Palestine Journal of Mathematics

Vol. 12(2)(2023) , 430–449

On the exponential admissibility of singular systems with multiply variable delays

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Communicated by Ayman Badawi

2020 AMS (MOS) Subject Classifications: 34A08; 34K40.

Keywords and phrases: Singular system, Multiple variable delays, Impulse-free, Exponential admissibility, LKF, LMI.

Abstract We investigate exponential admissibility of singular systems with multiply variable delays. We give new criteria such that the systems are regular, impulse-free (IF) and exponentially stable (ES). The techniques of the proofs are based on the Lyapunov- Krasovskiĭ functional (LKF), convex polyhedron methods and linear matrix inequalities (LMIs). Two numerical examples are given to illustrate the obtained results. Using MATLAB-Simulink software, the trajectories of the solutions are provided. This study generalizes and improves certain results in the present literature.

1 Introduction

As known from the relevant literature singular differential systems have wide real applications in electrical circuit networks, economic, biological systems, medicine, aerospace engineering, physics and more. Hence, stability, admissibility, etc., analysis of solutions of ordinary and time-delay of singular systems have been active research areas in recent years. Indeed, the stability and admissibility, etc., analyses of linear or non-linear singular systems are very important both theoretically and practically. However, finding analytical solutions of singular differential systems is a hard task. The Lyapunov function and LKF approaches are very important research techniques to have information on qualitative properties of solutions of ordinary differential equations, functional differential equations and that kind of systems without prior information of solutions of that kind systems (see, [1-16], [21-38] and references therein).

The aim of this paper to obtain a novel result on the exponential admissibility of singular systems with multiply variable delays and to provide two new examples as applications of the result.

First, we would like to summarize some related works in the literature.

In 2009, Haidar and Boukas [6], considered singular linear system with multiple time-varying delays:

$$E\dot{x}(t) = Ax(t) + \sum_{k=1}^{p} A_k x(t - d_k(t)).$$

By means of an LKF and LMIs, the authors obtained sufficient conditions for the exponential stability of the system above.

In 2011, Lin et al. [14] considered the following continuous-time singular delay system:

$$E\dot{x}(t) = Ax(t) + A_d x(t - d(t)).$$

By defining LKF and giving a tighter upper bound of its derivative, the authors established certain exponential admissibility criteria for this singular system in terms of LMI.

Chen and Hu [1] considered the below linear delay singular system:

$$E\dot{x}(t) = Ax(t) + A_d x(t - d(t)) + A_h x(t - h(t)).$$

By means of LKF and convex polyhedron method, the authors established some new delay dependent exponential stability criteria in terms of LMIs for this system.

Further, in 2015, Hien et al. [8] considered the following linear singular system with variable delay:

$$E\dot{x}(t) = Ax(t) + Dx(t - h(t)) + G\int_{t-d(t)}^{t} x(s)ds.$$

By help of an LKF and various efficient convex optimization algorithms, Hien et al. [8] obtained new delay-dependent exponential stability criteria for the system above in terms of LMIs.

For some recent results on qualitative properties of solutions of various non-singular systems, we also refer the readers to the papers ([17-20]).

Motivated by the results of Chen and Hu [1], Haidar and Boukas [6], Hien et al. [8], Lin et al. [14] and those in the references of this paper and the literature, we consider the following linear singular integro-differential system (SIDS) with two variable delays:

$$E\dot{x}(t) = Ax(t) + \sum_{n=1}^{2} A_{d_n}x(t - d_n(t)) + \sum_{n=1}^{2} A_{h_n}x(t - h_n(t)) + B\int_{t-\tau(t)}^{t} x(s)ds, \qquad (1.1)$$

$$x(t) = \phi(t) \in C([-\delta, 0], R^n), \delta > 0,$$

where $x(t) \in R^n$ is the state vector,$E \in R^{n\times n}$ is a singular constant matrix and we assume $rankE = r \le n$, $A \in R^{n\times n}$ is a negative definite real constant matrix and $A, A_{d_1}, A_{d_2}, A_{h_1}, A_{h_2}, B \in R^{n\times n}$ are real constant system matrices. The variable delays $d_n(t), h_n(t), \tau(t) : R^+ \to R^+$ are continuous satisfying

$$0 \le d_{n1} \le d_n(t) \le d_{n2}, \dot{d}_n(t) \le \mu_n,$$

$$0 \le h_{n1} \le h_n(t) \le h_{n2}, \dot{h}_n(t) \le \mu_{n+2}, \qquad (1.2)$$

$$0 \le \tau_1 \le \tau(t) \le \tau_2, (n = 1, 2),$$

where $d_{n1}, d_{n2}, h_{n1}, h_{n2}, \tau_1, \tau_2, \mu_1, \mu_2, \mu_3$ and μ_4 are constants and $\delta = max\{d_{12}, d_{22}, h_{12}, h_{22}, \tau_2\}$.

Let $\phi(t)$ be a continuous initial function defined on $[-\delta, 0]$.

Before proceeding to the proof of admissibility, we present some useful definitions and lemmas.

Definition 1.1 ([4]). If $det(sE-A) \neq 0$, then the pair (E, A) is called regular. If $deg(det(sE-A)) = rank(E)$, then the pair (E, A) is called IF.

Definition 1.2 ([18]). If the pair (E, A) is regular and IF, then system (1.1) is called regular and IF. System (1.1) is called admissible if it is regular, IF and stable.

Lemma 1.1 ([14]). Let $x, y \in R^n$ and $P > 0$ be a real matrix with an appropriate dimension. Then, the inequality $2x^T y \le x^T P^{-1} x + y^T P y$ holds.

