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# Difference monomial and its shift sharing a polynomial

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**Abstract** With the notion of weighted sharing, we study the uniqueness of meromorphic functions of finite order, sharing a polynomial with its difference monomial and its shift, obtain some improved and generalized results from those of Sujoy Majumder [12].

## 1 Introduction and definitions

A function $f(z)$ is meromorphic if it is analytic in the complex plane except at isolated poles; if there are no poles, then $f(z)$ reduces to an entire function. Throughout the rest of this paper, we assume that the reader is familiar with Nevanlinna's preliminary results and notation [6], [13]. Let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, where $\mathbb{C}$ denotes set of all complex numbers. For any non-constant meromorphic function $h(z)$ we define $S(r,h) = o\left(T(r,h)\right), (r \to \infty, r \notin E)$ where $E$ denote any set of positive real numbers having finite linear measure. Many authors have investigated the value distribution of difference polynomial.

**Definition 1.1.** [7] Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all $a$-points of $f$ where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that $f,\ g$ share the value $a$ with weight $k$.

**Definition 1.2.** [4] Let $k$ be a positive integer for any constant $a$ in the complex plane $\mathbb{C}$.
We denote

(i) By $N_{k)}\left(r, \frac{1}{f-a}\right)$ the counting function of $a$-points of $f(z)$ with multiplicity $\leq k$.

(ii) By $N_{(k}\left(r, \frac{1}{f-a}\right)$ the counting function of $a$-points of $f(z)$ with multiplicity $\geq k$. Similarly, the reduced counting functions $\overline{N}_{k)}\left(r, \frac{1}{f-a}\right)$ and $\overline{N}_{(k}\left(r, \frac{1}{f-a}\right)$ are defined.

**Definition 1.3.** By a difference product, we mean a difference monomial and its shifts, that is an expression of the type $\prod_{j=1}^{s} f(z+c_j)^{\mu_j}$.

In 2011, K. Liu, X. L. Liu, and T. B. Cao studied the uniqueness of the difference monomials and obtained the following results.

**Theorem A.** [10] Let $f$ and $g$ be two transcendental meromorphic functions with finite order. Suppose that $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$. If $n \geq 14$, $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share 1 CM, then $f \equiv tg$ or $fg \equiv t$, where $t^{n+1} = 1$.

**Theorem B.** [10] Let $f$ and $g$ be two transcendental meromorphic functions with finite order. Suppose that $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$. If $n \geq 26$, $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share 1 IM, then $f \equiv tg$ or $fg \equiv t$, where $t^{n+1} = 1$.

In 2015, Y.Liu, J. P. Wang, and F. H. Liu improved Theorems A and B and obtained the following results.

**Theorem C.** [11] Let $c \in \mathbb{C} \setminus \{0\}$ and let $f$ and $g$ be two transcendental meromorphic functions with finite order, and $n\,(\geq 14)$, $k\,(\geq 3)$ be two positive integers. If $E_k(1, f^n(z)f(z+c)) = E_k(1, g^n(z)g(z+c))$, then $f \equiv t_1 g$ or $fg \equiv t_2$ for some constants $t_1$ and $t_2$ satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.

**Theorem D.** [11] Let $c \in \mathbb{C} \setminus \{0\}$ and let $f$ and $g$ be two transcendental meromorphic functions with finite order, and $n\,(\geq 16)$ be a positive integer. If $E_2(1, f^n(z)f(z+c)) = E_2(1, g^n(z)g(z+c))$, then $f \equiv t_1 g$ or $fg \equiv t_2$, for some constants $t_1$ and $t_2$ satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.

**Theorem E.** [11] Let $c \in \mathbb{C} \setminus \{0\}$ and let $f$ and $g$ be two transcendental meromorphic functions with finite order, and $n\,(\geq 22)$ be a positive integer. If $E_1(1, f^n(z)f(z+c)) = E_1(1, g^n(z)g(z+c))$, then $f \equiv t_1 g$ or $fg \equiv t_2$, for some constants $t_1$ and $t_2$ satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.

In 2017, Sujoy Majumder replaced the sharing value 1 by a nonzero polynomial $p(z)$ in C, D, and E and obtained the following results.

**Theorem F.** [12] Let $f$ and $g$ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C}\setminus\{0\}$ and $n \in \mathbb{N}$ be such that $n \geq 14$. Let $p\,(\not\equiv 0)$ be a polynomial such that $\deg(p) < (n-1)/2$. If $f^n(z)f(z+c) - p(z)$ and $g^n(z)g(z+c) - p(z)$ share $(0, 2)$, then one of the following two cases holds:

1 $f \equiv tg$ for some constant $t$ such that $t^{n+1} = 1$,

2 $fg \equiv t$, where $p(z)$ reduces to a nonzero constant $c$ and $t$ is a constant such that $t^{n+1} = c^2$.

