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EQUIVALENCE OF PSEUDO-DISCRETE MODULES AND DISCRETE MODULES

Prajnan Kumar Bhagawati, Hiya Saharia and Helen K. Saikia

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Abstract Let M be a left $R-$module. In this paper, various results pertaining to the equivalence of discrete modules and pseudo-discrete modules are established. Furthermore, a new class of module termed as $(CD)_2$-module is introduced and certain characteristics of this new class of module, along with its relation with the above mentioned equivalence of discrete modules and pseudo- discrete modules are studied.

1 Introduction

Injective Modules constitute an important part of Module Theory. A weaker notion of injective module is 'quasi-injective module' whereas projective module is a dual interpretation of injective module. At this point we recall that the definition of $M-$projective and $M-$injective were first coined by G. Azumaya in his unpublised paper named "$M-$ projective and $M-$injective modules". The researcher made generalization of injectivity and named it as 'Continuous Module'. The concept of continuity in continuous module was originated from the book- "Continuous Geometry" by Von Neumann [7]. The continuous modules are defined in terms of C_1,C_2,C_3 and C_4 conditions. The concept of discrete module was introduced as a dual notion of continuous module, which was dualized in terms of d-continuous module by Mohamed and Singh in their paper [13]. At first the dual concept of continuous module was studied under various names namely perfect, dual continuous etc. By the motivation of Oshiro's theorem, which states that every quasi-discrete module is the direct sum of indecomposables, Mohamed and Müller termed it as 'Discrete Module'. Like the continuous modules, the dual notion of C_1,C_2,C_3 and C_4 modules were subsequently introduced namely D_1, D_2,D_3 and D_4 respectively.

In this article, our main objective is to show the equivalence of discrete and pseudo-discrete module. An attempt is made to show partly the equivalence between pseudo-discrete modules and pseudo-continuous modules. We denote:

Condition(*) : Equivalence of discrete modules and pseudo-discrete modules.

Condition(**): Equivalence of pseudo-discrete modules and pseudo-continuous modules.

In the first section, we consider the ring $End_R(M)$ and apply different conditions to obtain the structure of an R-module which satisfies condition (*). In the subsequent theorems the equivalence of discrete modules and pseudo-discrete modules is established in terms of the finite exchange property. Also, a new class of module called $(CD)_{(2,2)}$-module is introduced which satisfies both condition(*) and condition(**). In short, this module is denoted as $(CD)_2$-module. In theorem (3.15), it is shown that if M is a pseudo-discrete module with DCC on summands and if for any two summands D_1, D_2 with $M = D_1 + D_2$, there exists a $f \in Hom(D_1, D_2)$ such that $Imf \subseteq^{\oplus} M$, then M is a discrete module if and only if every epimorphism $M \longrightarrow M$ with small kernel is an isomorphism i.e. M satisfies condition(*). Finally in theorem (3.16), it is established that if M is a pseudo-discrete module with DCC on summands with $\bigtriangledown = J$ and $End(M)/\bigtriangledown$ is regular and then M is a discrete module under the assumption that for any two summands D_1, D_2 with $M = D_1 + D_2$, there exists a $f \in Hom(D_1, D_2)$ such that $Imf \subseteq^{\oplus} M$.

The second section deals with a brief discussion on $(CD)_2$-module. It is easy to verify that every $(CD)_2$-module is a C_2-module and a D_2-module. But every C_2-module is not a $(CD)_2$-module. Similarly, every D_2-module is not a $(CD)_2$-module.

Throughout this paper, M is considered as a left module over an associative ring R with unity; Imf and $Kerf$ denote the image and kernel of a homomorphism f respectively; $\bigtriangledown$ denotes the cosingular ideal of a module M and J denotes the Jacobson radical of $End_R(M)$. Moreover, the symbols $\leq$, $\subseteq^{\oplus}$ and $\subseteq^{ess}$ denote the submodule, direct summand and essential submodule of a module respectively.

2 Preliminaries

Let M be an R-module. A submodule X of M is said to be small in M if there is no proper submodule Y such that $M = X + Y$. It is denoted by $X \ll M$. M is said to be hollow if every submodule of M is small in M [6].
Let X and Y be two submodules of M. As in [12], Y is said to be supplement of X if $M = X+Y$ and $X \cap Y \ll Y$.

Theorem 2.1. *[11, Modularity condition] If X, Y, Z are submodules of M and $X \leq Y$, then $Y \cap (X + Z) = Y + (Z \cap Y)$.*

Theorem 2.2. *[12, lemma 4.2] Let A, B and C be submodules of M. Then :*

(i) If $A \ll B$ and $B \leq C$, then $A \ll C$;

(ii) If $A \ll M$, $A \leq B$ and $B \subseteq^{\oplus} M$, then $A \ll B$;

(iii) If $A \ll M$ and $f : M \to N$ is a homomorphism, then $f(A) \ll f(M)$.