Lemma 1.2 ([14]). Let $f(t)$ be positive continuous function such that $f(t) \le \eta_1 \sup_{t-d\le s\le t} f(s) + \eta_2 e^{-\lambda t}$, where $0 < \eta_1 < 1, \eta_1 e^{\lambda d} < 1, \lambda > 0$ and $\eta_2 > 0$. Then

$$f(t) \le \sup_{-d\le s\le 0} f(s)e^{-\lambda t} + \frac{\eta_2}{1 - \eta_1 e^{\lambda d}} e^{-\lambda t}.$$

Lemma 1.3 (Schur complement [18]). Given any real matrices P_1, P_2 and $P_3, P_1 = P_1^T$ and $P_3 > 0$. Then

$$P_1 + P_2 P_3^{-1} P_2^T < 0$$

if

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & -P_3 \end{bmatrix} < 0,$$

or equivalenty

$$\begin{bmatrix} -P_3 & P_2^T \\ P_2 & P_1 \end{bmatrix} < 0.$$

Lemma 1.4 ([1]). Assume that $\Omega, \Xi_{1i}, \Xi_{2i}, (i = 1, 2)$ are constant matrices of appropriate dimensions and $a, b \in [0, 1]$. If the inequalities

$$\Omega + \Xi_{11} + \Xi_{21} < 0, \Omega + \Xi_{11} + \Xi_{22} < 0, \Omega + \Xi_{12} + \Xi_{21} < 0$$

and

$$\Omega + \Xi_{12} + \Xi_{22} < 0$$

are held, then

$$\Omega + [a\Xi_{11} + (1 - a)\Xi_{12}] + [b\Xi_{21} + (1 - b)\Xi_{22}] < 0.$$

2 Admissibility criteria

A. Conditions

We suppose that the conditions below are held.

(A1) We have positive definite symmetric matrices $P, Q_i, W_i, Z_j, N, T, (i = 1, ..., 6; j = 1, ..., 8)$, with appropriate dimensions and some defined matrices:

$$K_1 = -\, e^{-2\alpha d_{12}} \begin{bmatrix} 0 & -I & 0_{n\times 3n} & I & 0_{n\times 9n} \end{bmatrix}^T E^T Z_3 E \begin{bmatrix} 0 & -I & 0_{n\times 3n} & I & 0_{n\times 9n} \end{bmatrix},$$
$$K_2 = -\, e^{-2\alpha d_{12}} \begin{bmatrix} 0 & -I & 0_{n\times 4n} & I & 0_{n\times 8n} \end{bmatrix}^T E^T Z_3 E \begin{bmatrix} 0 & -I & 0_{n\times 4n} & I & 0_{n\times 8n} \end{bmatrix},$$
$$K_3 = -\, e^{-2\alpha d_{22}} \begin{bmatrix} 0_{n\times 2n} & -I & 0_{n\times 4n} & I & 0_{n\times 7n} \end{bmatrix}^T E^T Z_4 E \begin{bmatrix} 0_{n\times 2n} & -I & 0_{n\times 4n} & I & 0_{n\times 7n} \end{bmatrix},$$
$$K_4 = -\, e^{-2\alpha d_{22}} \begin{bmatrix} 0_{n\times 2n} & -I & 0_{n\times 5n} & I & 0_{n\times 6n} \end{bmatrix}^T E^T Z_4 E \begin{bmatrix} 0_{n\times 2n} & -I & 0_{n\times 5n} & I & 0_{n\times 6n} \end{bmatrix},$$
$$U_1 = -\, e^{-2\alpha h_{12}} \begin{bmatrix} 0_{n\times 3n} & -I & 0_{n\times 5n} & I & 0_{n\times 5n} \end{bmatrix}^T E^T Z_7 E \begin{bmatrix} 0_{n\times 3n} & -I & 0_{n\times 5n} & I & 0_{n\times 5n} \end{bmatrix},$$
$$U_2 = -\, e^{-2\alpha h_{12}} \begin{bmatrix} 0_{n\times 3n} & -I & 0_{n\times 6n} & I & 0_{n\times 4n} \end{bmatrix}^T E^T Z_7 E \begin{bmatrix} 0_{n\times 3n} & -I & 0_{n\times 6n} & I & 0_{n\times 4n} \end{bmatrix},$$
$$U_3 = -\, e^{-2\alpha h_{22}} \begin{bmatrix} 0_{n\times 4n} & -I & 0_{n\times 6n} & I & 0_{n\times 3n} \end{bmatrix}^T E^T Z_8 E \begin{bmatrix} 0_{n\times 4n} & -I & 0_{n\times 6n} & I & 0_{n\times 3n} \end{bmatrix},$$
$$U_4 = -\, e^{-2\alpha h_{22}} \begin{bmatrix} 0_{n\times 4n} & -I & 0_{n\times 7n} & I & 0_{n\times 2n} \end{bmatrix}^T E^T Z_8 E \begin{bmatrix} 0_{n\times 4n} & -I & 0_{n\times 7n} & I & 0_{n\times 2n} \end{bmatrix}$$

such that the following LMIs hold:

$$\begin{aligned}
&\Omega + K_1 + K_3 + U_1 + U_3 < 0, \Omega + K_1 + K_3 + U_1 + U_4 < 0,\\
&\Omega + K_1 + K_3 + U_2 + U_3 < 0, \Omega + K_1 + K_3 + U_2 + U_4 < 0,\\
&\Omega + K_1 + K_4 + U_1 + U_3 < 0, \Omega + K_1 + K_4 + U_1 + U_4 < 0,\\
&\Omega + K_1 + K_4 + U_2 + U_3 < 0, \Omega + K_1 + K_4 + U_2 + U_4 < 0,\\
&\Omega + K_2 + K_3 + U_1 + U_3 < 0, \Omega + K_2 + K_3 + U_1 + U_4 < 0,\\
&\Omega + K_2 + K_3 + U_2 + U_3 < 0, \Omega + K_2 + K_3 + U_2 + U_4 < 0,\\
&\Omega + K_2 + K_4 + U_1 + U_3 < 0, \Omega + K_2 + K_4 + U_1 + U_4 < 0,\\
&\Omega + K_2 + K_4 + U_2 + U_3 < 0, \Omega + K_2 + K_4 + U_2 + U_4 < 0,
\end{aligned} \tag{2.1}$$