**Theorem G.** [12] Let $f$ and $g$ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C}\setminus\{0\}$ and $n \in \mathbb{N}$ be such that $n \geq 16$. Let $p\,(\not\equiv 0)$ be a polynomial such that $\deg(p) < (n-1)/2$. Suppose $f^n(z)f(z+c) - p(z)$ and $g^n(z)g(z+c) - p(z)$ share $(0, 1)$. Then the conclusion of Theorem F holds.

**Theorem H.** [12] Let $f$ and $g$ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C}\setminus\{0\}$ and $n \in \mathbb{N}$ be such that $n \geq 26$. Let $p\,(\not\equiv 0)$ be a polynomial such that $\deg(p) < (n-1)/2$. Suppose $f^n(z)f(z+c) - p(z)$ and $g^n(z)g(z+c) - p(z)$ share $(0, 0)$. Then conclusion of Theorem F holds.

It is quite natural to ask the following question.

**Question 1.** When two finite order meromorphic functions $f$ and $g$ share a nonzero polynomial, what can be said about their uniqueness if we consider the difference monomial and its shift instead of the difference polynomial?

Based on the possible answer to the above question, we obtain the following results that improve and generalize Theorems F - H of [12].

## 2 Main results

**Theorem 2.1.** *Let $f$ and $g$ be two transcendental meromorphic function of finite order and $n$ be a positive integer such that $n \geq 2s + 2\sigma + 6$. Suppose that $c_j \in \mathbb{C} \setminus \{0\}$ for $j = \{1, 2, 3, \ldots, s\}$. Let $f^n \prod_{j=1}^{s} f(z + c_j)^{\mu_j} - p(z)$ and $g^n \prod_{j=1}^{s} g(z + c_j)^{\mu_j} - p(z)$ share $(0, 2)$ where $p$ be a nonzero polynomial such that $2deg\,(p(z)) < n - \sigma$, then one of the following two cases holds:*

*(i) $f \equiv tg$ for some constant $t$ such that $t^{n+\sigma} = 1$,*

*(ii) $f(z)g(z) \equiv t$, where $p(z)$ reduces to a nonzero constant $c$ and $t$ is a constant such that $t^{n+\sigma} = c^2$.*

**Theorem 2.2.** *Let $f$ and $g$ be two transcendental meromorphic function of finite order and $n$ be a positive integer such that $n > \frac{5s+2\sigma+11}{2}$. Suppose that $c_j \in \mathbb{C} \setminus \{0\}$ for $j = \{1, 2, 3, ...s\}$. Let $f^n \prod_{j=1}^{s} f(z + c_j)^{\mu_j} - p(z)$ and $g^n \prod_{j=1}^{s} g(z + c_j)^{\mu_j} - p(z)$ share $(0, 1)$ where $p$ be a nonzero polynomial such that $2deg\,(p(z)) < n - \sigma$, then the conclusion of Theorem 2.1 holds.*

**Theorem 2.3.** *Let $f$ and $g$ be two transcendental meromorphic function of finite order and $n$ be a positive integer such that $n \geq 5s + \sigma + 9$. Suppose that $c_j \in \mathbb{C} \setminus \{0\}$ for $j = \{1, 2, 3, ...s\}$. Let $f^n \prod_{j=1}^{s} f(z + c_j)^{\mu_j} - p(z)$ and $g^n \prod_{j=1}^{s} g(z + c_j)^{\mu_j} - p(z)$ share $(0, 0)$ where $p$ be a nonzero polynomial such that $2deg\,(p(z)) < n - \sigma$, then the conclusion of Theorem 2.1 holds.*

**Remark 2.4.** Theorem 2.1 - 2.3 is an improvement of Theorem F - H respctively.
Taking $s = 1$ we get $\sigma = \sum_{j=1}^{s} \mu_j = 1$, $f^n \prod_{j=1}^{s} f(z+c_j)^{\mu_j}$ and $g^n \prod_{j=1}^{s} g(z+c_j)^{\mu_j}$ reduces to $f^n f(z+c)$ and $g^n g(z+c)$, then

(i) In Theorem 2.1 we obtain $n \geq 10$ which is an improvement of Theorem F,

(ii) In Theorem 2.2 we obtain $n > 9$ which is an improvement of Theorem G,

(iii) In Theorem 2.3 we obtain $n \geq 15$ which is an improvement of Theorem H.

