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Estimation of η-Einstein solitons on Lorentzian concircular structure manifolds-[$(LCS)_n$]

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Abstract: The goal of this research is to investigate the 3-dimensional Lorentzian concircular structure manifolds, abbreviated as $(LCS)_n$-manifolds admitting the η-Einstein solitons. A symmetric second order covariant tensor in a $(LCS)_n$-manifold is proven to be a constant multiple of the metric tensor and η-Einstein soliton on a $(LCS)_n$-manifold is satisfying some symmetric conditions $R.S = 0$ and $S.R = 0$. Furthermore, we deduce the result for gradient η-Einstein soliton when the potential vector field ξ is of gradient type, $\xi = grad(\psi)$, and if the soliton vector field is a torqued vector field τ. Finally, in the region where the $(LCS)_n$-manifold is expanding, an example of η-Einstein soliton $(LCS)_n$-manifolds is given.

1 Introduction

Einstein manifolds are important in both Riemannian geometry and the general theory of relativity (GTR). The equation of Einstein soliton in relation to string theory has also been studied by theoretical physicists. The investigation of exact solutions to the Einstein field equations, as well as considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces, led to the discovery of quasi-Einstein manifolds. In GTR, quasi-Einstein manifolds play a role. The Robertson-Walker spacetimes, for example, are quasi-Einstein manifolds. Furthermore, quasi-Einstein manifolds can be used to simulate perfect fluid spacetime in GTR. In [16], Mantica and Molinari have proved that the $(LCS)_n$-manifold coincides with *generalized Robertson-Walker* (GRW) *spacetime*, which was introduced by Alías, Romero and Sánchez [3] in 1995. The geometry of semi-Riemannian submersions has risen in popularity in present geometric evaluation to its involvement in physics, specially in the general theory of relativity (GTR) such as Yang-Mills theory, String theory, Kaluza-Klein theory, and Hodge theory, etc. Moreover, $(LCS)_4$-Lorentzian concircular sapcetime manifold is one of the suitable framework for 4-dimensional spacetime.

Shaikh [21] proposed the geometry of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds) in 2003, which generalizes the notion of LP-Sasakian manifolds established by Matsumoto [15] and Mihai and Rosca [17]. Then, [22], Shaikh and Baishya examined the applications of $(LCS)n$-manifolds to the general theory of relativity, solitons, and cosmology. Atceken [2] also investigated the $(LCS)_n$-manifolds. Shaikh et al. [23, 24, 25] also explored some interesting properties of $(LCS)_n$-manifolds, which are correlated with this research note.

In 1988, Hamilton [13] established the Ricci solitons move under the Ricci flow merely by diffeomorphisms of the initial metric, implying that they are stationary points of the Ricci flow determined by

$$\frac{\partial g}{\partial t} = -2S(g). \tag{1.1}$$

Definition 1.1. A Ricci soliton (g, V, λ) on a Riemannian manifold (M, g) is a generalization of Einstein metric such that

$$\mathcal{L}_V g + 2S + 2\lambda = 0, \tag{1.2}$$

where S is the Ricci tensor, $\mathcal{L}_V$ is the Lie derivative along the vector field V on M and λ is a real scalar. Ricci soliton is said to be shrinking, steady or expanding according as $\lambda < 0, \lambda = 0$ or $\lambda > 0$, respectively.

If the vector field V is the gradient of a potential function -ψ , where ψ is some smooth function $\psi : M \to \mathbb{R}$, then g is called a gradient Ricci soliton, and equation (1.1) assumes the form

$$\nabla\nabla\psi = S + \lambda g. \tag{1.3}$$

It is well known that the quantity $a(g, \psi) := R + \left|\nabla\psi^2\right| - \psi$ must be constant on M and it is often called the *auxiliary constant*. When ψ is constant, the gradient Ricci soliton is simply an *Einstein manifold*. Thus Ricci solitons are natural extensions of Einstein metrics, an Einstein manifold with constant potential function is called a trivial gradient Ricci soliton. Gradient Ricci solitons play an important role in Hamilton Ricci flow [13] as they correspond to self-similar solutions, and often arise as singularity models. They are also related to smooth metric measure spaces, since equation (1.2) is equivalent to ∞-Bakry-Emery Ricci tensor $Ric\psi = 0$. In physics, a smooth metric space $(M, g, e^{\psi}, dvol)$ with $Ric\psi = \lambda$g is called quasi-Einstein manifold. Therefore, it is important to study the geometry and topology of gradient Ricci solitons and their classifications.

In general, one cannot expect potential function ψ to grow or decay linearly along all directions at infinity, because of the product property: the product of any two gradient steady Ricci solitons is also a gradient steady Ricci soliton. Consider for example (R, g, ψ), where g is the standard Euclidean metric, $\psi(x_1, x_2) = x_1\psi$ is constant along x_2 direction, so, without additional conditions, ψ may not have linear growth at infinity.

In the same way as the Ricci solitons generate self-similar solutions to the Ricci flow, the Einstein solitons, which generate self-similar solutions to the Einstein flow.