As in [6], a module M is said to have the n-exchange property if, for any module A and any internal direct sum decompositions of A given by

$$A = M' \bigoplus N = \bigoplus_I A_i$$

for modules M', N, A_i where $M \simeq M'$ and card$(I)\leq n$, can be exchanged at M'. If M has the $n-$exchange property for every positive integer n, then M has the finite exchange property. A module M has the finite exchange property if $End_R(M)$ is an exchange ring. As defined in[5] by G.S.Monk, a homomorphism f of M into K is divided on M, where M is a submodule of K, if $M = M_1 \bigoplus M_2$ and $K = K_1 \bigoplus K_2$ such that $f(M_1) \simeq K_1, f(M_2) \leq K_2$ and $1 - f : M_2 \longrightarrow K$ splits.
Monk characterized a module M with the finite exchange property as follows:

Theorem 2.3. *[5, Theorem 1] The module M has the finite exchange property if and only if for $f \in End_R(M)$, f is divided on M.*

We call a module M a D_i-module (i=1,2,3,4) as follows:

(i) D_1 (Lifting Module): For every submodule X of M, there is a decomposition, $M = M_1 \bigoplus M_2$ such that $M_1 \leq X$ and $X \cap M_2 \ll M$.

(ii) D_2 (Direct Projective Module): For every submodule $X \leq M$ such that $M/X \simeq Y \subseteq^{\oplus} M$, then $X \subseteq^{\oplus} M$.

(iii) D_3 ($\cap$-Direct Projective Module): For any two direct summands M_1, M_2 of M with $M = M_1 + M_2$, then $M_1 \cap M_2 \subseteq^{\oplus} M$.

(iv) D_4: If $M = M_1 \bigoplus M_2$ and $f : M_1 \longrightarrow M_2$ is a homomorphism with $Imf \subseteq^{\oplus} M_2$, then $Kerf \subseteq^{\oplus} M_1$.

An R-module is called a discrete module if it is both D_1 and D_2-module. A quasi-discrete module is both D_1 and D_3-module. Since a D_2-module is again a D_3-module, so a discrete module is also a quasi-discrete module. A module M which is both D_1 and D_4 is called a pseudo-discrete module as in [10]. In [10, proposition 2.1] it has been shown that a module M is a D_4-module iff for any two submodules X and Y of M, with $X \subseteq Y$ and $M/Y \simeq X \subseteq^{\oplus} M$, then $Y \subseteq^{\oplus} M$. If M is a module with the finite internal exchange property, then M is a D_4

module iff M is a D_3 module [10, proposition 2.23]. We have used the above characterization (2.3) of a module with the finite exchange property, given by Monk, to show the equivalence between D_2 and D_4-module. Direct sum of D_4-module need not be a D_4-module. In [10], it has been proved that every direct sum of two D_4-modules over a ring R is a D_4-module iff R is semisimple artinian. Some important results for D_i-module ($i = 1, 2, 3, 4$) are as follows:

Theorem 2.4. *[6, 22.3 Characterization of lifting modules] The following are equivalent for a module M:*

(i) M has (D_1);

(ii) Every submodule X of M can be written as $X = Y \bigoplus S$ with $Y \subseteq^{\oplus} M$ and $S \ll M$;

(iii) M is amply supplemented and every supplement submodule of M is a summand.

Theorem 2.5. *[12, lemma 4.22] Let M be a quasi-discrete module. If $M = \sum_{i\in I} M_i$ is an irredundant sum of indecomposable submodules M_i, then $M = \bigoplus_{i\in I} M_i$.*

Theorem 2.6. *[12, Theorem 4.15] Any quasi-discrete module M has a decomposition $M = \bigoplus_{i\in I} H_i$ where each H_i is hollow.*

Theorem 2.7. *[10, proposition 2.11] The following statements hold:*

(i) A direct summand of a D_4-module is again a D_4-module.

(ii) If $M \bigoplus M$ is a D_4-module, then M is a D_2-module.

(iii) If $M_1 \bigoplus M_2$ is a D_4-module and there exists an epimorphism $f : M_1 \longrightarrow M_2$, then M_2 is a D_2-module.

A module M is said to be dual-square-free(DSF) if M has no proper submodule X and Y with $M = X + Y$ and $M/X \cong M/Y$ as in [15]. Direct summand and homomorphic image of a DSF-module is also a DSF-module. Factor module of a module is a DSF-module provided the module itself is a DSF-module [2]. Instead of submodules, if X and Y are summands of M, then M is called summand square free module (SDSF)[15].

Theorem 2.8. *[10, Lemma 5.5] If M is a lifting module, then M is an SDSF-module if and only if M is a DSF-module.*

Two left R-modules A and B are said to be factor orthogonal if no non zero factor module of A is isomorphic to a factor module of B [15].

Theorem 2.9. *[15, Lemma 2.7] If $M = X \bigoplus Y$ is a DSF-module, then X and Y are factor orthogonal.*

Theorem 2.10. *[10, Lemma 4.9] Let $X = N \bigoplus M$ be a pseudo-discrete module. If $f : N \to N/B$ is the canonical homomorphism and $g : M \to N/B$ is an epimorphism with $B \ll N$, then there exists a homomorphism $h : M \to N$ such that $fh = g$.*

Theorem 2.11. *[10, Theorem 5.6] If M is a pseudo-discrete module with DCC on summands, then $M = Q \bigoplus D$ with Q, a quasi projective module and D, a DSF-module.*

Now, whenever $M \bigoplus X \cong M \bigoplus Y$ implies $X \cong Y$, then M is said to have the property of cancellation. M is said to have the internal cancellation property if whenever $M = A_1 \bigoplus B_1 \cong A_2 \bigoplus B_2$ with $A_1 \cong A_2$ implies $B_1 \cong B_2$[12].