with

$$\begin{aligned}
\Omega_{1,1} =& A^T P + PA + 2\alpha E^T P + \sum_{n=1}^{2}(Q_n + Q_{n+2} + Q_{n+4} + W_n + W_{n+2} + W_{n+4}) \\
&+ A^T SA - \sum_{n=1}^{2} e^{-2\alpha d_{n1}} E^T Z_n E - \sum_{n=1}^{2} e^{-2\alpha h_{n1}} E^T Z_{n+4} E + \tau_1^2 N + (\tau_2 - \tau_1)^2 T, \\
\Omega_{1,2} =& PA_{d_1} + A^T SA_{d_1}, \Omega_{1,3} = PA_{d_2} + A^T SA_{d_2}, \Omega_{1,4} = PA_{h_1} + A^T SA_{h_1}, \\
\Omega_{1,5} =& PA_{h_2} + A^T SA_{h_2}, \Omega_{1,6} = e^{-2\alpha d_{11}} E^T Z_1 E, \Omega_{1,8} = e^{-2\alpha d_{21}} E^T Z_2 E, \\
\Omega_{1,10} =& e^{-2\alpha h_{11}} E^T Z_5 E, \Omega_{1,12} = e^{-2\alpha h_{21}} E^T Z_6 E, \Omega_{1,14} = PB + A^T SB, \\
\Omega_{2,2} =& -e^{-2\alpha d_{12}}(1-\mu_1)Q_3 - 2e^{-2\alpha d_{12}} E^T Z_3 E + A_{d_1}^T SA_{d_1}, \Omega_{2,3} = A_{d_1}^T SA_{d_2}, \\
\Omega_{2,4} =& A_{d_1}^T SA_{h_1}, \Omega_{2,5} = A_{d_1}^T SA_{h_2}, \Omega_{2,6} = \Omega_{2,7} = e^{-2\alpha d_{12}} E^T Z_3 E, \Omega_{2,14} = A_{d_1}^T SB, \\
\Omega_{3,3} =& -e^{-2\alpha d_{22}}(1-\mu_2)Q_4 - 2e^{-2\alpha d_{22}} E^T Z_4 E + A_{d_2}^T SA_{d_2}, \Omega_{3,4} = A_{d_2}^T SA_{h_1}, \\
\Omega_{3,5} =& A_{d_2}^T SA_{h_2}, \Omega_{3,8} = \Omega_{3,9} = e^{-2\alpha d_{22}} E^T Z_4 E, \Omega_{3,14} = A_{d_2}^T SB, \\
\Omega_{4,4} =& -e^{-2\alpha h_{12}}(1-\mu_3)W_3 - 2e^{-2\alpha h_{12}} E^T Z_7 E + A_{h_1}^T SA_{h_1}, \Omega_{4,5} = A_{h_1}^T SA_{h_2}, \\
\Omega_{4,10} =& \Omega_{4,11} = e^{-2\alpha h_{12}} E^T Z_7 E, \Omega_{4,14} = A_{h_1}^T SB, \\
\Omega_{5,5} =& -e^{-2\alpha h_{22}}(1-\mu_4)W_4 - 2e^{-2\alpha h_{22}} E^T Z_8 E + A_{h_2}^T SA_{h_2} \\
\Omega_{5,12} =& \Omega_{5,13} = e^{-2\alpha h_{22}} E^T Z_8 E, \Omega_{5,14} = A_{h_2}^T SB, \\
\Omega_{6,6} =& -e^{-2\alpha d_{11}} Q_1 - e^{-2\alpha d_{11}} E^T Z_1 E - e^{-2\alpha d_{12}} E^T Z_3 E, \\
\Omega_{7,7} =& -e^{-2\alpha d_{12}} Q_5 - e^{-2\alpha d_{12}} E^T Z_3 E, \\
\Omega_{8,8} =& -e^{-2\alpha d_{21}} Q_2 - e^{-2\alpha d_{21}} E^T Z_2 E - e^{-2\alpha d_{22}} E^T Z_4 E, \\
\Omega_{9,9} =& -e^{-2\alpha d_{22}} Q_6 - e^{-2\alpha d_{22}} E^T Z_4 E, \\
\Omega_{10,10} =& -e^{-2\alpha h_{11}} W_1 - e^{-2\alpha h_{11}} E^T Z_5 E - e^{-2\alpha h_{12}} E^T Z_7 E, \\
\Omega_{11,11} =& -e^{-2\alpha h_{12}} W_5 - e^{-2\alpha h_{12}} E^T Z_7 E, \\
\Omega_{12,12} =& -e^{-2\alpha h_{21}} W_2 - e^{-2\alpha h_{21}} E^T Z_6 E - e^{-2\alpha h_{22}} E^T Z_8 E, \\
\Omega_{13,13} =& -e^{-2\alpha h_{22}} W_6 - e^{-2\alpha h_{22}} E^T Z_8 E, \\
\Omega_{14,14} =& -e^{-2\alpha \tau_2} T + B^T SB, \Omega_{14,15} = e^{-2\alpha \tau_2} T, \\
\Omega_{15,15} =& -e^{-2\alpha \tau_1} N - e^{-2\alpha \tau_2} T, \rho_n = d_{n2} - d_{n1}, r_n = h_{n2} - h_{n1}, (n = 1, 2), \\
S =& \sum_{n=1}^{2} \{d_{n1}^2 Z_n + \rho_n^2 Z_{(n+2)} + h_{n1}^2 Z_{(n+4)} + r_n^2 Z_{(n+6)}\},
\end{aligned}$$

where $\alpha \in R, \alpha > 0$ and I is the identity matrix of appropriate dimension. In addition, Ω is a negative definite symmetric matrix in the form of $\Omega = (\Omega_{k,m})_{15\times 15}$ and its unexpressed components are zero.