In 2016, Catino and Mazzieri introduced the notion of Einstein solitons [9], which generate self-similar solutions to Einstein flow

$$\frac{\partial g}{\partial t} = -2\left(S - \frac{scal}{2}g\right). \tag{1.4}$$

The interest in studying this equation from different points of view arises from concrete physical problems. On the other hand, gradient vector fields play a central role in Morse-Smale theory. In what follows, after characterizing the manifold of constant scalar curvature via the existence of η-Einstein solitons, we focus on the case when the potential vector field ξ is of gradient type (i.e., ξ = $grad\psi$), for ψ a nonconstant smooth function on M and give the Poisson equation satisfied by ξ.

η-Einstein solitons are natural extension of Einstein solitons and Ricci soliton which include quasi-Einstein metrics. Perturbing the equation (1.4) that defines this kind of solitons by a multiple of a certain $(0, 2)$-tensor field $\eta \otimes \eta$, we obtain a slightly more general notion, namely η-Einstein solitons satisfying the equation ([4])

$$\mathcal{L}_\xi g + 2S + (2\lambda - scal)\, g + 2\mu\eta \otimes \eta = 0, \tag{1.5}$$

in particular if $\mu = 0$ then η-Einstein soliton reduces to Einstein soliton.

In 1925, Levy [14] obtained the necessary and sufficient conditions for the existence of such tensors. In [1], Ali and Ahsan examined the 4-dimensional spacetime Lorentzian manifolds in terms of Ricci soliton. Moreover, Sharma [20] initiated the study of Ricci solitons in contact Riemannian geometry. After that, others like Bagewadi et al. [6] extensively studied Ricci soliton in Lorentzian Sasakian manifolds. In 2009, Cho and Kimura [8] introduced the notion of η-Ricci solitons and gave a classification of real hypersurfaces in non-flat complex space forms [7] admitting η-Ricci solitons. In [18] Prakasha and Hadimani discussed η-Ricci solitons on para-Sasakian manifolds. Blaga also has studied a more notion named η -Einstein soliton [4, 5]. Moreover, Siddiqi et al. [26, 27, 28, 29, 30, 31] have also studied some properties of the η -Einstein soliton which is closely related to this paper. Therefore, in the present paper, the author has studied the η-Einstein soliton in Lorentzian concircular structure manifolds (briefly $(LCS)_n$-manifolds).

2 Preliminaries

A 3-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ is a non-degenerate inner product of signature $(-,+,....,+)$, where T_pM denotes the tangent vector space of M at p and $\mathbb{R}$ is the real number space. A non-zero vector $v \in T_pM$ is said to be timelike (*resp.* non-spacelike, null, spacelike) if it satisfies $g_p(v,v) < 0$ (*resp.* $\leq 0, = 0, \geq 0$) [19]. The category into which a given vector falls is called its causal character.

Definition 2.1. In a Lorentzian manifold (M,g) [32, 23] a vector field P defined by

$$g(X,P) = A(X)$$

for any $X \in \chi(M)$ is said to be concircular vector field if

$$(\nabla_X A)Y = \alpha\left\{g(X,Y) + \omega(X)A(Y)\right\},$$

where α is a non-zero scalar and ω is a closed 1-form.

Let M be a Lorentzian 3- dimensional manifold admitting a unit timelike concircular vector field ξ, called the characteristics vector field of the manifold. Then we have

$$g(\xi,\xi) = -1. \tag{2.1}$$

Since ξ is a unit concircular vector filed, it follows that there exists a non-zero 1-form η such that for

$$g(X,\xi) = \eta(X), \tag{2.2}$$

the equation of the following form holds.

$$(\nabla_X \eta)Y = \alpha[g(X,Y) + \eta(X)\eta(Y)], \qquad (\alpha \neq 0), \tag{2.3}$$

$$\nabla_X \xi = \alpha\left\{X + \eta(X)\xi\right\} \qquad (\alpha \neq 0) \tag{2.4}$$

for all vector fields X, Y, where ∇ denotes the operator of the covariant differentiation with respect to the Lorentzian metric g and α is non-zero scalar function satisfies

$$\nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{2.5}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$ if we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{2.6}$$

then from (2.3) and (2.6) we have

$$\phi X = X + \eta(X)\xi, \tag{2.7}$$

$$g(\phi X, Y) = g(X, \phi Y), \tag{2.8}$$

from which it follows that ϕ is a symmetric $(1,1)$ tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold M together with unit time like concircular vector field ξ, it associated 1-form η and $(1,1)$ tensor filed ϕ is said to be Lorentzian concircular structure manifold or briefly called $(LCS)_n$-manifold [22]. Moreover, a $(LCS)_n$-manifold is a *generalized Robertson-Walker spacetime* if and only if it admits a unit timelike vector field which is also an eigenvector of the Ricci tensor. Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure [32] of the following relations hold [21]

$$\phi^2 X = X + \eta(X)\xi, \tag{2.9}$$

$$\eta(\xi) = -1, \quad \phi(\xi) = 0, \quad \eta(\phi) = 0, \tag{2.10}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.11}$$

$$\eta(R(X, Y)Z) = (\alpha^2 - \rho)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{2.12}$$

$$R(X, Y)Z = \phi R(X, Y)Z + (\alpha^2 - \rho)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\xi, \tag{2.13}$$