A ring R is said to be a von Neumann regular ring if every element of R is regular[8]. A ring R is regular iff every principal ideal is generated by an idempotent element. If $End_R(M)$ is a von Neumann regular ring, then M is an endoregular module.

Theorem 2.12. *[11, Theorem 37.7] Let M be an R-module and $S = End_R(M)$. Let $f \in S$. Then the following properties are equivalent:*

(i) There exists $g \in S$ with $fgf = f$;

(ii) $Kerf$ and Imf are direct summands of M.

As in [6], the cosingular ideal of two modules M and N is defined as:

$$\bigtriangledown(M, N) = \{f \in Hom(M, N) \mid Imf \ll N\}$$

If $M = N$, then $\bigtriangledown(M) = \bigtriangledown(M, N)$.

In the following theorem, a discrete module M is characterized in terms of Jacobson radical and cosingular ideal.

Theorem 2.13. *[13, Theorem 3.10] Let M be a discrete module and let J denote the Jacobson radical of $\mathrm{End}(M)$. Then:*

(i) $End(M)/J$ is a (von neumann) regular ring;

(ii) $J = \bigtriangledown$;

(iii) Idempotents modulo J can be lifted.

Let M be a left R-module. Then M is said to be a rickart module if $Kerf$ is a direct summand of M for all $f \in End_R(M)$[4]. Dual of rickart module is d-rickart module. If Imf is a direct summand of a module M for all $f \in End_R(M)$, then M is called a d-rickart module.

Theorem 2.14. *[3, Proposition 2.3] The following are equivalent for a module M:*

(i) M is an endoregular module;

(ii) M is a rickart and d-rickart module.

Theorem 2.15. *[4, Proposition 2.11] A module M is rickart iff M is a D_2-module and there is an isomorphism between Imf and a direct summand of M for all $f \in End_R(M)$.*

Theorem 2.16. *[6, Lemma 11.2] Let M_1, M_2 and N be modules with $M_1 \leq N \leq M_1 \bigoplus M_2$. Then $N = M_1 \bigoplus N_1$ where $N_1 = N \cap M_2$.*

Theorem 2.17. *[10, proposition 2.23] If M has the finite internal exchange property, then M is a D_4-module iff M is a D_3-module.*

Theorem 2.18. *[1, problem 7.4.40] Each element of a commutative ring R is either a unit or a nilpotent element iff the ring R has exactly one prime ideal.*

A module M over a ring R is said to be an $LE-module$ if its endomorphism ring $End_R(M)$ is local [6].

A left R-module M is called an Utumi module (U-module) if, whenever A and B are submodules of M with $A \cong B$ and $A \cap B = 0$, there exists two summands K and L of M such that $A \subseteq^{ess} K$, $B \subseteq^{ess} L$ and $K \oplus L \subseteq^{\oplus} M$ [17].

A left R-module M is called a Dual-Utumi-Module (DU-module) if for any two proper submodules A and B of M with $M/A \cong M/B$ and $A + B \leq M$, there exist two summands K and L of M such that A lies over K, B lies over L and $K \cap L \subseteq^{\oplus} M$. Dual-U-modules are strict generalizations of quasi-discrete, pseudo-discrete and dual-square-free modules [16].

3 Main Results

3.1 Discussion on Equivalence

Lemma 3.1. *Let M be an R-module with the finite exchange property and $End_R(M)$ satisfies the left cancellation property. Then M is a D_2-module.*

Proof. Let X be a submodule of M such that $M/X \simeq Y \subseteq^{\oplus} M$. Thus, there is an $f \in End_R(M)$ with $Imf = Y$ and $Kerf = X$. Now, it is enough to show that there exists a $g \in End_R(M)$ such that $fg = 1_M$ i.e, f is a retraction.

Since M has the finite exchange property, so f is divided on M. Thus there exists $g \in End_R(M)$ such that $gfg = g$, which implies that $fg = 1_M$. Thus $Kerf$ is a direct summand of M. □

Following this lemma it is easy to conclude the theorem as below:

Theorem 3.2. *Let M be a left R-module with the finite exchange property such that $End_R(M)$ has left cancellation property. Then the following conditions are equivalent:*

(i) M is a discrete module;

(ii) M is a pseudo-discrete module.

Proof. $1) \Leftrightarrow 2)$ follows from lemma (3.1). □

Remark 3.3. It has already been mentioned that R is a semisimple artinian ring if and only if every direct sum of two D_4-modules over a ring R is again a D_4-module [10]. Then using the result of (2.7), it can be concluded that if M is a module over a semisimple artinian ring, then M is a pseudo-discrete module if and only if M is a discrete module.