(A2)

$$E^T P = PE \geq 0, \tag{2.2}$$

$$Q_3 \leq Q_4 \leq W_3 \leq W_4 \tag{2.3}$$

inequalities are held.

Theorem 2.1. If conditions (A1), (A2) and (1.2) are held, then SIDS (1.1) is exponentially admissible.

proof. As the first step, we prove that SIDS (1.1) is regular and IF. Let $G \in R^{n\times n}$ and $H \in R^{n\times n}$ be two invertible matrices such that

$$\overline{E} = GEH = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \tag{2.4}$$

and define

$$\overline{A} = GAH = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \overline{A_{d_n}} = GA_{d_n}H = \begin{bmatrix} A_{d_n1} & A_{d_n2} \\ A_{d_n3} & A_{d_n4} \end{bmatrix},$$

$$\overline{A_{h_n}} = GA_{h_n}H = \begin{bmatrix} A_{h_n1} & A_{h_n2} \\ A_{h_n3} & A_{h_n4} \end{bmatrix}, \overline{B} = GBH = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix},$$

$$\overline{P} = G^{-T}PH = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}, \overline{Z_j} = G^{-T}Z_jG^{-1} = \begin{bmatrix} Z_{j1} & Z_{j2} \\ Z_{j3} & Z_{j4} \end{bmatrix}, (j = 1, 2, ..., 8),$$

$$\overline{Q_i} = H^TQ_iH = \begin{bmatrix} Q_{i1} & Q_{i2} \\ Q_{i3} & Q_{i4} \end{bmatrix}, \overline{W_i} = H^TW_iH = \begin{bmatrix} W_{i1} & W_{i2} \\ W_{i3} & W_{i4} \end{bmatrix}, (i = 1, ..., 6),$$

$$\overline{N} = H^TNH = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix}, \overline{T} = H^TTH = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}. \tag{2.5}$$

In view of the estimates (2.1) and (2.5), doing some mathematical calculations, we have $A_4^TP_4 + P_4A_4 < 0$ and thus $\overline{A_4}$ is nonsingular. By virtue of this result, the pair (E, A) is regular and IF (see [18]). According to Definition 1.2, we arrive that SIDS (1.1) is regular and IF.

Let

$$\eta(t) = \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} = H^{-1}x(t),$$

where $\eta_1(t) \in R^r$ and $\eta_2(t) \in R^{n-r}$.

Then, we can write SIDS (1.1) in the form:

$$\overline{E}\dot{\eta}(t) = \overline{A}\eta(t) + \sum_{n=1}^{2}\overline{A_{d_n}}\eta(t - d_n(t)) + \sum_{n=1}^{2}\overline{A_{h_n}}\eta(t - h_n(t)) + \overline{B}\int_{t-\tau(t)}^{t}\eta(s)ds. \tag{2.6}$$

Hence, this system can be decomposed to the following system:

$$\begin{aligned}
\dot{\eta_1}(t) =& A_1\eta_1(t) + A_2\eta_2(t) + \sum_{n=1}^{2}A_{d_n1}\eta_1(t - d_n(t)) + \sum_{n=1}^{2}A_{d_n2}\eta_2(t - d_n(t)) \\
&+ \sum_{n=1}^{2}A_{h_n1}\eta_1(t - h_n(t)) + \sum_{n=1}^{2}A_{h_n2}\eta_2(t - h_n(t)) \\
&+ B_1\int_{t-\tau(t)}^{t}\eta_1(s)ds + B_2\int_{t-\tau(t)}^{t}\eta_2(s)ds, \\
0 =& A_3\eta_1(t) + A_4\eta_2(t) + \sum_{n=1}^{2}A_{d_n3}\eta_1(t - d_n(t)) + \sum_{n=1}^{2}A_{d_n4}\eta_2(t - d_n(t)) \\
&+ \sum_{n=1}^{2}A_{h_n3}\eta_1(t - h_n(t)) + \sum_{n=1}^{2}A_{h_n4}\eta_2(t - h_n(t)) \\
&+ B_3\int_{t-\tau(t)}^{t}\eta_1(s)ds + B_4\int_{t-\tau(t)}^{t}\eta_2(s)ds.
\end{aligned} \tag{2.7}$$