$$S(X, Y) = \left[\frac{r}{2} - (\alpha^2 - \rho)\right] g(X, Y) \tag{2.14}$$

$$- \left[\frac{r}{2} - 3(\alpha^2 - \rho)\right] \eta(X)\eta(Y),$$

$$QX = \left[\frac{r}{2} - (\alpha^2 - \rho)\right] X - \left[\frac{r}{2} - 3(\alpha^2 - \rho)\right] \eta(X)\xi \tag{2.15}$$

$$(\nabla_X \phi)Y = \alpha \left\{g(X, Y)\xi + 2\eta(X)\eta(Y) + \eta(Y)X\right\} \tag{2.16}$$

$$(X\rho) = d\rho(X) = \beta\eta(X) \tag{2.17}$$

Using (2.13) and (2.14), for constants α and ρ , we have

$$R(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{2.18}$$

$$R(\xi, Y)Z = (\alpha^2 - \rho)[g(Y, Z)\xi - \eta(Z)Y], \tag{2.19}$$

$$S(X, \xi) = [2(\alpha^2 - \rho)]\eta(X), \tag{2.20}$$

$$Q\xi = [2(\alpha^2 - \rho)]\xi, \tag{2.21}$$

where R is curvature tensor, while Q is the Ricci operator given by $S(X, Y) = g(QX, Y)$.

Definition 2.2. A $(LCS)_n$-manifold (M^n, g) is said to be η-Einstein if its Ricci tensor S of the type $(0, 2)$ is of the form

$$S = ag(X, Y) + b\eta(X)\eta(Y), \tag{2.22}$$

where a and b are smooth function on M.

Again, from the definition of Lie derivative, we have

$$(\mathcal{L}_\xi g)(X, Y) = (\nabla_X g)(X, Y) + 2\alpha g(X, Y) + 2\alpha\eta(X)\eta(Y) \tag{2.23}$$

$$= 2\alpha[g(X, Y) + \eta(X)\eta(Y)].$$

3 η-Einstein solitons on $(M, \phi, \xi, \eta, g, \alpha)$

In the study of the η-Einstein soliton equation, we will consider certain assumptions, one essential condition being $\nabla\xi = \alpha[I_\xi(M) + \eta \otimes \xi]$ which naturally arises in different geometry of $(LCS)_n$- manifolds.

An important geometrical object in studying η-Einstein solitons is a symmetric $(0, 2)$- tensor field which is parallel with respect to the Levi-Civita connection.

Fix h a symmetric tensor field of $(0, 2)$-type which we suppose to be parallel with respect to the Levi-Civita connection ∇ that is $\nabla h = 0$. Applying the Ricci commutation identity [12]

$$\nabla^2 h(X, Y; Z, W) - \nabla^2 h(X, Y; W, Z) = 0, \tag{3.1}$$

we obtain the relation

$$h(R(X, Y)Z, W) + h(Z, R(X, Y)W) = 0. \tag{3.2}$$

Replacing $Z = W = \xi$ in (3.2) and also use the symmetricness of h, we have

$$h(\xi, R(\xi, Y)\xi) = 0. \tag{3.3}$$

Using (2.19) in (3.3) we get

$$(\alpha^2 - \rho)[h(Y, \xi) + \eta(Y)h(\xi, \xi)] = 0. \tag{3.4}$$

Since $\alpha^2 - \rho \neq 0$ we have

$$h(Y, \xi) + \eta(Y)h(\xi, \xi) = 0. \tag{3.5}$$

Differentiating (3.5) covariantly along X, we get

$$g(\nabla_X Y, \xi)h(\xi, \xi) + g(Y, \nabla_X \xi)h(\xi, \xi) + 2g(Y, \xi)h(\nabla_X \xi, \xi) \tag{3.6}$$

$$+h(\nabla_X Y, \xi) + h(Y, \nabla_X \xi) = 0.$$

Putting $Y = \nabla_X Y$ in (3.5) we obtain

$$g(\nabla_X Y, \xi)h(\xi, \xi) + h(\nabla_X Y, \xi) = 0. \tag{3.7}$$

In view of (3.7) it follows from (3.6) that

$$g(Y, \nabla_X \xi)h(\xi, \xi) + 2g(Y, \xi)h(\nabla_X \xi, \xi) + h(Y, \nabla_X \xi) = 0. \tag{3.8}$$

Using (2.6) in (3.8) we get

$$g(Y, \phi X)h(\xi, \xi) + 2\eta(Y)h(\phi X, \xi) + h(Y, \phi X) = 0, \quad since \quad \alpha \neq 0. \tag{3.9}$$

Replacing X by ϕX in (3.9) and then using (2.9) and (3.5) we obtain

$$h(X, Y) = -h(\xi, \xi)g(X, Y). \tag{3.10}$$

Differentiating (3.10) covariantly along any vector filed on M, it can be easily shown that $h(\xi, \xi)$ is constant. Now, we can give the conclusion:

Theorem 3.1. *Let $(M, \phi, \xi, \eta, g, \alpha)$ be a $(LCS)_n$- manifold with non-vanishing ξ-sectional curvature and endowed with a tensor field $h \in \gamma(T_2^0(M))$ which is symmetric and ϕ-skew-symmetric. If h is parallel with respect to ∇ then it is a constant multiple of the metric tensor g.*

Definition 3.2. Let $(M, \phi, \xi, \eta, g, \alpha)$ be a $(LCS)_n$-manifold. consider the equation

$$\mathcal{L}_\xi g + 2S + (2\lambda - scal)\, g + 2\mu\eta \otimes \eta = 0, \tag{3.11}$$

where $\mathcal{L}_\xi$ is the Lie derivative operator along the vector field ξ, S is the Ricci curvature tensor field of the metric g, and λ and μ are real constants. For $\mu \neq 0$, the data (g, ξ, λ, μ) will be called η-Einstein soliton.