Theorem 3.4. *Let M be an R-module such that the ring $End_R(M)$ is commutative. If $End_R(M)$ has exactly one prime ideal with degree of each nilpotent element atmost 2, then M is a discrete module iff M is a pseudo-discrete module.*

Proof. Discrete $\Rightarrow$ pseudo-discrete is trivial.

Let X be a submodule of M such that $M/X \cong Y \subseteq^{\oplus} M$. Therefore there exists an endomorphism $f : M \to M$ with $Imf = Y$ and $Kerf = X$. From (2.18), f is either a unit or a nilpotent. Now, if f is a unit, then it is obvious. If f is a nilpotent element with degree 2, then $f^2 = 0$. For $x \in Imf$ implies $x \in Kerf$. Since M is a pseudo-discrete module, this implies $Kerf \subseteq^{\oplus} M$. Hence M is a discrete module. □

If we consider $End_R(M)$ as a commutative ring with identity, then as a consequence of theorem (3.4), we get the following about the equivalence:

Corollary 3.5. *Let M be an $LE-module$ such that the ring $End_R(M)$ is commutative with identity. If $End_R(M)$ is a principal ideal domain with each nilpotent element of degree atmost 2, then M is a pseudo-discrete module iff M is a discrete module.*

Proof. Let M be a pseudo-discrete module. The ring $End_R(M)$ has exactly one maximal ideal. Since $End_R(M)$ is commutative ring with unity, it implies that it has exactly one prime ideal. Now from (3.4), M is a discrete module. □

Corollary 3.6. *If M is a simple R-module, then M is a pseudo-discrete module iff M is a discrete module.*

Proof. Since M is a simple R-module, so $End_R(M)$ is a division ring. The rest is easy to prove. □

Theorem 3.7. *Let M be an endoregular module. Then M is a pseudo-discrete module iff M is a discrete module.*

Proof. Follows from theorem (2.14) and theorem (2.15). □

A module M is said to be a $(CD)_2$ module if for every submodule A and B of M such that $A \cong C \subseteq^{\oplus} M$ and $M/B \cong D \subseteq^{\oplus} M$ implies $A \subseteq^{\oplus} M$ and $B \subseteq^{\oplus} M$. It is easy to show that if M is a $(CD)_2$-module, then M is a C_i-module iff M is a D_i-module, for $i = 2, 4$. Thus if M is a $(CD)_2$-module with C_1, D_1 condition, then M satisfies both condition (*) and condition (**).

Lemma 3.8. *Let M be a D_4 module. Then M is indecomposable provided M is an SDSF-module.*

Proof. Assume to the contrary that M is decomposable i.e. there exists two summands D_1 and D_2 such that $M = D_1 \bigoplus D_2$. Let $f : D_1 \to D_2$ be a homomorphism such that $Imf \subseteq^{\oplus} D_2$. Thus $D_2 = Imf \bigoplus K$ for $K \leq D_2$. Then $D_2/K \cong Imf$. Since M is a D_4-module, thus $Kerf \subseteq^{\oplus} D_1$ and we have $D_1/Kerf \cong D_2/K$. Hence D_1 and D_2 are not factor orthogonal which is a contradiction (using theorem 2.9). Thus M is indecomposable. □

Lemma 3.9. *Let M be a pseudo-discrete module. Then M is an amply supplemented module and $M = X \oplus Y$ for any two submodules X and Y which are supplements of each other such that $X \cong Y$.*

Proof. Consider M is a pseudo-discrete module. By (2.12), M is an amply supplemented module. Since X, Y are supplements of each other in M, it implies that $M = X+Y$ and $X \cap Y \ll M$. Again by (2.12), X, Y are direct summands of M. Now $M = X + Y$ and $X \cong Y$, hence by (D_4), $X \cap Y \subseteq^{\oplus} M$ and $X \cap Y = 0$. □

Lemma 3.10. *Let Y be a supplement of a submodule X in a pseudo-discrete module M and Z be a supplement submodule of M contained in X. Assume that for any two summands D_1, D_2 with $M = D_1 + D_2$, there exists a homomorphism $f \in Hom(D_1, D_2)$ such that $Imf \subseteq^{\oplus} M$, then $Y \cap Z = 0$ and $Y \bigoplus Z \subseteq^{\oplus} M$.*

Proof. Since M is a pseudo-discrete module, by (2.4) both Y and Z are direct summands. Consider $M = Z \bigoplus Z^*$. By the modularity condition, $X = X \cap Z^* \bigoplus Z$, it implies that $M = X \cap Z^* + Z + Y$. Again by (2.4), $X \cap Z^*$ contains a supplement U of $Z + Y$. Clearly $U \bigoplus Z \subseteq^{\oplus} M$.Now writing $M = (U \bigoplus Z) + Y$, consider a map $f : U \bigoplus Z \to Y$ such that $f(k) = m - k$, where $m = y + k, m \in M, y \in Y$ and $k \in U \bigoplus Z$. Clearly this map is well-defined and a homomorphism. For $k \in Y, m = k$ implies that $Kerf = (U \bigoplus Z) \cap Y$. By assumption, $Imf \subseteq^{\oplus} M$ and M is a pseudo-discrete module, hence $Kerf \subseteq^{\oplus} M$. Now $(U \bigoplus Z) \cap Y \leq X \cap Y \ll M$, so $(Z \bigoplus U) \cap Y = 0$. Thus we have $Y \cap Z = 0$ and $Y \bigoplus Z \subseteq^{\oplus} M$. □