## 3 Some Lemmas

In this section, we present few lemmas needed in the sequel. Let $F$ and $G$ be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. We denote by $H$ the function

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{3.1}$$

**Lemma 3.1.** *[3] Let $f$ be a meromorphic function of finite order $\rho$, and let $c \in \mathbb{C} \setminus \{0\}$ be fixed. Then for each $\epsilon > 0$, we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O(r^{\rho-1+\epsilon}) = S(r,f).$$

*The following lemma has few modifications of the original version [3, Corollary 2.5]*

**Lemma 3.2.** *[13] Let $f$ be a non-constant meromorphic function and let $a_n(z)(\not\equiv 0), a_{n-1}(z), \ldots, a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r,f)$ for $i = 0, 1, 2, ..., n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0) = nT(r,f) + S(r,f).$$

**Lemma 3.3.** *[5] Let $f$ be a non-constant meromorphic function of finite order and $c \in \mathbb{C}$. Then*

$$N(r, 0; f(z+c)) \leq N(r, 0; f(z)) + S(r,f), \quad N(r, \infty; f(z+c)) \leq N(r, \infty; f) + S(r,f),$$

$$\overline{N}(r, 0; f(z+c)) \leq \overline{N}(r, 0; f(z)) + S(r,f), \quad \overline{N}(r, \infty; f(z+c)) \leq \overline{N}(r, \infty; f) + S(r,f).$$

**Lemma 3.4.** *[3] Let $f$ be a transcendental meromorphic function of finite order, $c \in \mathbb{C} \setminus \{0\}$ be fixed. Then*

$$T(r, f(z+c)) = T(r,f) + S(r,f).$$

**Lemma 3.5.** *[13] Let $f$ and $g$ be two non-constant meromorphic functions. Then*

$$N\left(r, \infty; \frac{f}{g}\right) - N\left(r, \infty; \frac{g}{f}\right) = N(r, \infty; f) + N(r, 0; g) - N(r, \infty; g) - N(r, 0; f).$$

**Lemma 3.6.** *Let $f$ be a transcendental meromorphic function of finite order and let $F = f^n \prod_{j=1}^{s} (f(z+c_j))^{\mu_j}$, where $n$ is positive integer. Then*

$$(n - \sigma)T(r,f) \leq T(r,F) + S(r,f).$$

*Proof.* From Lemma 3.2, 3.1 and first fundamental theorem, we obtain

$$
\begin{aligned}
(n+\sigma)T(r,f) &= T(r,f^{n+\sigma}) + S(r,f),\\
&\leq T\left(r, \frac{f^{\sigma}(z)F}{\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}}\right) + S(r,f),\\
&\leq T(r,F) + T\left(r, \frac{f^{\sigma}(z)}{\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}}\right) + S(r,f),\\
&\leq T(r,F) + N\left(r, \frac{\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}}{f^{\sigma}(z)}\right) + S(r,f),\\
&\leq T(r,F) + 2\sigma T(r,f) + S(r,f),\\
(n-\sigma)T(r,f) &\leq T(r,F) + S(r,f).
\end{aligned}
$$

This completes the proof of Lemma. □

**Lemma 3.7.** *Let $f$ and $g$ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C}\setminus\{0\}$ and $n \in \mathbb{N}$ such that $n > \sigma$. Let $p$ be a nonzero polynomial such that $2deg\,(p) < n-\sigma$. Then*

*(i) if $deg(p) \geq 1$, then $f^n \prod_{j=1}^{s}(f(z+c_j))^{\mu_j} . g^n \prod_{j=1}^{s}(g(z+c_j))^{\mu_j} \not\equiv p^2(z)$,*

*(ii) if $p(z) = c \in \mathbb{C}\setminus\{0\}$,*
*then the relation $f^n \prod_{j=1}^{s}(f(z+c_j))^{\mu_j}\, g^n \prod_{j=1}^{s}(g(z+c_j))^{\mu_j} \equiv p^2(z)$, always implies that $f.g = t$, where $t$ is a constant such that $t^{n+\sigma} = c^2$.*

*Proof.* Suppose

$$\left(f^n \prod_{j=1}^{s}(f(z+c_j))^{\mu_j}\right)\left(g^n \prod_{j=1}^{s}(g(z+c_j))^{\mu_j}\right) \equiv p^2(z). \tag{3.2}$$