Remark that if the scalar curvature $scal$ of the manifold is constant, then the η-Einstein soliton $(g, \xi, \lambda, \frac{scal}{2}, \mu)$ reduces to an η-Ricci soliton and, moreover, if $\mu = 0$, to a Ricci soliton $(g, \xi, \lambda, \frac{scal}{2})$. Therefore, the two concepts of η-Einstein soliton [9] and η-Ricci soliton are distinct on manifolds of nonconstant scalar curvature. Writing $\mathcal{L}_\xi g$ in terms of the Levi-Civita connection ∇, we obtain [9]:

$$2S(X,Y) = -g(\nabla_X \xi, Y) - g(X, \nabla_X \xi) - (2\lambda - scal)g(X,Y) - 2\mu\eta(X)\eta(Y), \tag{3.12}$$

for any $X, Y \in \chi(M)$.

Definition 3.3. The data (g, ξ, λ, μ) which satisfy the equation (3.11) is said to be η- *Einstein soliton* on M [8], and it is called shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$ respectively [8].

Now, from (2.4) , the equation (3.11) becomes:

$$S(X,Y) = -\left(\lambda - \frac{scal}{2} + \alpha\right) g(X,Y) - (\mu + \alpha)\eta(X)\eta(Y). \tag{3.13}$$

The above equations yields

$$S(X,\xi) = -\left(\lambda - \frac{scal}{2} + \mu\right) \eta(X) \tag{3.14}$$

$$QX = -\left(\lambda - \frac{scal}{2} + \alpha\right) X + (\mu + \alpha)\xi \tag{3.15}$$

$$Q\xi = -\left(\lambda - \frac{scal}{2} + \mu\right) \xi \tag{3.16}$$

Note that on $(LCS)_n$-manifold, the existence of an η-Einstein soliton implies that the characteristic vector ξ is an eigenvector if the Ricci operator corresponding to the eigenvalue $-\left(\lambda - \frac{scal}{2} + \mu\right)$.

Now we shall apply the previous results on η-Einstein soliton.

Theorem 3.4. *Let $(M, \phi, \xi, \eta, g, \alpha)$ be a $(LCS)_n$-manifold. Assume that the symmetric (0,2)-tensor field $h = \mathcal{L}_\xi g + 2S + \mu\eta \otimes \eta$ is parallel with respect to the Levi-Civita connection associated to g. Then (g, ξ, μ) yield an η-Einstein soliton.*

Proof. Compute $h(\xi, \xi)$ and from (3.11) we obtain

$$h(\xi,\xi) = (\mathcal{L}_\xi g)(\xi,\xi) + 2S(\xi,\xi) + 2\eta(\xi)\eta(\xi) = -2\lambda + scal + 2\alpha + 4\mu,$$

so $\lambda = -\frac{1}{2}[h(\xi,\xi) + \frac{scal}{2} - 2\mu - \alpha]$. From (3.10) we conclude that

$$h(X,Y) = [2\lambda - scal + 2\alpha + 4\mu]g(X,Y)$$

for any $X, Y \in \chi(M)$. Therefore

$$\mathcal{L}_\xi g + 2S + 2\mu\eta \otimes \eta = (2\lambda - scal + 2\alpha + 4\mu)g.$$

□

Conversely, we shall study the consequence of the existence of the η-Einstein soliton on a $(LCS)_n$-manifold. From (3.13) we deduce:

Theorem 3.5. *If (3.11) defines an η-Einstein soliton on the $(LCS)_n$-manifold $(M, \phi, \xi, \eta, g, \alpha)$. Then (M, g) is quasi-Einstein.*

Recall that the manifold is called *quasi-Einstein* [11] if the Ricci curvature tensor field S is a linear combination (with real scalars λ and μ respectively, with $\mu \neq 0$) of g and the tensor product of a non-zero 1-from η satisfying $\eta = g(X, \xi)$, for ξ a unit vector field and respectively, *Einstein* if S is collinear with g.

In what follows we shall consider η-Einstein solitons requiring for the curvature to satisfy $R(\xi, X).S = 0$, $S.R(\xi, X) = 0$, and also, we study the two cases (i) when the potential vector field ξ is of gradient type and second (ii) the potential vector field ξ is a torqued vector filed τ [10].

Definition 3.6. The vector filed ξ which is nowhere zero vector filed, is called the torqued vector filed τ if it satisfies

$$\nabla_X \tau = \phi X + \gamma(X)\tau, \qquad \gamma(\tau) = 0, \tag{3.17}$$

where the function ϕ is called torqued function and 1-form γ is called the torqued of τ [10].