As for the next step, we show that subsystem (2.7) is the ES. Define an LKF by

$$V(t,\eta)=\sum_{i=1}^{5}V_i(t,\eta),$$

where

$$\begin{aligned}
V_1(t,\eta)=&\eta^T(t)\overline{E}^T\overline{P}\eta(t),\\
V_2(t,\eta)=&\sum_{n=1}^{2}\int_{t-d_{n_1}}^{t}e^{2\alpha(s-t)}\eta^T(s)\overline{Q_n}\eta(s)ds+\sum_{n=1}^{2}\int_{t-d_n(t)}^{t}e^{2\alpha(s-t)}\eta^T(s)\overline{Q_{n+2}}\eta(s)ds\\
&+\sum_{n=1}^{2}\int_{t-d_{n_2}}^{t}e^{2\alpha(s-t)}\eta^T(s)\overline{Q_{n+4}}\eta(s)ds+\sum_{n=1}^{2}\int_{t-h_{n_1}}^{t}e^{2\alpha(s-t)}\eta^T(s)\overline{W_n}\eta(s)ds\\
&+\sum_{n=1}^{2}\int_{t-h_n(t)}^{t}e^{2\alpha(s-t)}\eta^T(s)\overline{W_{n+2}}\eta(s)ds+\sum_{n=1}^{2}\int_{t-h_{n_2}}^{t}e^{2\alpha(s-t)}\eta^T(s)\overline{W_{n+4}}\eta(s)ds,\\
V_3(t,\eta)=&\sum_{n=1}^{2}d_{n_1}\int_{-d_{n_1}}^{0}\int_{t+\theta}^{t}e^{2\alpha(s-t)}(E\dot\eta(s))^T\overline{Z_n}(E\dot\eta(s))dsd\theta\\
&+\sum_{n=1}^{2}(d_{n_2}-d_{n_1})\int_{-d_{n_2}}^{-d_{n_1}}\int_{t+\theta}^{t}e^{2\alpha(s-t)}(E\dot\eta(s))^T\overline{Z_{n+2}}(E\dot\eta(s))dsd\theta\\
&+\sum_{n=1}^{2}h_{n_1}\int_{-h_{n_1}}^{0}\int_{t+\theta}^{t}e^{2\alpha(s-t)}(E\dot\eta(s))^T\overline{Z_{n+4}}(E\dot\eta(s))dsd\theta\\
&+\sum_{n=1}^{2}(h_{n_2}-h_{n_1})\int_{-h_{n_2}}^{-h_{n_1}}\int_{t+\theta}^{t}e^{2\alpha(s-t)}(E\dot\eta(s))^T\overline{Z_{n+6}}(E\dot\eta(s))dsd\theta,\\
V_4(t,\eta)=&\tau_1\int_{-\tau_1}^{0}\int_{t+\theta}^{t}e^{2\alpha(s-t)}\eta^T(s)\overline{N}\eta(s)dsd\theta,\\
V_5(t,\eta)=&(\tau_2-\tau_1)\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t}e^{2\alpha(s-t)}\eta^T(s)\overline{T}\eta(s)dsd\theta.
\end{aligned}$$

By the derivative of the LKF $V(t,\eta)$ along system (2.7), we derive that

$$\dot V(t,\eta)=\sum_{i=1}^{5}\dot V_i(t,\eta), \tag{2.8}$$

where

$$\begin{aligned}
\dot V_1(t,\eta)=&\dot\eta^T(t)\overline{E}^T\overline{P}\eta(t)+\eta^T(t)\overline{PE}\dot\eta(t)\\
=&\eta^T(t)[\overline{A}^T\overline{P}+\overline{PA}+2\alpha\overline{E}^T\overline{P}]\eta(t)+\sum_{n=1}^{2}\eta^T(t)\overline{PA_{d_n}}\eta(t-d_n(t))\\
&+\sum_{n=1}^{2}\eta^T(t-d_n(t))\overline{A_{d_n}}^T\overline{P}\eta(t)+\sum_{n=1}^{2}\eta^T(t)\overline{PA_{h_n}}\eta(t-h_n(t))\\
&+\sum_{n=1}^{2}\eta^T(t-h_n(t))\overline{A_{h_n}}^T\overline{P}\eta(t)+\eta^T(t)\overline{PB}\int_{t-\tau(t)}^{t}\eta(s)ds\\
&+(\int_{t-\tau(t)}^{t}\eta(s)ds)^T\overline{B}^T\overline{P}\eta(t)-2\alpha V_1(t,\eta),
\end{aligned}\tag{2.9}$$

$$\begin{aligned}\dot{V}_2(t,\eta) \leq & \sum_{n=1}^{2}\eta^T(t)\overline{Q_n}\eta(t) - \sum_{n=1}^{2}e^{-2\alpha d_{n1}}\eta^T(t-d_{n1})\overline{Q_n}\eta(t-d_{n1}) \\ & + \sum_{n=1}^{2}\eta^T(t)\overline{Q_{n+2}}\eta(t) - \sum_{n=1}^{2}e^{-2\alpha d_n(t)}(1-\dot{d}_n(t))\eta^T(t-d_n(t)) \\ & \times \overline{Q_{n+2}}\eta(t-d_n(t)) + \sum_{n=1}^{2}\eta^T(t)\overline{Q_{n+4}}\eta(t) \\ & - \sum_{n=1}^{2}e^{-2\alpha d_{n2}}\eta^T(t-d_{n2})\overline{Q_{n+4}}\eta(t-d_{n2}) \\ & + \sum_{n=1}^{2}\eta^T(t)\overline{W_n}\eta(t) - \sum_{n=1}^{2}e^{-2\alpha h_{n1}}\eta^T(t-h_{n1})\overline{W_n}\eta(t-h_{n1}) \\ & + \sum_{n=1}^{2}\eta^T(t)\overline{W_{n+2}}\eta(t) - \sum_{n=1}^{2}e^{-2\alpha h_n(t)}(1-\dot{h}_n(t))\eta^T(t-h_n(t)) \\ & \times \overline{W_{n+2}}\eta(t-h_n(t)) + \sum_{n=1}^{2}\eta^T(t)\overline{W_{n+4}}\eta(t) \\ & - \sum_{n=1}^{2}e^{-2\alpha h_{n2}}\eta^T(t-h_{n2})\overline{W_{n+4}}\eta(t-h_{n2}) - 2\alpha V_2(t,\eta), \end{aligned} \tag{2.10}$$

Consider the last term in the inequality (2.10). When $0 \leq \mu_n < 1, (n=1,2)$, we have

$$\begin{aligned} & \sum_{n=1}^{2}\{-e^{-2\alpha d_n(t)}(1-\dot{d}_n(t))\eta^T(t-d_n(t))\overline{Q_{n+2}}\eta(t-d_n(t)) \\ & \leq -e^{-2\alpha d_{n2}}(1-\mu_n)\eta^T(t-d_n(t))\overline{Q_{n+2}}\eta(t-d_n(t))\}. \end{aligned} \tag{2.11}$$