Let $h_1 = fg$ then by 3.2, we have

$$h_1^n(z) = \frac{p^2(z)}{\prod_{j=1}^{s} h_1(z+c_j)^{\mu_j}}. \tag{3.3}$$

We now consider the following two cases,

**Case 1.** Suppose $h_1$ is a transcendental meromorphic function. Now by Lemma 3.1, 3.2 and 3.3, we get

$$
\begin{aligned}
nT(r,h_1) &= T(r,h_1^n) + S(r,h_1),\\
&= T\left(r, \frac{p^2(z)}{\prod_{j=1}^{s} h_1(z+c_j)^{\mu_j}}\right) + S(r,h_1),\\
&\leq N\left(r,0;\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right) + N\left(r,0;\prod_{j=1}^{s} g(z+c_j)^{\mu_j}\right) + S(r,h_1),\\
&\leq \sigma\left[T(r,f) + T(r,g)\right] + S(r,h_1),
\end{aligned}
$$

from $h_1 = fg$, which implies $T(r, h_1) = T(r, f) + T(r, g)$, we get

$$n\left[T(r, f) + T(r, g)\right] \leq \sigma\left[T(r, f) + T(r, g)\right] + S\left(r, h_1\right),$$

which is a contradiction.
**Case 2.** Suppose $h_1$ is a rational function. Let

$$h_1 = \frac{h_2}{h_3} \tag{3.4}$$

where $h_2$, $h_3$ are two non-zero relatively prime polynomials. By 3.3, we have

$$T(r, h_1) = \max\{deg(h_2), deg(h_3)\} \log r + O(1). \tag{3.5}$$

Now by 3.3 - 3.5, we have

$$\begin{aligned} \max\{deg(h_2), deg(h_3)\} \log r &= T\left(r, h_1^n\right) + O(1), \\ &\leq \sigma\left[T(r, f) + T(r, g)\right] + 2T(r, p) + O(1), \end{aligned}$$

$$\max\{deg(h_2), deg(h_3)\} \log r \leq \sigma \max\{deg(h_2),\ deg(h_3)\} \log r + 2T(r, p) + O(1). \tag{3.6}$$

We see that $\max\{deg(h_2), deg(h_3)\} \geq 1$. Now by 3.6, we deduce that $(n - \sigma) \leq 2deg\left(p(z)\right)$. Which contradicts our assumption that $2deg\left(p(z)\right) < (n - \sigma)$. Hence $h_1$ must be a non-zero constant.
Let

$$h_1 = t \in \mathbb{C} \setminus \{0\}. \tag{3.7}$$

Now when $deg(p) \geq 1$ by 3.3 and 3.7, we arrive at a contradiction.
In this case we have $\left(f^n \prod_{j=1}^{s} \left(f(z + c_j)\right)^{\mu_j}\right) \left(g^n \prod_{j=1}^{s} \left(g(z + c_j)\right)^{\mu_j}\right) \not\equiv p^2(z)$.
Suppose $p(z) = c \in \mathbb{C} \setminus \{0\}$. So by 3.2, we see that $h_1^{n+\sigma} \equiv c^2$.
By 3.7, we get $t^{n+\sigma} \equiv c^2$.
This completes the proof. □

**Lemma 3.8.** *Let $f$ and $g$ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C}\setminus\{0\}$ be finite complex constant such that $f(z+c_j) \neq f(z)$ for $i = 0, 1, 2, \ldots, s$. and $g(z+c_j) \neq g(z)$ and let $n$ be an integer such that $n \geq 3(s+1) + \sigma$. Let $F(z) = \frac{f^n \prod_{j=1}^{s}(f(z+c_j))^{\mu_j}}{p(z)}$ and $G(z) = \frac{g^n \prod_{j=1}^{s}(g(z+c_j))^{\mu_j}}{p(z)}$, where $p(z)$ is nonzero polynomial. If $H \equiv 0$ then either*

*(i)* $f^n \prod_{j=1}^{s} \left(f(z + c_j)\right)^{\mu_j} . g^n \prod_{j=1}^{s} \left(g(z + c_j)\right)^{\mu_j} \equiv p^2(z)$,
*where* $f^n \prod_{j=1}^{s} \left(f(z + c_j)\right)^{\mu_j} - p(z)$ *and* $g^n \prod_{j=1}^{s} \left(g(z + c_j)\right)^{\mu_j} - p(z)$ *share* 0 *CM,*

*(ii)* $f(z) = tg(z)$ *for a constant* $t$ *with* $t^{n+\sigma} = 1$.

*Proof.*

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F - 1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G - 1}\right).$$

Since $H \equiv 0$, by integration, we get

$$\frac{1}{F - 1} = \frac{BG + A - B}{G - 1} \tag{3.8}$$

where $A \neq 0$, $B$ are constant. From 3.7 it is clear that $F$ and $G$ share $(1, \infty)$. We now consider the following cases.