4 η-Einstein solition on $(LCS)_n$-manifold satisfying $R.(\xi, X).S = 0$

Now we consider a $(LCS)_n$-manifold satisfying the condition

$$S(R(\xi, X)Y, Z) + S(Y, (\xi, X)Z) = 0, \tag{4.1}$$

for any $X, Y \in \chi(M)$.

Replacing the expression of S from (3.13) and from the symmetries of R we get

$$(\alpha^2 - \rho)(\mu + \alpha)[\eta(Y)g(X, Z) + \eta(Z)g(X, Y) + 2\eta(X)\eta(Y)\eta(Z)] = 0, \tag{4.2}$$

for any $X, Y \in \chi(M)$.
For $Z = \xi$ we have

$$-(\alpha^2 - \rho)(\mu + \alpha)g(\phi X, \phi Y) = 0, \tag{4.3}$$

for any $X, Y \in \chi(M)$.
Also, for $X = Y = Z = \xi$ in (4.2) we obtain

$$(\alpha^2 - \rho)(\mu + \alpha)[\eta(\xi)]^2[\eta(\xi) + 1] = 0, \quad since \quad (\alpha^2 - \rho) \neq 0, \tag{4.4}$$

which implies $\mu = -\alpha$
Hence we can state the following theorem:

Theorem 4.1. *If (g, ξ, λ, μ) is an η-Einstein soliton on a $(LCS)_n$-manifold M and satisfies $R(\xi, X).S = 0$, then $\mu = -\alpha$ and the manifold (M, g) is an Einstein manifold.*

For $\mu = 0$, we deduce:

Corollary 4.2. *On a $(LCS)_n$-manifold satisfying $R(\xi, X).S = 0$, there is no Ricci soliton with the potential vector field ξ.*

5 η-Einstein soliton on $(LCS)_n$-manifold satisfying $S.R(\xi, X) = 0$

In this section we consider $(LCS)_n$-manifold satisfying the condition $S.R(\xi, X) = 0$.

$$S(X, R(Y, Z)W)\xi - S(\xi, R(Y, Z)W)X + S(X, Y)R(\xi, Z)W- \tag{5.1}$$

$$-S(\xi, Y)R(X, Z)W + S(X, Z)R(Y, \xi)W - S(\xi, Z)R(Y, X)W+$$

$$+S(X, W)R(Y, Z)\xi - S(\xi, W)R(Y, Z)X = 0$$

for any $X, Y, Z, W \in \chi(M)$.

Taking the inner product with ξ, the equation (5.1) becomes

$$S(X, R(Y, Z)W)\,|\xi|^2 - S(\xi, R(Y, Z)W)\eta(X) + S(X, Y)\eta(R(\xi, Z)W) - \tag{5.2}$$

$$-S(\xi, Y)\eta(R(X, Z)W) + S(X, Z)\eta(R(Y, \xi)W) - S(\xi, Z)\eta(R(Y, X)W) +$$

$$+S(X, W)\eta(R(Y, Z)\xi) - S(\xi, W)\eta(R(Y, Z)X) = 0$$

for any $X, Y, Z, W \in \chi(M)$.

For $W = \xi$ and from the symmetries of R we get

$$S(X, R(Y, Z)W)\,|\xi|^2 - S(\xi, R(Y, Z)W)\eta(X) + S(\xi, \xi)\eta(R(Y, Z)\xi, X) = 0 \tag{5.3}$$

for any $X, Y, Z, W \in \chi(M)$.

Replacing the expression of S from (3.13), we get

$$(\alpha^2 - \rho)\,|\xi|^2\,[2\lambda + 2\alpha - scal + (\mu + \alpha)\,|\xi|^2][\eta(Y)g(X, Z) - \eta(Z)g(X, Y)] = 0. \tag{5.4}$$

For $W = \xi$ we have

$$(\alpha^2 - \rho)\,|\xi|^2\,[2\lambda + 2\alpha - scal + (\mu + \alpha)\,|\xi|^2][\eta(X)\eta(Y) - |\xi|^2\,g(X, Y)] = 0. \tag{5.5}$$

for any $X, Y \in \chi(M)$, and we obtain

$$(\alpha^2 - \rho)[2\lambda + 2\alpha - scal + (\mu + \alpha)\,|\xi|^2]g(X, R(Y, Z)\xi) = 0. \tag{5.6}$$

Also, we get

$$(\alpha^2 - \rho)[2\lambda + 2\alpha - scal + (\mu + \alpha)\,|\xi|^2] \tag{5.7}$$

for any $X, Y \in \chi(M)$. We can state the following:

Theorem 5.1. *If $(\phi, \xi, \eta, g, \alpha)$ is a $(LCS)_n$-manifold M, (g, ξ, λ, μ) is an η-Einstein soliton on M and $S(\xi, X).R = 0$, then $(\alpha^2 - \rho)[2\lambda + 2\alpha - scal + (\mu + \alpha)\,|\xi|^2] = 0$.*

For $\mu = 0$ follows $\lambda = \frac{|\xi|^2 + scal}{2} - \alpha$, so;

