}(\phi\xi_a,N_a) + \frac{(c+3)}{4}\sum_{l=2s+1}^{r} g(\phi\xi_l,Z)g(X,\phi N_l) \\ &+\sum_{a=1}^{r}\overline{g}g(A_{N_a}\xi_a,h^l(X,Z)) - \sum_{a=1}^{r}\overline{g}g(A_{N_a}X,h^l(\xi_a,Z)) \\ &-\sum_{a=1}^{r}\overline{g}(D^s(\xi_a,N_a),h^s(X,Z)) + \sum_{a=1}^{r}\overline{g}(D^s(X,N_a),h^s(\xi_a,Z)) \\ &+\sum_{k=1}^{2p}\overline{g}(h^l(E_k,Z),h^*(X,E_k)) - \sum_{k=1}^{2p}\overline{g}(h^l(X,Z),h^*(E_k,E_k)) \\ &+\sum_{k=1}^{2p}\overline{g}(h^s(E_k,Z),h^s(X,E_k)) - \sum_{k=1}^{2p}\overline{g}(h^s(X,Z),h^s(E_k,E_k)) \\ &-\sum_{l=2s+1}^{r} g(A_{h^l(X,Z)}\phi\xi_l,\phi N_l) + \sum_{l=2s+1}^{r} g(A_{h^l(\phi\xi_l,Z)}X,\phi N_l) \\ &-\sum_{l=2s+1}^{r} g(A_{h^s(X,Z)}\phi\xi_l,\phi N_l) + \sum_{l=2s+1}^{r} g(A_{h^s(\phi\xi_l,Z)}X,\phi N_l), \end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
R^{(0,2)}_{\overline{D}}(Z,W) &= \frac{(c+3)(1-q)}{4}g(Z,W) - \sum_{i=1}^{q}\overline{g}(h^l(Z,W),h^*(F_i,F_i)) \\
&+ \sum_{i=1}^{q}\overline{g}(h^l(F_i,W),h^*(Z,F_i)) - \sum_{i=1}^{q}\overline{g}(h^s(F_i,F_i),h^s(Z,W)) \\
&+ \sum_{i=1}^{q}\overline{g}(h^s(F_i,W),h^s(Z,F_i)).
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
R^{(0,2)}_{D}(Z,W) &= -\left\{\frac{(c+3)(m-q)}{4} + \frac{(c-1)}{2}\right\}g(Z,W) \\
&+ \frac{(c+3)}{4}\sum_{l=2s+1}^{r} g(\phi\xi_l,W)g(Z,\phi N_l) \\
&+ \sum_{a=1}^{r}\overline{g}(A_{N_a}\xi_a,h^l(Z,W)) - \sum_{a=1}^{r}\overline{g}(A_{N_a}Z,h^l(\xi_a,W)) \\
&- \sum_{a=1}^{r}\overline{g}(D^s(\xi_a,N_a),h^s(Z,W)) + \sum_{a=1}^{r}\overline{g}(D^s(Z,N_a),h^s(\xi_a,W)) \\
&+ \sum_{k=1}^{2p}\overline{g}(h^l(E_k,W),h^*(Z,E_k)) - \sum_{k=1}^{2p}\overline{g}(h^l(Z,W),h^*(E_k,E_k)) \\
&+ \sum_{k=1}^{2p}\overline{g}(h^s(E_k,W),h^s(Z,E_k)) - \sum_{k=1}^{2p}\overline{g}(h^s(Z,W),h^s(E_k,E_k)) \\
&- \sum_{l=2s+1}^{r} g(A_{h^l(Z,W)}\phi\xi_l,\phi N_l) + \sum_{l=2s+1}^{r} g(A_{h^l(\phi\xi_l,W)}Z,\phi N_l) \\
&- \sum_{l=2s+1}^{r} g(A_{h^s(Z,W)}\phi\xi_l,\phi N_l) + \sum_{l=2s+1}^{r} g(A_{h^s(\phi\xi_l,W)}Z,\phi N_l).
\end{aligned} \tag{4.17}$$

Thus from (4.12) to (4.17), we have the following results.

Theorem 4.1. *Let M be a totally geodesic GCR-lightlike submanifold of an indefinite Sasakian space form $\overline{M}(c)$. Then*

$$\begin{aligned}
R^{(0,2)}_{D}(X,Y) &= -\frac{(c+3)(m-q)+(3c-5)}{4}g(X,Y) - \frac{(c-1)(m-q+1)}{4}\epsilon \\
&+ \frac{3(c-1)}{4}g(fX,fY) + \frac{(c-1)}{4}\sum_{a=1}^{r} g(\phi Y,X)\overline{g}(\phi\xi_a,N_a),
\end{aligned}$$

$$R^{(0,2)}_{D}(X,Z) = \frac{(c-1)}{4}\sum_{a=1}^{r} g(\phi Z,X)\overline{g}(\phi\xi_a,N_a) + \frac{(c+3)}{4}\sum_{l=2s+1}^{r} g(\phi\xi_l,Z)g(X,\phi N_l),$$