**Case 1.** Let $B \neq 0$ and $A \neq B$, if $B = -1$ then from 3.8, we have $F = \frac{-A}{G-A-1}$.
Therefore $\overline{N}\left(r, \frac{1}{G-(A+1)}\right) = \overline{N}\left(r, F\right) \leq N\left(r, \frac{1}{p}\right) \leq T(r,p) = S(r,g)$.
So in view of Lemma 3.6 and the Second fundamental theorem , we get

$$
\begin{aligned}
(n-\sigma)T(r,g) &\leq T\left(r, g^n \prod_{j=1}^{s} (g(z+c_j))^{\mu_j}\right) + S(r,g),\\
&\leq T(r,G) + S(r,g),\\
&\leq \overline{N}\left(r,\infty;G\right) + \overline{N}\left(r,0;G\right) + \overline{N}\left(r,(A+1);G\right) + S(r,g),\\
&\leq 2(s+1)T(r,g) + S(r,g)
\end{aligned}
$$

which contradicts with $n > 2(s+1) + \sigma$.
If $B \neq 1$, from 3.8 we get that

$$
F - \left(1 + \frac{1}{B}\right) = \frac{-A}{B^2\left(G + \frac{A-B}{B}\right)}.
$$

So $\overline{N}\left(r, \frac{B-A}{B}; G\right) = \overline{N}(r,\infty;F) \leq S(r,g)$. Using Lemma 3.6 and the same argument as used in the case when $B = -1$ we can get a contradiction.
**Case 2.** Let $B \neq 0$ and $A = B$. If $B = -1$, then from 3.8, we have

$$
f^n \prod_{j=1}^{s} f(z+c_j)^{\mu_j} g^n \prod_{j=1}^{s} g(z+c_j)^{\mu_j} = p^2(z),
$$

when $\left[f^n \prod\limits_{j=1}^{s} f(z+c_j)^{\mu_j}\right] - p(z)$ and $\left[g^n \prod\limits_{j=1}^{s} g(z+c_j)^{\mu_j}\right] - p(z)$ share 0 CM.
If $B \neq 1$, from 3.8 we have

$$
\frac{1}{F} = \frac{BG}{(1+B)G - 1}.
$$

Therefore $\overline{N}\left(r,(1+B);G\right) = \overline{N}(r,\infty;F)$. So in view of Lemma 3.8 and the Second fundamental theorem, we get

$$
\begin{aligned}
(n-\sigma)T(r,g) &\leq T(r,G) + S(r,g),\\
&\leq \overline{N}\left(r,\infty;G\right) + \overline{N}\left(r,0;G\right) + \overline{N}\left(r,(1+B);G\right) + S(r,g),\\
&\leq 2(s+1)T(r,g) + (1+s)T(r,f) + S(r,f) + S(r,g),
\end{aligned}
$$

so for $r \in I$, we have

$$
\left(n - \sigma - 3(s+1)\right)T(r,g) \leq S(r,g).
$$

Which is contardiction, since $n > 3(s+1) + \sigma$.
**Case 3.** Let $B = 0$. From 3.8 we obtain

$$
F = \frac{G + A - 1}{A} \tag{3.9}
$$

if $A \neq 1$, then from 3.9, we obtain $\overline{N}\left(r,(1-A);G\right) = \overline{N}\left(r,0;F\right)$.
We can similarly deduce a contradiction as in case 2. Therefore $A = 1$ and from 3.9, we obtain , $F(z) \equiv G(z)$.
That is

$$
f^n \prod_{j=1}^{s} f(z+c_j)^{\mu_j} \equiv g^n \prod_{j=1}^{s} g(z+c_j)^{\mu_j}. \tag{3.10}
$$

Let $h = \frac{f}{g}$ and then substituting $f = gh$ in 3.10, we deduce

$$
h^n(z) = \frac{1}{\prod\limits_{j=1}^{s} h(z+c_j)^{\mu_j}} \tag{3.11}
$$

$$h^{n+1}=\frac{f}{g\prod\limits_{j=1}^{s}h(z+c_j)^{\mu_j}},$$

if $h$ is not a constant, then we have

$$\begin{aligned}(n+1)T(r,h)&\leq T\left(r,\frac{f}{\prod\limits_{j=1}^{s}f(z+c_j)^{\mu_j}}\right)+T\left(r,\frac{\prod\limits_{j=1}^{s}g(z+c_j)^{\mu_j}}{g}\right)+S(r,f)+S(r,g),\\&\leq(1+\sigma)\left[T(r,f)+T(r,g)\right]+S(r,f)+S(r,g),\end{aligned}$$

we obtain

$$(n-\sigma)\left[T(r,f)+T(r,g)\right]\leq S(r,f)+S(r,g),$$

which is impossible.
Therefore $h$ is constant, then substituting $f=gh$ in 3.10, we have $h^{n+\sigma}\equiv 1$.
Therefore $f=tg$ where $t$ is a constant with $t^{n+\sigma}=1$. □