Similarly $0 \leq \mu_{n+2} < 1, (n=1,2)$, we can write

$$\begin{aligned} & \sum_{n=1}^{2}\{-e^{-2\alpha h_n(t)}(1-\dot{h}_n(t))\eta^T(t-h_n(t))\overline{W_{n+2}}\eta(t-h_n(t)) \\ & \leq -e^{-2\alpha h_{n2}}(1-\mu_{n+2})\eta^T(t-h_n(t))\overline{W_{n+2}}\eta(t-h_n(t))\}. \end{aligned} \tag{2.12}$$

As for the next step, differentiating the LKF $V_3(t,\eta)$ and performing some elementary calculations, we obtain

$$\begin{aligned}\dot{V}_3(t,\eta) = & \sum_{n=1}^{2}d_{n1}^2\dot{\eta}^T(t)\overline{E}^T\overline{Z_nE}\dot{\eta}(t) - \sum_{n=1}^{2}d_{n1}\int_{t-d_{n1}}^{t}e^{2\alpha(s-t)}(\overline{E}\dot{\eta}(s))^T\overline{Z_n}(\overline{E}\dot{\eta}(s))ds \\ & + \sum_{n=1}^{2}\rho_n^2\dot{\eta}^T(t)\overline{E}^T\overline{Z_{n+2}E}\dot{\eta}(t) - \sum_{n=1}^{2}\rho_n\int_{t-d_{n2}}^{t-d_{n1}}e^{2\alpha(s-t)}(\overline{E}\dot{\eta}(s))^T\overline{Z_{n+2}}(\overline{E}\dot{\eta}(s))ds \\ & + \sum_{n=1}^{2}h_{n1}^2\dot{\eta}^T(t)\overline{E}^T\overline{Z_{n+4}E}\dot{\eta}(t) - \sum_{n=1}^{2}h_{n1}\int_{t-h_{n1}}^{t}e^{2\alpha(s-t)}(\overline{E}\dot{\eta}(s))^T\overline{Z_{n+4}}(\overline{E}\dot{\eta}(s))ds \\ & + \sum_{n=1}^{2}r_n^2\dot{\eta}^T(t)\overline{E}^T\overline{Z_{n+6}E}\dot{\eta}(t) - \sum_{n=1}^{2}r_n\int_{t-h_{n2}}^{t-h_{n1}}e^{2\alpha(s-t)}(\overline{E}\dot{\eta}(s))^T\overline{Z_{n+6}}(\overline{E}\dot{\eta}(s))ds \end{aligned}$$

$$
\begin{aligned}
&\leq \dot{\eta}^T(t)\overline{E}^T\sum_{n=1}^{2}\{d_{n1}^2\overline{Z_n}+\rho_n^2\overline{Z_{n+2}}+h_{n1}^2\overline{Z_{n+4}}+r_n^2\overline{Z_{n+6}}\}\overline{E}\dot{\eta}(t)\\
&\quad-\sum_{n=1}^{2}d_{n1}\int_{t-d_{n1}}^{t}e^{2\alpha(s-t)}(\overline{E}\dot{\eta}(s))^T\overline{Z_n}(\overline{E}\dot{\eta}(s))ds\\
&\quad-\sum_{n=1}^{2}\rho_n\int_{t-d_{n2}}^{t-d_{n1}}e^{2\alpha(s-t)}(\overline{E}\dot{\eta}(s))^T\overline{Z_{n+2}}(\overline{E}\dot{\eta}(s))ds\\
&\quad-\sum_{n=1}^{2}h_{n1}\int_{t-h_{n1}}^{t}e^{2\alpha(s-t)}(\overline{E}\dot{\eta}(s))^T\overline{Z_{n+4}}(\overline{E}\dot{\eta}(s))ds\\
&\quad-\sum_{n=1}^{2}r_n\int_{t-h_{n2}}^{t-h_{n1}}e^{2\alpha(s-t)}(\overline{E}\dot{\eta}(s))^T\overline{Z_{n+6}}(\overline{E}\dot{\eta}(s))ds\\
&\leq \dot{\eta}^T(t)\overline{E}^TS\overline{E}\dot{\eta}(t)-\sum_{n=1}^{2}e^{-2\alpha d_{n1}}d_{n1}\int_{t-d_{n1}}^{t}(\overline{E}\dot{\eta}(s))^T\overline{Z_n}(\overline{E}\dot{\eta}(s))ds\\
&\quad-\sum_{n=1}^{2}e^{-2\alpha d_{n2}}\rho_n\int_{t-d_{n2}}^{t-d_{n1}}(\overline{E}\dot{\eta}(s))^T\overline{Z_{n+2}}(\overline{E}\dot{\eta}(s))ds\\
&\quad-\sum_{n=1}^{2}e^{-2\alpha h_{n1}}h_{n1}\int_{t-h_{n1}}^{t}(\overline{E}\dot{\eta}(s))^T\overline{Z_{n+4}}(\overline{E}\dot{\eta}(s))ds\\
&\quad-\sum_{n=1}^{2}e^{-2\alpha h_{n2}}r_n\int_{t-h_{n2}}^{t-h_{n1}}(\overline{E}\dot{\eta}(s))^T\overline{Z_{n+6}}(\overline{E}\dot{\eta}(s))ds-2\alpha V_3(t,\eta).
\end{aligned}
\tag{2.13}
$$