Corollary 5.2. *On a $(LCS)_n$-manifold satisfies equation (3.11) and $S(\xi, X).R = 0$, then $\lambda = \frac{|\xi|^2 + scal}{2} - \alpha$.*

6 η-Einstein solition on $(LCS)_n$-manifold with the potential vector of type $\xi = grad(\psi)$

Consider the equation

$$\mathcal{L}_\xi g + 2S + 2(\lambda + scal)g + 2\mu\eta \otimes \eta = 0. \tag{6.1}$$

Writing explicitly the Lie derivative $\mathcal{L}_\xi g$ we get

$$(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \tag{6.2}$$

and form (6.1) we obtain

$$S(X, Y) = -(\lambda - \frac{scal}{2})g(X, Y) - \mu\eta(X)\eta(Y) - \frac{1}{2}[g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)], \tag{6.3}$$

for any $X, Y \in \chi(M)$.

Contracting (6.3) we get

$$r = -(\lambda - \frac{scal}{2})dim(M) + \mu - div(\xi). \tag{6.4}$$

Let (M^n, g) be a 3-dimensional $(LCS)_n$-manifold and (g, ξ, λ, μ) be a η-Einstein soliton in M. From (2.14) and (6.3) we obtain

$$\left[-(\alpha^2-\rho)-\lambda\right] g(X,Y) - [3\lambda+\mu+4\alpha-3(\alpha^2-\rho)+\mu]\eta(X)\eta(Y) \tag{6.5}$$

$$+\frac{1}{2}[g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi)] = 0.$$

for any $X, Y \in \chi(M)$.

Consider $\{e_i\}_{1\leq i\leq 3}$ an orthonormal frame field and $\xi = \sum_{i=1}^{3}\xi^i e_i$. We have $\sum_{i=1}^{3}\varepsilon_{ii}(\xi^i)^2 = -1$ and $\eta(e_i) = \varepsilon_{ii}\xi^i$.

Multiplying (6.5) by ε_{ii} and summing over i for $X = Y = e_i$, we get

$$7\lambda - 2\mu = (\alpha^2-\rho) - 4\alpha - div(\xi). \tag{6.6}$$

Writing (6.5) for $X = Y = \xi$, we obtain

$$4\lambda - \mu = 4\alpha - 2(\alpha^2-\rho) \tag{6.7}$$

Therefore

$$\begin{cases} \lambda = 4\alpha + (\alpha^2-\rho) - \frac{div(\xi)}{3} \\ \mu = 8\alpha + 6(\alpha^2-\rho) - 4div(\xi) \end{cases} \tag{6.8}$$

Using (6.8) we can state the following results

Theorem 6.1. *Let (M^n, g) be a 3-dimensional $(LCS)_n$ -manifold and η be the g-dual 1-form of the gradient vector field $\xi = grad(\psi)$ with $g(\xi,\xi) = -1$. If (6.1) defines an η-Einstein soliton in M^n, then the Poisson equation satisfied by ψ becomes*

$$\Delta(\psi) = -\frac{1}{2}\left[\frac{\mu}{2} - 4\alpha - 3(\alpha^2-\rho)\right], \tag{6.9}$$

where Δ is the Laplace operator, and ψ is the solution to be determined. The Laplace equation plays a fundamental role in Physics; also well known for its importance in Electrostatics, Biophysics, and Engineering. Moreover, in case if ψ will be a harmonic function manifold M, then we can also explore some harmonic aspects of η-Einstein soliton on manifold M.

7 Application of torqued vector field to η-Einstein soliton on $(LCS)_n$-manifold

In this section, we are going to study the case of η-Einstein soliton on $(LCS)_n$-manifolds in which the potential vector filed ξ is torqued vector filed τ. From the definition of Lie-derivative and in the view of equation (3.17) we have

$$(\mathcal{L}_\tau g)(X,Y) = g(\nabla_X\tau, Y) + (X, \nabla_Y\tau) = \gamma(X)g(\tau, Y) + \gamma(Y)g(\tau, X) \tag{7.1}$$

for X and Y being tangent to M.

In view of (3.11) and (7.1), we get

$$S(X,Y) = -\left(\lambda - \frac{scal}{2}\right) g(X,Y) - \mu\eta(X)\eta(Y) - \frac{1}{2}[\gamma(X)\eta(Y) + \gamma(Y)\eta(X)]. \tag{7.2}$$

Here we denote that the dual 1-form of τ by η. Hence the manifold M is in the form of the generalized quasi-Einstein manifold such that [11]

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + c[A(X)B(Y) + A(Y)B(X)]. \tag{7.3}$$

where $a = -\left(\lambda - \frac{scal}{2}\right)$, $b = \mu$ and $c = -\frac{1}{2}$.

Putting $X = Y = e_i$ in (7.2), where $\{e_i\}$ is the orthonormal basis of the tangent space and summing over i, we get

$$\lambda = \frac{5}{2}r + \mu + 1 \tag{7.4}$$

So, we assert the following.