Lemma 3.11. *Let X be a summand and Y be an indecomposable summand of a pseudo- discrete module M. Assume that for any two summands D_1, D_2 with $M = D_1 + D_2$, there exists a homomorphism $f \in Hom(D_1, D_2)$ such that $Imf \subseteq^{\oplus} M$.Then either $X + Y \subseteq^{\oplus} M$ and $X \cap Y = 0$ or $X + Y = X \bigoplus Z$ with $Z \ll M$ and Y is isomorphic to a summand of X.*

Proof. Let Z be a supplement of $X+Y$ in M. Therefore $M = Z+X+Y$ and $Z \cap (X+Y) \ll Z$. By (3.10), $X \cap Z = 0$. If $Y \nleq X \bigoplus Z$, then $Y \cap (X + Z) \ll Y$ and hence Y is a supplement of $X + Z$. By (3.10), $X \cap Y = 0$ and $X \bigoplus Y \subseteq^{\oplus} M$.
If $Y \leq X \bigoplus Z$, then $M = X \bigoplus Z$.By the modularity condition, $X + Y = X \bigoplus ((X + Y) \cap Z)$ where $Z \cap (X + Y) \ll M$. Writing $M = Y \bigoplus Y'$, we have $M = X + Y + Y' = X + ((X + Y) \cap Z) + Y' = X + Y'$. Since M is an amply supplemented, so by (3.9) $M = X' \bigoplus Y'$ where X' is a supplement of Y' contained in X. Therefore $Y \cong X'$ which is a summand of X. □

Lemma 3.12. *Let M be a pseudo-discrete module. Assume that for any two summands D_1, D_2 with $M = D_1 + D_2$, there exists a homomorphism $f \in Hom(D_1, D_2)$ such that $Imf \subseteq^{\oplus} M$. If $M = \sum_{i \in I} M_i$ is an irredundant sum of indecomposable submodules M_i, then $M = \bigoplus_{i \in I} M_i$*

Proof. It follows from theorem (2.5) and theorem (3.11). □

Lemma 3.13. *Let M be a pseudo-discrete module with DCC on summands and assume that for any two summands D_1, D_2 with $M = D_1 + D_2$, there exists a homomorphism $f \in Hom(D_1, D_2)$ such that $Imf \subseteq^{\oplus} M$. Let A and B be two direct summands of M. If $\frac{A}{X} \cong \frac{B}{Y}$ with $X \ll A$ and $Y \ll B$, then $A \cong B$.*

Proof. Since M is a pseudo-discrete module with DCC on summands, so A and B are also pseudo-discrete modules with DCC on summands. We can write $A = Q \bigoplus D$ where Q is a quasi-projective module and D is a DSF-module (by 2.11). By (2.6), we can write $Q = \bigoplus_{i \in I} H_i$, where H_i's are hollow. Thus $A = (\bigoplus_{i \in I} H_i) \bigoplus D$.

Let $A' = \frac{A}{X}$ and $B' = \frac{B}{Y}$. Let $f : A' \to B'$ is an isomorphism. Then $B' = f\left(\frac{A}{X}\right)$.

$$B' = f\left(\frac{A}{X}\right)$$
$$\Rightarrow B' = f\left(\frac{\oplus H_i \oplus D}{X}\right)$$
$$\Rightarrow B' = f\left(\oplus_{i\in I}\frac{H_i}{H_i\cap X} \oplus \frac{D}{D\cap X}\right)$$
$$\Rightarrow B' = \sum_{i\in I} f\left(\frac{H_i}{H_i\cap X}\right) + f\left(\frac{D}{D\cap X}\right)$$

Consider $H_i' = H_i/(H_i \cap X)$ and $D' = D/(D\cap X)$. Then we have $B' = \sum_{i\in I} f(H_i') + f(D')$. Here, $f(D')$ and D' are also DSF-modules. Since f is an isomorphism and $X \ll A$, so this sum is irredundant.

Let $f(H_i') = C_i' \leq B'$ for $i \in I$, where $C_i' = C_i/(C_i \cap Y)$ with $C_i \leq B$. Clearly $H_i' \cong C_i'$ and C_i''s are hollow. Also, $f(D') = E' \leq B'$ where $E' = E/(E\cap Y)$, $E \leq B$. Similarly $D' \cong E'$ and E' is itself a DSF-module. It is easy to prove that $B = \sum C_i + E$ where C_i's are hollow and E is a DSF-module. This sum is irredundant.