Using the Jensen inequality [17], we have

$$
\begin{aligned}
&\sum_{n=1}^{2}\{-e^{-2\alpha d_{n1}}d_{n1}\int_{t-d_{n1}}^{t}(\overline{E}\dot{\eta}(s))^T\overline{Z_n}(\overline{E}\dot{\eta}(s))ds\leq -e^{-2\alpha d_{n1}}[\eta(t)-\eta(t-d_{n1})]^T\\
&\quad\times\overline{E}^T\overline{Z_nE}[\eta(t)-\eta(t-d_{n1})]\leq -e^{-2\alpha d_{n1}}\eta^T(t)\overline{E}^T\overline{Z_nE}\eta(t)\\
&\quad+e^{-2\alpha d_{n1}}\eta^T(t)\overline{E}^T\overline{Z_nE}\eta(t-d_{n1})+e^{-2\alpha d_{n1}}\eta^T(t-d_{n1})\overline{E}^T\overline{Z_nE}\eta(t)\\
&\quad-e^{-2\alpha d_{n1}}\eta^T(t-d_{n1})\overline{E}^T\overline{Z_nE}\eta(t-d_{n1})\}
\end{aligned}
\tag{2.14}
$$

and

$$
\begin{aligned}
&\sum_{n=1}^{2}\{-e^{-2\alpha h_{n1}}h_{n1}\int_{t-h_{n1}}^{t}(\overline{E}\dot{\eta}(s))^T\overline{Z_{n+4}}(\overline{E}\dot{\eta}(s))ds\leq -e^{-2\alpha h_{n1}}[\eta(t)-\eta(t-h_{n1})]^T\\
&\quad\times\overline{E}^T\overline{Z_{n+4}E}[\eta(t)-\eta(t-h_{n1})]\leq -e^{-2\alpha h_{n1}}\eta^T(t)\overline{E}^T\overline{Z_{n+4}E}\eta(t)\\
&\quad+e^{-2\alpha h_{n1}}\eta^T(t)\overline{E}^T\overline{Z_{n+4}E}\eta(t-h_{n1})+e^{-2\alpha h_{n1}}\eta^T(t-h_{n1})\overline{E}^T\overline{Z_{n+4}E}\eta(t)\\
&\quad-e^{-2\alpha h_{n1}}\eta^T(t-h_{n1})\overline{E}^T\overline{Z_{n+4}E}\eta(t-h_{n1})\}.
\end{aligned}
\tag{2.15}
$$

As for the next step, we obtain

$$\begin{aligned}&\sum_{n=1}^{2}\{-e^{-2\alpha d_{n2}}\rho_n\int_{t-d_{n2}}^{t-d_{n1}}(\overline{E}\dot{\eta}(s))^T\overline{Z_{n+2}}(\overline{E}\dot{\eta}(s))ds=-e^{-2\alpha d_{n2}}[d_{n2}-d_n(t)]\\&\times\int_{t-d_{n2}}^{t-d_n(t)}(\overline{E}\dot{\eta}(s))^T\overline{Z_{n+2}}(\overline{E}\dot{\eta}(s))ds-e^{-2\alpha d_{n2}}[d_{n2}-d_n(t)]\int_{t-d_n(t)}^{t-d_{n1}}(\overline{E}\dot{\eta}(s))^T\\&\times\overline{Z_{n+2}}(\overline{E}\dot{\eta}(s))ds-e^{-2\alpha d_{n2}}[d_n(t)-d_{n1}]\int_{t-d_{n2}}^{t-d_n(t)}(\overline{E}\dot{\eta}(s))^T\overline{Z_{n+2}}(\overline{E}\dot{\eta}(s))ds\\&-e^{-2\alpha d_{n2}}[d_n(t)-d_{n1}]\int_{t-d_n(t)}^{t-d_{n1}}(\overline{E}\dot{\eta}(s))^T\overline{Z_{n+2}}(\overline{E}\dot{\eta}(s))ds\}.\end{aligned}$$

Let $\beta_n=\frac{d_{n2}-d_n(t)}{d_{n2}-d_{n1}}$. Then,

$$\begin{aligned}&\sum_{n=1}^{2}\{-e^{-2\alpha d_{n2}}[d_{n2}-d_n(t)]\int_{t-d_n(t)}^{t-d_{n1}}(\overline{E}\dot{\eta}(s))^T\overline{Z_{n+2}}(\overline{E}\dot{\eta}(s))ds\\&-e^{-2\alpha d_{n2}}[d_n(t)-d_{n1}]\int_{t-d_{n2}}^{t-d_n(t)}(\overline{E}\dot{\eta}(s))^T\overline{Z_{n+2}}(\overline{E}\dot{\eta}(s))ds\\&\le-e^{-2\alpha d_{n2}}\beta_n[\eta(t-d_{n1})-\eta(t-d_n(t))]^T\overline{E}^T\overline{Z_{n+2}E}[\eta(t-d_{n1})\\&-\eta(t-d_n(t))]-e^{-2\alpha d_{n2}}(1-\beta_n)[\eta(t-d_n(t))-\eta(t-d_{n2})]^T\\&\times\overline{E}^T\overline{Z_{n+2}E}[\eta(t-d_n(t))-\eta(t-d_{n2})]\}\end{aligned}$$

and

$$\begin{aligned}&\sum_{n=1}^{2}\{-e^{-2\alpha d_{n2}}\rho_n\int_{t-d_{n2}}^{t-d_{n1}}(\overline{E}\dot{\eta}(s))^T\overline{Z_{n+2}}(\overline{E}\dot{\eta}(s))ds\le-e^{-2\alpha d_{n2}}[\eta(t-d_n(t))\\&-\eta(t-d_{n2})]^T\overline{E}^T\overline{Z_{n+2}E}[\eta(t-d_n(t))-\eta(t-d_{n2})]-e^{-2\alpha d_{n2}}[\eta(t-d_{n1})\\&-\eta(t-d_n(t))]^T\overline{E}^T\overline{Z_{n+2}E}[\eta(t-d_{n1})-\eta(t-d_n(t))]-e^{-2\alpha d_{n2}}\beta_n[\eta(t-d_{n1})\\&-\eta(t-d_n(t))]^T\overline{E}^T\overline{Z_{n+2}E}[\eta(t-d_{n1})-\eta(t-d_n(t))]-e^{-2\alpha d_{n2}}(1-\beta_n)\\&\times[\eta(t-d_n(t))-\eta(t-d_{n2})]^T\overline{E}^T\overline{Z_{n+2}E}[\eta(t-d_n(t))-\eta(t-d_{n2})]\}.\end{aligned}\tag{2.16}$$