Theorem 7.1. *Let $(g, \xi, \lambda - \frac{scal}{2}, \mu)$ be an η-Einstein soliton on a 3-dimensional $(LCS)_n$ manifold. If the potential vector field ξ is a torqued vector field τ, then M is a generalized quasi-Einstein manifold.*

Corollary 7.2. *Let $(g, \xi, \lambda - \frac{scal}{2}, \mu)$ be an η-Einstein soliton on a 3-dimensional $(LCS)_n$ manifold . If the potential vector field ξ is a torqued vector field τ, then such soliton is expanding.*

8 Example of 3-dimensional $(LCS)_n$-manifold admitting η-Einstein soliton

Example 8.1. Consider the three dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$, where (x, y, z) are the Cartesian coordinates in $\mathbb{R}^3$ and let the linearly independent vector fields are

$$e_1 = z^3 \frac{\partial}{\partial x}, \quad e_2 = z^3 \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z},$$

where e_1, e_2, e_3 are linearly independent at each point of M. Let g be the Lorentzian metric defined by
$g(e_1, e_1) = g(e_2, e_2) = -g(e_3, e_3) = 1, g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0,$

Let η be the 1-form defined by $\xi = e_3$, $\eta(X) = g(X, e_3)$ for any vector field X on M and ϕ be the (1,1) tensor field defined by

$$\phi(e_1) = e_1, \quad \phi(e_2) = e_2, \quad \phi(e_3) = 0$$

Then by using the linearity of ϕ and g, we have $\phi^2 X = X + \eta(X)\xi$, with $\xi = e_3$.

Further $g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$ for any vector fields X and Y on M. Hence for $e_3 = \xi$, the structure defines an $(LCS)_3$- structure in $\mathbb{R}^3$.

Let ∇ be the Levi-Civita connection with respect to the metric g, then we have

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]),$$

which is known as Koszul's formula.

$$\nabla_{e_1} e_1 = -\frac{3}{z} e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -\frac{3}{z} e_1,$$

$$\nabla_{e_2} e_2 = -\frac{3}{z} e_2, \quad \nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_3 = -\frac{3}{z} e_2,$$

$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0,$$

using the above relation, for any vector X on M, we have $\nabla_X \xi = \alpha[X + \eta(X)\xi)]$, where $\alpha = -\frac{3}{z}$. Hence $(\phi, \xi, \eta, g, \alpha)$ structure defines the $(LCS)_3$-structure in $\mathbb{R}^3$.

Example 8.2. On the $(LCS)_3$-manifold $(M, g, \xi, \eta, \phi, \alpha)$ considered in Example (8.1), the data (g, ξ, λ, μ) for $\lambda = \frac{3(z+1)}{z^2}$ and $\mu = \frac{3(z+7)}{z^2}$ defines an almost η-Ricci soliton.

Indeed,the Riemann and the Ricci curvature tensor fields are computed as follows:

$$R(e_1, e_2)e_2 = \frac{6}{z^2} e_1, \quad R(e_1, e_3)e_3 = -\frac{9}{z^2} e_1, \quad R(e_2, e_1)e_1 = \frac{6}{z^2} e_2,$$

$$R(e_2, e_3)e_3 = -\frac{9}{z^2} e_2, \quad R(e_3, e_1)e_1 = \frac{6}{z^2} e_3, \quad R(e_3, e_2)e_2 = \frac{6}{z^2} e_3,$$

From the above expression of the curvature tensor we can also obtain Ricci tensor

$$S(e_1, e_1) = S(e_2, e_2) = -\frac{3}{z^2}, \quad S(e_3, e_3) = -\frac{18}{z^2}$$

Also, $\alpha = -\frac{3}{z}$, $\rho = -\frac{3}{z^2}$.
Therefore,

$$r = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) = -\frac{24}{z^2} \tag{8.1}$$

scalar curvature $scal = -\frac{24}{z^2}$. By the definition of almost η- Einstein soliton and using (3.12) we obtain

$$2\alpha[g(e_i, e_i) + \eta(e_i)\eta(e_i)] + 2S(e_i, e_i) + (2\lambda - scal)g(e_i, e_i) + 2\mu\eta(e_i)\eta(e_i) = 0$$

for all $i \in \{1, 2, 3\}$, and we have

$$2(1 + \delta_{i3}) - 2\frac{3}{z^2} + (2\lambda + \frac{24}{z^2}) + 2\mu\delta_{i3} = 0$$

for all $i \in \{1, 2, 3\}$.
Therefore $\lambda = \frac{3(z-3)}{z^2}$ and $\mu = \frac{3(z+3)}{z^2}$ the data $(g, \xi, \lambda, \frac{scal}{2}, \mu)$ is an η-Einstein soliton on 3-dimensional $(LCS)_n$-manifolds. Since the $\lambda > 0$, so, it is expanding.

References

[1] Ali, M, and Ahsan, Z., Ricci Solitons and Symmetries of Space time manifold of general relativity, Journal of Advanced Research on Classical and Modern Geometries, Vol.1(2),75- 84, (2014).

[2] Atceken, M., On geometry of submanifolds of $(LCS)_n$-manifolds, Int. J. Math. Math. Sci. 2012 (2012), Art. ID 304647, 11 pp.

[3] Alías, L. J., Romero, A. and Sánchez, M., Uniqueness of complete spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetimes, Gen. Relativity Gravitation 27 (1995), no. 1, 71-84.