Since B is lifting, $C_i = B_i \bigoplus K_i$ where $B_i \subseteq^{\oplus} B$ and $K_i \ll B$. Since C_i''s are hollow, so either $C_i' = B_i/(B_i\cap Y)$ or $C_i' = K_i/(K_i \cap Y)$. If $C_i' = K_i/(K_i\cap Y)$, then it is easy to show that $C_i \ll B$, which is a contradiction to the fact that $B = \sum C_i + E$ is irredundant. Therefore, the only possibility is $C_i' = B_i/(B_i \cap Y) \cong (B_i + Y)/Y$ which implies that $C_i = B_i + Y$. Similarly $D' \cong E'$.

Claim: D' is indecomposable.

If possible, consider $D/(D\cap X) \cong D_1/(D_1\cap X) \bigoplus D_2/(D_2 \cap X)$, implies $D = (D_1 \bigoplus D_2) + (D\cap X)$ where $D \cap X \ll D$. Therefore $D_1 \bigoplus D_2 = D$. But D is indecomposable in A. Thus D' is indecomposable and E' is indecomposable. Now let $E = F \bigoplus L$ where $F \subseteq^{\oplus} B$ and $L \ll B$. Since E' is indecomposable, so either $E' = F/(F\cap Y)$ or $E' = L/(L\cap Y)$. Now it is easy to show that $E' = L/(L\cap Y)$ is not possible and $E = F + Y$. Therefore

$$B = \textstyle\sum_{i\in I} C_i + E = \sum_{i\in I}(B_i + Y) + (F + Y) = \sum_{i\in I} B_i + F.$$

and this sum is irredundant.

Claim: Each B_i, for $i \in I$, is hollow and F is a DSF-module.

Assume $B_i = U + V$. Then $B_i' = B_i/(B_i \cap Y) = U/(U\cap Y) + V/(V\cap Y) = U' + V'$.Then $B_i' = U'$ or $B_i' = V'$. Now it is easy to show that B_i' is hollow for $i \in I$. As F is indecomposable, so F is an SDSF-module i.e. a DSF-module.

Now using theorem (3.12), we can write $B = (\bigoplus_{i\in I} B_i) \bigoplus F$.

Claim: $H_i \cong B_i$ for $i \in I$ and $D \cong F$.

Since H_i and B_i are hollow summands of M, so using theorem (3.11), either $H_i \cong B_i$ or $H_i + B_i$ is direct and it is a summand of M. For the second case, $H_i \bigoplus B_i$ is a pseudo-discrete module. Using theorem (2.10), there exists an epimorphism $\alpha_i : H_i \to B_i$ such that the following diagram is commutative:

[node distance=3cm] (A) H_i; (B) [right of=A] H_i'; (C) [below of=A] B_i; (D) [right of=C] B_i'; [-»] (A)to node (B); [->] (A) to node [left] α_i (C); [-»] (C)to node (D); [->] (B) to node [right] $\cong$ (D);

Since $H_i \bigoplus B_i$ is pseudo-discrete and H_i and B_i are hollow, so α_i is an isomorphism. Hence $H_i \cong B_i$. Similarly, we can show that $D \cong F$.

Therefore,

$$(\textstyle\bigoplus_{i\in I} H_i) \bigoplus D \cong (\bigoplus_{i\in I} B_i) \bigoplus F \Rightarrow A \cong B.$$

□

In [10], the decomposition theorem for pseudo-discrete module has already been proved. We now restate the decomposition theorem for pseudo-discrete module as follows:

Theorem 3.14 (**Decomposition Theorem**). *Let M be a pseudo-discrete module with DCC on summands. Then $M = \bigoplus_{i \in I} H_i$ where each H_i is hollow.*

Proof. It follows from theorem (2.11) and theorem (3.8). □

Theorem 3.15. *Let M be a pseudo-discrete module with DCC on summands. Assume that for any two summands D_1, D_2 with $M = D_1 + D_2$, there exists a homomorphism $f \in Hom(D_1, D_2)$ such that $Imf \subseteq^{\oplus} M$. Then M is a discrete module if and only if every epimorphism $M \longrightarrow M$ with small kernel is an isomorphism.*

Proof. Let $M \xrightarrow{\text{f}} M$ be an epimorphism with small kernel K. Then $M/K \cong M$. As M is a pseudo-discrete module, so $K \subseteq^{\oplus} M$. Let $M = K \bigoplus L$ where L is a summand of M. From this we can conclude that $M = L$ and $K = 0$. Thus f is a monomorphism, which implies that f is an isomorphism.

To prove M is discrete,it is enough to show that M is a D_2 module. Let N be a submodule of M such that $M/N \cong K \subseteq^{\oplus} M$. We need to show that N is also a direct summand of M.