$$\begin{aligned}
R^{(0,2)}_{D}(Z,W) &= -\left\{\frac{(c+3)(m-q)}{4} + \frac{(c-1)}{2}\right\}g(Z,W) \\
&+ \frac{(c+3)}{4}\sum_{l=2s+1}^{r} g(\phi\xi_l,W)g(Z,\phi N_l),
\end{aligned}$$

$$R^{(0,2)}_{\overline{D}}(X,Y) = -\frac{q(c+3)}{4}g(X,Y) + \frac{q(c-1)\epsilon}{4}, \quad R^{(0,2)}_{\overline{D}}(X,Z) = 0,$$

and

$$R^{(0,2)}_{\overline{D}}(Z,W) = \frac{(c+3)(1-q)}{4}g(Z,W),$$

for any $X, Y \in \Gamma(D)$ *and* $Z, W \in \Gamma(\overline{D})$.

Theorem 4.2. *Let* M *be a* D*-geodesic GCR-lightlike submanifold of an indefinite Sasakian space form* $\overline{M}(c)$*. Then*

$$\begin{aligned} R^{(0,2)}_{D}(X,Y) &= -\frac{(c+3)(m-q)+(3c-5)}{4}g(X,Y) - \frac{(c-1)(m-q+1)}{4}\epsilon \\ &\quad + \frac{3(c-1)}{4}g(fX,fY) + \frac{(c-1)}{4}\sum_{a=1}^{r} g(\phi Y, X)\overline{g}(\phi\xi_a, N_a), \end{aligned}$$

and

$$\begin{aligned} R^{(0,2)}_{\overline{D}}(X,Y) &= -\frac{q(c+3)}{4}g(X,Y) + \frac{q(c-1)\epsilon}{4} + \sum_{i=1}^{q}\overline{g}(h^l(F_i,Y), h^*(X,F_i)) \\ &\quad + \sum_{i=1}^{q}\overline{g}(h^s(F_i,Y), h^s(X,F_i)), \end{aligned}$$

for any $X, Y \in \Gamma(D)$.

Theorem 4.3. *Let* M *be a* $\overline{D}$*-geodesic GCR-lightlike submanifold of an indefinite Sasakian space form* $\overline{M}(c)$*. Then*

$$\begin{aligned} R^{(0,2)}_{D}(Z,W) &= -\left\{\frac{(c+3)(m-q)}{4} + \frac{(c-1)}{2}\right\}g(Z,W) + \frac{(c+3)}{4}\sum_{l=2s+1}^{r} g(\phi\xi_l, W)g(Z,\phi N_l) \\ &\quad - \sum_{a=1}^{r}\overline{g}(A_{N_a}Z, h^l(\xi_a,W)) + \sum_{a=1}^{r}\overline{g}(D^s(Z,N_a), h^s(\xi_a,W)) \\ &\quad + \sum_{k=1}^{2p}\overline{g}(h^l(E_k,W), h^*(Z,E_k)) + \sum_{k=1}^{2p}\overline{g}(h^s(E_k,W), h^s(Z,E_k)) \\ &\quad + \sum_{l=2s+1}^{r} g(A_{h^l(\phi\xi_l,W)}Z, \phi N_l) + \sum_{l=2s+1}^{r} g(A_{h^s(\phi\xi_l,W)}Z, \phi N_l), \end{aligned}$$

and

$$R^{(0,2)}_{\overline{D}}(Z,W) = \frac{(c+3)(1-q)}{4}g(Z,W),$$

for any $Z, W \in \Gamma(\overline{D})$.

Theorem 4.4. *Let* M *be a mixed-geodesic GCR-lightlike submanifold of an indefinite Sasakian space form* $\overline{M}(c)$*. Then*

$$R^{(0,2)}_{D}(X,Z) = \frac{(c-1)}{4}\sum_{a=1}^{r} g(\phi Z, X)g(\phi\xi_a, N_a) + \frac{(c+3)}{4}\sum_{l=2s+1}^{r} g(\phi\xi_l, Z)g(X,\phi N_l),$$

and

$$R^{(0,2)}_{\overline{D}}(X,Z) = \sum_{i=1}^{q}\overline{g}(h^l(F_i,Z), h^*(X,F_i)) + \sum_{i=1}^{q}\overline{g}(h^s(F_i,Z), h^s(X,F_i)),$$

for any $X \in \Gamma(D)$ *and* $Z \in \Gamma(\overline{D})$.

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Author information

Varun Jain, Department of Mathematics, Multani Mal Modi College, Patiala 147 001, Punjab, India.
E-mail: varun82jain@gmail.com

Satvinder Singh Bhatia, Department of Mathematics, Thapar Institute of Engineering and Technology, Patiala 147 001, Punjab, India.
E-mail: ssbhatia@thapar.edu

Rakesh Kumar, Department of Mathematics, Punjabi University, Patiala 147 002, Punjab, India.
E-mail: dr_rk37c@yahoo.co.in (Corresponding Author)

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