[4] Blaga, A. M., On gradient η-Einstein soliton, Kragujevac J. Math., 42 (2), (2018), 229-237.

[5] Blaga, A. M., η-Ricci solitons on Lorentzian para-Sasakian manifolds, Filomat 30 (2016), no. 2, 489-496.

[6] Bagewadi, C. S. and Ingalahalli, G., Ricci Solitons in Lorentzian α-Sasakian Manifolds, Acta Math. Acad. Paedagog. Nyhzi. (N.S.) 28(1) (2012), 59-68.

[7] Calin, C. and Crasmareanu, M., η-Ricci solitons on Hopf Hypersurfaces in complex space forms, Rev. Roumaine Math. Pures Appl. 57 (2012), no. 1, 55-63.

[8] Cho, J. T. and Kimura, M., Ricci solitons and Real hypersurfaces in a complex space form, Tohoku math.J., 61(2009), 205-212.

[9] Catino, G. and Mazzieri, L., Gradient Einstein solitons, Nonlinear Anal. 132 (2016), 66-74.

[10] Chen, B. Y., Classification of torqued vector fields and its applications to Ricci solitons,Kragujevac J. Math.41(2017), no. 2, 239-250.

[11] Chaki, M, C., On generalized quasi-Einstein manifolds, Publ. Math. Debrecen, 58(2001), 683-691.

[12] Eisenhart, L. P., Symmetric tensors of the second order whose first covariant derivatives are zero, Trans. Amer. Math. Soc., 25(2) (1923), 297-306.

[13] Hamilton, R. S., The Ricci flow on surfaces, Mathematics and general relativity, (Santa Cruz. CA, 1986), Contemp. Math. 71, Amer. Math. Soc., (1988), 237-262.

[14] Levy, H. Symmetric tensors of the second order whose covariant derivatives vanish, Ann. Math. 27(2) (1925), 91-98.

[15] Matsumoto, K., On Lorentzian paracontact manifolds, Bull. Yamagata Univ. Nat. Science, 2(1989), 151-156.

[16] Mantica, C. A. and Molinari, L. G., A notes on concircular structure spacetimes, Commun. Korean Math. Soc. 34 (2019), No. 2, pp. 633-635.

[17] Mihai, I and R. Rosca, R., On Lorentzian para-Sasakian manifolds, Classical Anal., World Sci. Publ., Singapore, 1992. 155-169.

[18] Prakasha, D. G. and B. S. Hadimani, η-Ricci solitons on para-Sasakian manifolds, J. Geom., 108 (2), (2017), 383-392.

[19] O'Neill, B., Semi Riemannian Geometry with Applications to Relativity, Academic Press, New York, 1983.

[20] Sharma, R., Certain results on K-contact and (k, μ)-contact manifolds, J. Geom., 89(1-2) (2008), 138-147.

[21] Shaikh, A. A., On Lorentzian almost paracontact manifolds with a structure of the concircular type, Kyungpook Math. J. 43 (2003), no. 2, 305-314.

[22] Shaikh, A. A and Baishya, K. K., On concircular structure spacetimes, J. Math. Stat. 1 (2005), no. 2, 129-132.

[23] Shaikh, A. A., Some results on $(LCS)_n$-manifolds, J. Korean Math. Soc. 46 (2009), no. 3, 449-461.

[24] Shaikh, A. A., Datta, B. R., Ali. A. and Alkhaldi, A. H., $(LCS)_n$-manifolds and Ricci solitons, Int. J. Geom. Methods Mod. Phys., 18(9) (2021), 2150138 (17 pages).

[25] Shaikh, A. A. and Helaluddin, A., Some transformations on $(LCS)_n$-manifolds, Tskuba J. Math., 38(1) (2014), 1-24.

[26] Siddiqi, M. D., η-Einstein soliton in a δ- Lorentzian Trans Sasakian manifolds, Mathematical Advances in Pure and Applied Sciences. 1(1) (2018) 27-38.

[27] Siddiqi, M. D., Chaubey, S. K., η-Einstein soliton on (ε)-Kenmotsu manifolds, Kyungpook Math. J., 60(4) (2020), 805-819.

[28] Siddiqi, M.D. Chaubey, S.K. Khan, M.N.I. $f(R,T)$-Gravity Model with Perfect Fluid Admitting Einstein Solitons. Mathematics 2022, 10, 82. https://doi.org/10.3390/ math10010082.

[29] Siddiqi, M. D, Chaubey, S. K., Almost Conformal η-Ricci solitons in three dimensional Lorentzian Concircular structures, Konuralp Journal of Mathematics, 8(1) (2020), 70-78.

[30] Siddiqi, M. D, η-Einstein solitons in an (ε)-Kenmotsu manifolds with a semi-symmetric metric connection, Annales, Univ. Sci. Budapest, 62 (2019), 5-25.

[31] Siddiqi, M. D, Chaubey, S. K., Almost η-Conformal Ricci solitons in $(LCS)_3$-manifolds, Sarajevo Journal of Mathematics, 16(29), no.2 (2020), 1-15.

[32] Yano, K., Concircular geometry I. Concircular transformations, Proc. Imp. Acad. Tokyo 16 (1940), 195-200

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