**Lemma 3.9.** *[1] If $f$, $g$ be two non-constant meromorphic functions such that they share$(1,1)$. Then*

$$2\overline{N}_L(r,1;f)+2\overline{N}_L(r,1;g)+\overline{N}_E^{(2}(r,1;f)-\overline{N}_{f>2}(r,1;g)\leq N(r,1;g)-\overline{N}(r,1;g).$$

**Lemma 3.10.** *[2] Let $f$ and $g$ share $(1,1)$. Then*

$$\overline{N}_{f>2}(r,1;g)\leq\frac{1}{2}\overline{N}(r,0;f)+\frac{1}{2}\overline{N}(r,\infty;f)-\frac{1}{2}N_0(r,0;f^{'})+S(r,f),$$

*where $N_0(r,0;f')$ is the counting function of the zeros of $f'$ which are not the zero of $f(f-1)$.*

**Lemma 3.11.** *[2] Let $f$, $g$ share $(1,0)$. Then*

*(i)* $\overline{N}_{f>1}(r,1;g)\leq\overline{N}(r,0;f)+\overline{N}(r,\infty;f)-N_0(r,0;f^{'})+S(r,f)$

*(ii)* $\overline{N}_{g>1}(r,1;f)\leq\overline{N}(r,0;g)+\overline{N}(r,\infty;g)-N_0(r,0;g^{'})+S(r,f)$.

**Lemma 3.12.** *[9] Let $f$, $g$ share $(1,0)$. Then*

$$\overline{N}_L(r,1;f)\leq\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+S(r,f).$$

**Lemma 3.13.** *[2] Let $f$ and $g$ be two non-constant meromorphic functions sharing $(1,0)$. Then*

$$\overline{N}_L(r,1;f)+2\overline{N}_L(r,1;g)+\overline{N}_E^{(2}(r,1;f)-\overline{N}_{f>1}(r,1;g)-\overline{N}_{g>1}(r,1;f)\leq N(r,1;g)-\overline{N}(r,1;g).$$

**Lemma 3.14.** *[8] If $N(r,0;f^{(k)}|f\neq 0)$ denotes the counting function of those zeros of $f^{(k)}(z)$ which are not the zeros of $f(z)$, where a zero of $f^{(k)}(z)$ is counted according to its multiplicity, then*

$$N(r,0;f^{(k)}|f\neq 0)\leq k\overline{N}(r,\infty;f)+N(r,0;f|<k)+k\overline{N}(r,0;f|\geq k)+S(r,f).$$

## 4 Proof of main Theorems

***Proof of Theorem 2.1.*** Let $F=\frac{f^n\prod\limits_{j=1}^{s}f(z+c_j)^{\mu_j}}{p(z)}$ and $G=\frac{g^n\prod\limits_{j=1}^{s}g(z+c_j)^{\mu_j}}{p(z)}$. It follows that $F$ and $G$ share $(1,2)$ except the zeors of $p(z)$.
**Case 1.** Let $H\not\equiv 0$ from 3.1, we obtain

$$\begin{aligned}N(r,\infty;H)\leq\overline{N}_*(r,1;F,G)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_0(r,0;F^{'})\\+\overline{N}_0(r,0;G^{'}).\end{aligned}\tag{4.1}$$

$\overline{N}_0(r,0;F^{'})$ is the reduced counting function of those zero's of $F^{'}$ which are not the zero's of $F(F-1)$, and $\overline{N}_0(r,0;G^{'}$ is similarly defined.
Let $z_0$ be a simple zero of $F-1$ such that $p(z_0)\neq 0$. Then $z_0$ is a simple zero of $G-1$ and a zero of $H$. So

$$N(r,1;F|=1)\leq N(r,0;H)\leq N(r,\infty;H)+S(r,f)+S(r,g). \tag{4.2}$$

From 4.1 and 4.2

$$\begin{aligned}\overline{N}(r,1;F)\leq \overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_*(r,1;F,G)+\overline{N}(r,1;F|\geq 2)\\ +\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g).\end{aligned} \tag{4.3}$$

Now in the view of Lemma 3.14, we get

$$\begin{aligned}\overline{N}_0(r,0;G^{'})+\overline{N}(r,1;F|\geq 2)+\overline{N}_*(r,1;F,G)&\leq \overline{N}_0(r,0;G^{'})+\overline{N}(r,1;F|\geq 2)\\ &\quad+\overline{N}(r,1;F|\geq 3)\\ &\leq N(r,0;G^{'}|G\neq 0)\leq \overline{N}(r,0;G)+S(r,g).\end{aligned} \tag{4.4}$$