Let $f : M \to K$ is an epimorphism with $Kerf = N$. Since M is lifting, so let $M = A \bigoplus B$ with $A \ll N$ and $B \cap N \ll B$. Thus $M = N + B$ and by (2.16), we get $N = A \bigoplus (B \cap N)$. Now $K \cong M/N \cong (N + B)/N \cong B/(N \cap B)$. K and B are both summands and $N \cap B \ll B$. By (3.13), we can write $K \cong B$. Let $g : N \to B$ be an isomorphism. Then we get an epimorphism as follows:

$$M = A \bigoplus B \overset{1 \oplus f \mid B}{\longrightarrow} A \bigoplus N \overset{1 \oplus g}{\longrightarrow} A \bigoplus B = M$$

where its kernel is $B \cap N$ which is small. Thus $B \cap N = 0$ and $N = A \subseteq^{\oplus} M$. □

Theorem 3.16. *Let M be a pseudo-discrete module with DCC on summands. Assume that for any two summands D_1, D_2 with $M = D_1 + D_2$, there exists a homomorphism $f \in Hom(D_1, D_2)$ such that $Imf \subseteq^{\oplus} M$. If $\bigtriangledown = J$ and $End(M)/\bigtriangledown$ is regular, then M is a discrete module.*

Proof. Let $f \in End(M)$ be an epimorphism with small kernel. To prove M is a discrete module, it is enough to show that f is a monomorphism.

Since $End(M)/\bigtriangledown$ is regular, so there exist $g \in End(M)$ such that $f - fgf \in \bigtriangledown$. We prove that f is a monomorphism. Consider $(1 - gf)M + A = M$. Now $f(1 - gf)M + f(A) = M$ implies $f(A) = M$. Thus $M = A + Kerf$ from which we get $M = A$. Now we have $(1 - gf)M \ll M$.Thus $(1 - gf) \in \bigtriangledown = J$ i.e. $1 - (1 - gf)$ is unit. This implies that f is a monomorphism. Thus by (3.15), we can conclude that M is a discrete module. □

3.2 $(CD)_2$-modules

Definition 3.17. A module M is said to be a $(CD)_2$-module if for every submodule A, B of M such that $A \cong C \subseteq^{\oplus} M$ and $M/B \cong D \subseteq^{\oplus} M$ implies $A \subseteq^{\oplus} M$ and $B \subseteq^{\oplus} M$.

In short, we can interpret it as a module which is both C_2 and D_2-module.

Example 3.18. (i) Every endoregular module is a $(CD)_2$-module. To see this, first consider that M is an endoregular module and A, B are two submodules of M such that $A \cong C \subseteq^{\oplus} M$ and $M/B \cong D \subseteq^{\oplus} M$. Now, since M is an endoregular module, so $A \cong f(M)$ where $f^2 = f \in End_R(M)$. Let $g : f(M) \to A$ be an isomorphism. Then $Imgf = gf(M) = A \subseteq^{\oplus} M$. Again since $M/B \cong D \subseteq^{\oplus} M$, there is an $f \in End_R(M)$ such that $Kerf = B$, a summand of M. But converse is not true. The converse is true if M is a $(CD)_2$-module and Imf is isomorphic to a direct summand of M for all $f \in End_R(M)$. [3, Proposition 2.3].

(ii) Every semi simple module is a $(CD)_2$-module.

(iii) A $(CD)_2$-module is a C_2-module, but the converse is not true in general. For example, Z_{p^∞} is a C_2 module but not a $(CD)_2$-module. This example also proves that every injective module is not a $(CD)_2-$module as Z-module Z_{p^∞} is an injective module.

(iv) A $(CD)_2$-module is also a D_2-module, but the converse is not true in general. For example, Z-module Z is a D_2-module, but not a $(CD)_2$-module. This example also works for projective module which is not $(CD)_2$-module.

(v) Submodule of a $(CD)_2$-module need not be a $(CD)_2$-module. For example:Z-module Q is a $(CD)_2$-module but its submodule Z is not a $(CD)_2$-module. In fact it is not even a C_2-module.

(vi) A module M is called an SSP-module (res. SIP-module) if the sum (res. intersection) of any two direct summands of M is again a direct summand [10]. Now it is easy to show that if M is a $(CD)_2$-module, then M is an SSP-module iff M is an $SIP-$module.

(vii) Direct sum of $(CD)_2-$module need not be a $(CD)_2$-module.

(viii) Fully invariant submodule of a $(CD)_2$-module need not be a $(CD)_2$-module.

Theorem 3.19. *Let M be a $(CD)_2$-module over R. Then it satisfies the following properties:*

(i) Every direct summand of M is also a $(CD)_2$-module.

(ii) Every submodule A of M with $M/A \cong B \leq M$, where B is isomorphic to some summand, then A is a direct summand of M.

(iii) Every submodule A of M with $A \cong B \leq M$, where B is kernel of some epimorphism $\phi : M \to C \subseteq^{\oplus} M$, then A is a direct summand of M.

Proof. (i) $\Rightarrow$ Consider A is a direct summand of M. Assume A_1, B_1 are two submodules of M such that $A_1 \cong C$ and $A/B_1 \cong D$, where C, D are direct summands of M. Then $(A \oplus A')/(B_1 \oplus A') \cong D$ implies $B_1 \oplus A' \subseteq^{\oplus} M$.Then $B_1 \subseteq^{\oplus} A$, which is easy to show. At the same time, it is also easy to show that $A_1 \subseteq^{\oplus} A$.Therefore, clearly A is also a $(CD)_2$ module.