Using equations 4.3, 4.4 and Lemma 3.6, we get from Second fundamental theorem that

$$\begin{aligned}(n-\sigma)T(r,f)&\leq T(r,F)+S(r,f),\\ &\leq \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,1;F)-N_0(r,0;F^{'})+S(r,f),\\ &\leq \overline{N}(r,\infty;F)+N_2(r,0;F)+N_2(r,0;G)+S(r,f)+S(r,g),\\ &\leq \overline{N}(r,\infty;f)+\overline{N}\left(r,\infty;\prod_{j=1}^{s}f(z+c_j)^{\mu_j}\right)+2\overline{N}(r,0;f)\\ &\quad+N\left(r,0;\prod_{j=1}^{s}f(z+c_j)^{\mu_j}\right)+N_2\left(r,0;g^n\prod_{j=1}^{s}g(z+c_j)^{\mu_j}\right)\\ &\quad+S(r,f)+S(r,g),\\ &\leq (3+s+\sigma)\,T(r,f)+(2+s)T(r,g)+S(r,f)+S(r,g).\end{aligned} \tag{4.5}$$

Similarly, we can obtain

$$(n-\sigma)T(r,g)\leq (3+s+\sigma)T(r,g)+(2+s)T(r,f)+S(r,f)+S(r,g). \tag{4.6}$$

Combining 4.5 and 4.6 we have

$$(n-2\sigma-2s-5)\left[T(r,f)+T(r,g)\right]\leq S(r,f)+S(r,g). \tag{4.7}$$

Since $n>2\sigma+2s+5$, equation 4.7 leads to contradiction.
**Case 2.** Let $H\equiv 0$. Then the theorem follows from Lemmas 3.7 and 3.8. This completes the proof. □

***Proof of Theorem 2.2.*** Let $F(z)=\frac{f^n(z)\prod_{j=1}^{s}f(z+c_j)^{\mu_j}}{p(z)}$ and $G(z)=\frac{g^n(z)\prod_{j=1}^{s}g(z+c_j)^{\mu_j}}{p(z)}$. Then $F$ and $G$ share (1,1) except for the zeros of $p(z)$. We now consider the following two cases.

**Case 1.** $H\not\equiv 0$.

Using Lemmas 3.6, 3.8, 3.9 and equations 4.1 and 4.2, we get

$$\begin{aligned}
\overline{N}(r,1;F) &\le N(r,1;F|=1)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F),\\
&\le \overline{N}(r,0;F|\ge 2)+\overline{N}(r,0;G|\ge 2)+\overline{N}_*(r,1;F,G)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)\\
&\quad +\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g),\\
&\le \overline{N}(r,0;F|\ge 2)+\overline{N}(r,0;G|\ge 2)+2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)\\
&\quad +\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g),\\
&\le \overline{N}(r,0;F|\ge 2)+\overline{N}(r,0;G|\ge 2)+\overline{N}_{F>2}(r,1;G)+N(r,1;G)-\overline{N}(r,1;G)\\
&\quad +\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g),\\
&\le \overline{N}(r,0;F|\ge 2)+\frac{1}{2}\overline{N}(r,0;F)+\overline{N}(r,0;G|\ge 2)+N(r,0;G^{'}|G\ne 0)+\overline{N}_0(r,0;F^{'})\\
&\quad +S(r,f)+S(r,g),
\end{aligned}$$

$$\overline{N}(r,1;F)\le \overline{N}(r,0;F|\ge 2)+\frac{1}{2}\overline{N}(r,0;F)+N_2(r,0;G)+\overline{N}_0(r,0;F^{'})+S(r,f)+S(r,g). \tag{4.8}$$

Hence by using Second fundamental theorem, 4.8, Lemmas 3.1 and 3.6 we get

$$\begin{aligned}
(n-\sigma)T(r,f) &\le T(r,F)+S(r,f),\\
&\le \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,1;F)-N_0(r,0;F')+S(r,f),\\
&\le \overline{N}(r,\infty;F)+\frac{1}{2}\overline{N}(r,0;F)+N_2(r,0;F)+N_2(r,0;G)+S(r,f)+S(r,g),\\
&\le \overline{N}(r,\infty;f)+\overline{N}\Big(r,\infty;\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\Big)+\frac{1}{2}\overline{N}\Big(r,0;f^n\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\Big)\\
&\quad +2\overline{N}(r,0;f)+N\Big(r,0;\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\Big)+N_2\left(r,0;g^n\prod_{j=1}^{s} g(z+c_j)^{\mu_j}\right)\\
&\quad +S(r,f)+S(r,g),
\end{aligned}$$

$$(n-\sigma)T(r,f)\le\left(\frac{7+3s+2\sigma}{2}\right)T(r,f)+(2+s)T(r,g)+S(r,f)+S(r,g). \tag{4.9}$$

Similarly, we obtain

$$(n-\sigma)T(r,g)\le\left(\frac{7+3s+2\sigma}{2}\right)T(r,g)+(2+s)T(r,f)+S(r,f)+S(r,g). \tag{4.10}$$

Combining 4.9 and 4.10, we get

$$\left(n-\frac{5s+4\sigma+11}{2}\right)[T(r,f)+T(r,g)]\le S(r,f)+S(r,g). \tag{4.11}$$

Since $n>\frac{5s+4\sigma+11}{2}$, equation 4.11 leads to contradiction.

**Case 2.** Let $H\equiv 0$.
Then the theorem follows from Lemmas 3.7 and 3.8.
This completes the proof. □

***Proof of Theorem 2.3.*** Let $F(z)=\frac{f^n(z)\prod_{j=1}^{s} f(z+c_j)^{\mu_j}}{p(z)}$ and $G(z)=\frac{g^n(z)\prod_{j=1}^{s} g(z+c_j)^{\mu_j}}{p(z)}$. Then $F$ and $G$ share $(1,0)$ except for the zeros of $p(z)$.
In this case equation 4.2 changes to

$$N_E^{1)}(r,1;F) \le N(r,0;H) \le N(r,\infty;H) + S(r,F) + S(r,G). \tag{4.12}$$

Using Lemmas 3.11 - 3.14 and equations 4.2 and 4.12, we get

$$\begin{aligned}
\overline{N}(r,1;F) &\le N_E^{1)}(r,1;F) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F)\\
&\le \overline{N}_*(r,1;F,G) + \overline{N}(r,0;F|\ge 2) + \overline{N}(r,0;G|\ge 2) + \overline{N}_L(r,1;F)\\
&\quad + \overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F) + \overline{N}_0(r,0;F^{'}) + \overline{N}_0(r,0;G^{'}) + S(r,f) + S(r,g),\\
&\le \overline{N}(r,0;F|\ge 2) + \overline{N}(r,0;G|\ge 2) + 2\overline{N}_L(r,1;F) + 2\overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F)\\
&\quad + \overline{N}_0(r,0;F^{'}) + \overline{N}_0(r,0;G^{'}) + S(r,f) + S(r,g),\\
&\le \overline{N}(r,0;F|\ge 2) + \overline{N}(r,0;G|\ge 2) + \overline{N}_{F>1}(r,1;G) + \overline{N}_{G>1}(r,1;F) + \overline{N}_L(r,1;F)\\
&\quad + N(r,1;G) - \overline{N}(r,1;G) + \overline{N}_0(r,0;F^{'}) + \overline{N}_0(r,0;G^{'}) + S(r,f) + S(r,g),\\
&\le N_2(r,0;F) + \overline{N}(r,0;F) + N_2(r,0;G) + N(r,0;G^{'}|G\ne 0) + \overline{N}_0(r,0;F^{'}) + S(r,f)\\
&\quad + S(r,g),
\end{aligned}$$

$$\overline{N}(r,1;F) \le N_2(r,0;F)+\overline{N}(r,0;F)+N_2(r,0;G)+\overline{N}(r,0;G)+\overline{N}_0(r,0;F^{'})+S(r,f)+S(r,g). \tag{4.13}$$

Hence using 4.13, Lemmas 3.1 and 3.6, we obtain from Second fundamental theorem that

$$\begin{aligned}
(n-\sigma)T(r,f) &\le T(r,F) + S(r,f),\\
&\le \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,1;F) - N_0(r,0;F^{'}) + S(r,f),\\
&\le \overline{N}(r,\infty;F) + 2N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,0;G) + S(r,f) + S(r,g),\\
&\le \overline{N}(r,\infty;f) + \overline{N}\left(r,\infty;\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right) + 4\overline{N}(r,0;f)\\
&\quad + 2N\left(r,0;\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right) + N_2\left(r,0;g^n\prod_{j=1}^{s} g(z+c_j)^{\mu_j}\right)\\
&\quad + \overline{N}\left(r,0;g^n\prod_{j=1}^{s} g(z+c_j)^{\mu_j}\right) + S(r,f) + S(r,g),
\end{aligned}$$

$$(n-\sigma)T(r,f) \le (5+3s)T(r,f) + (3+2s)T(r,g) + S(r,f) + S(r,g). \tag{4.14}$$

Similarly, we obtain

$$(n-\sigma)T(r,g) \le (5+3s)T(r,g) + (3+2s)T(r,f) + S(r,f) + S(r,g). \tag{4.15}$$

Combining 4.14 and 4.15, we get

$$(n-\sigma-5s-8)\left[T(r,f)+T(r,g)\right] \le S(r,f) + S(r,g). \tag{4.16}$$

Since $n > 5s+\sigma+8$, 4.16 leads to contradiction.
**Case 2.** Let $H \equiv 0$. Then the theorem follows from Lemmas 3.7 and 3.8.
This completes the proof.

□

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