(ii) $\Rightarrow$ Consider an arbitrary submodule A of M with $M/A \cong B \leq M$ and B is isomorphic to some summand of M. Since M is a $(CD)_2$-module, B is itself a direct summand. But by D_2 condition, A is itself a direct summand.

(iii) $\Rightarrow$ Consider an arbitrary submodule A of M with $A \cong B \leq M$ and $\phi : M \to C \subseteq^{\oplus} M$. Since M is a $(CD)_2$-module, B is a direct summand of M. Therefore A is a direct summand of M.

□

Remark 3.20. (i) If M is a $(CD)_2$-module, then for any two direct summands A, B of M and $A \cap B = 0$ implies $M = A_1 \bigoplus B = A \bigoplus B_1$ for submodules $A_1 \geq A$ and $B_1 \geq B$.

(ii) If M is a $(CD)_2$-module, then for any two direct summands $A, B \subseteq^{\oplus} M$ and $M = A + B$ implies $M = A_1 \bigoplus B = A \bigoplus B_1$ for submodules $A_1 \leq A$ and $B_1 \leq B$.

(iii) If M is a $(CD)_2$-module, then for any two summands $A, B \subseteq^{\oplus} M$ and $A \cap B \subseteq^{\oplus} M$ implies $A + B \subseteq^{\oplus} M$.

(iv) If M be a $(CD)_2$-module, then for any two summands $A, B \subseteq^{\oplus} M$ and $A + B \subseteq^{\oplus} M$ implies $A \cap B \subseteq^{\oplus} M$.

(v) $(CD)_2 + D_1 \Rightarrow$ Dual Utumi Module.

(vi) $(CD)_2 + C_1 \Rightarrow$ Utumi Module.

Proposition 3.21. *Let M be a $(CD)_2$-module. Then the following conditions are equivalent:*

(i) For any two direct summand $A, B \subseteq^{\oplus} M$ and $A \cap B = 0$ implies $M = A_1 \bigoplus B = A \bigoplus B_1$ for submodules $A_1 \geq A$ and $B_1 \geq B$;

(ii) for any two direct summand $A, B \subseteq^{\oplus} M$ and $M = A + B$ implies $M = A_1 \bigoplus B = A \bigoplus B_1$ for submodules $A_1 \leq A$ and $B_1 \leq B$.

Proof. (1) $\Rightarrow$ (2) $A, B \subseteq^{\oplus} M$ such that $A \cap B = 0$.Clearly $A \bigoplus B \subseteq^{\oplus} M$ implies $M = A \bigoplus B \bigoplus T$,for some $T \subseteq^{\oplus} M$.There are two cases:

- Case 1: When T is indecomposable, the proof is trivial.
- Case 2: When T is decomposable, M can be written as $M = (A \bigoplus K_1) + (B \bigoplus K_2)$ for some $K_1, K_2 \subseteq^{\oplus} M \Rightarrow M = C \bigoplus D, C := A \bigoplus K_1, D := B \bigoplus K_2$. Then by assumption there exists $C_1 := A \bigoplus K_1'$ such that $M = C_1 \bigoplus D = (A \bigoplus K_1') \bigoplus (B \bigoplus K_2) = (A \bigoplus K_1' \bigoplus K_2) \bigoplus B = A_1 \bigoplus B$ with $A_1 = A \bigoplus K_1' \bigoplus K_2$. Similarly there exists $D_1 := B \bigoplus K_2' \leq D$ such that $M = C \bigoplus D_1 = (A \bigoplus K_1) \bigoplus (B \bigoplus K_2') = (A \bigoplus K_1 \bigoplus K_2') \bigoplus B = A \bigoplus B_1$ with $B_1 = A \bigoplus K_1 \bigoplus K_2'$

$(2) \Rightarrow (1)$Consider $A, B \subseteq^{\oplus} M$ with $M = A + B$. Then $A \cap B \subseteq^{\oplus} M$ implies $A \setminus B$ and $B \setminus A$ are direct summands such that $(A \setminus B) \cap (B \setminus A) = 0$. By assumption, there exists $C := A \geq (A \setminus B)$ such that $M = C \bigoplus (B \setminus A) = A \bigoplus B_1$ with $B_1 := (B \setminus A) \leq B$. Similarly, there exists $D := B \geq (B \setminus A)$ such that $M = (A \setminus B) \bigoplus D = A_1 \bigoplus B$ with $A_1 := (A \setminus B) \leq B$. □

Remark 3.22. Let M be a $(CD)_2$-module and $M = A_1 \bigoplus A_2$. If there is an R-homomorphism from A_1 to A_2, then $Kerf \subseteq^{\oplus} A_1$ if and only if $Imf \subseteq^{\oplus} A_2$.

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Author information

Prajnan Kumar Bhagawati, Department of Mathematics, Gauhati University, Guwahati-781014, INDIA.
E-mail: `bhagawatipu@gmail.com`

Hiya Saharia, Department of Mathematics, Gauhati University, Guwahati-781014, INDIA.
E-mail: `hiyasaharia123@gmail.com`

Helen K. Saikia, Department of Mathematics, Gauhati University, Guwahati-781014, INDIA.
E-mail: hsaikia@yahoo.com

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