Next,let $\gamma_n=\frac{h_{n2}-h_n(t)}{h_{n2}-h_{n1}}$. Hence,

$$\begin{aligned}&\sum_{n=1}^{2}\{-e^{-2\alpha h_{n2}}r_n\int_{t-h_{n2}}^{t-h_{n1}}(\overline{E}\dot{\eta}(s))^T\overline{Z_{n+6}}(\overline{E}\dot{\eta}(s))ds\le-e^{-2\alpha h_{n2}}[\eta(t-h_n(t))\\&-\eta(t-h_{n2})]^T\overline{E}^T\overline{Z_{n+6}E}[\eta(t-h_n(t))-\eta(t-h_{n2})]-e^{-2\alpha h_{n2}}[\eta(t-h_{n1})\\&-\eta(t-h_n(t))]^T\overline{E}^T\overline{Z_{n+6}E}[\eta(t-h_{n1})-\eta(t-h_n(t))]-e^{-2\alpha h_{n2}}\gamma_n[\eta(t-h_{n1})\\&-\eta(t-h_n(t))]^T\overline{E}^T\overline{Z_{n+6}E}[\eta(t-h_{n1})-\eta(t-h_n(t))]-e^{-2\alpha h_{n2}}(1-\gamma_n)\\&\times[\eta(t-h_n(t))-\eta(t-h_{n2})]^T\overline{E}^T\overline{Z_{n+6}E}[\eta(t-h_n(t))-\eta(t-h_{n2})]\}.\end{aligned}\tag{2.17}$$

Next, by the derivative of the LKF $V_4(t,\eta)$ and the Jensen inequality, we find:

$$\begin{aligned}\dot{V}_4(t,\eta) =& \tau_1^2\eta^T(t)\overline{N}\eta(t) - \tau_1\int_{t-\tau_1}^{t} e^{2\alpha(s-t)}\eta^T(s)\overline{N}\eta(s)ds\\ =& \tau_1^2\eta^T(t)\overline{N}\eta(t) - e^{-2\alpha\tau_1}\tau_1\int_{t-\tau_1}^{t}\eta^T(s)\overline{N}\eta(s)ds\\ \le& \tau_1^2\eta^T(t)\overline{N}\eta(t) - e^{-2\alpha\tau_1}(\int_{t-\tau_1}^{t}\eta(s)ds)^T\overline{N}(\int_{t-\tau_1}^{t}\eta(s)ds)\\ &- 2\alpha V_4(t,\eta). \end{aligned} \tag{2.18}$$

As similar before, calculating the derivative of the LKF $V_5(t,\eta)$ and using some elementary calculations, we get:

$$\begin{aligned}\dot{V}_5(t,\eta) =& (\tau_2-\tau_1)^2\eta^T(t)\overline{T}\eta(t) - (\tau_2-\tau_1)\int_{t-\tau_2}^{t-\tau_1} e^{2\alpha(s-t)}\eta^T(s)\overline{T}\eta(s)ds\\ =& (\tau_2-\tau_1)^2\eta^T(t)\overline{T}\eta(t) - e^{-2\alpha\tau_2}(\tau_2-\tau_1)\int_{t-\tau(t)}^{t-\tau_1}\eta^T(s)\overline{T}\eta(s)ds\\ \le& (\tau_2-\tau_1)^2\eta^T(t)\overline{T}\eta(t) - e^{-2\alpha\tau_2}(\int_{t-\tau(t)}^{t-\tau_1}\eta(s)ds)^T\overline{T}(\int_{t-\tau(t)}^{t-\tau_1}\eta(s)ds)\\ \le& (\tau_2-\tau_1)^2\eta^T(t)\overline{T}\eta(t) - e^{-2\alpha\tau_2}[\int_{t-\tau(t)}^{t}\eta(s)ds - \int_{t-\tau_1}^{t}\eta(s)ds]^T\\ &\times\overline{T}[\int_{t-\tau(t)}^{t}\eta(s)ds - \int_{t-\tau_1}^{t}\eta(s)ds] - 2\alpha V_5(t,\eta). \end{aligned} \tag{2.19}$$

Next, by the virtue of the estimates (2.9)-(2.19), we obtain:

$$\begin{aligned}\dot{V}(t,\eta) + 2\alpha V(t,\eta) \le& \xi^T(t)[\overline{\Omega} + \sum_{n=1}^{2}\{(1-\beta_n)\overline{K_{2n-1}} + \beta_n\overline{K_{2n}}\}\\ &+ \sum_{n=1}^{2}\{(1-\gamma_n)\overline{U_{2n-1}} + \gamma_n\overline{U_{2n}}\}]\xi(t),\end{aligned}$$

where

$$\begin{aligned}\xi(t) = [&\eta^T(t), \eta^T(t-d_1(t)), \eta^T(t-d_2(t)), \eta^T(t-h_1(t)), \eta^T(t-h_2(t)),\\ &\eta^T(t-d_{11}), \eta^T(t-d_{12}), \eta^T(t-d_{21}), \eta^T(t-d_{22}), \eta^T(t-h_{11}),\\ &\eta^T(t-h_{12}), \eta^T(t-h_{21}), \eta^T(t-h_{22}), \int_{t-\tau(t)}^{t}\eta(s)ds, \int_{t-\tau_1}^{t}\eta(s)ds]^T\end{aligned}$$

$\overline{\Omega}$ is defined by

$$\overline{A}, \overline{A_{d_n}}, \overline{A_{h_n}}, \overline{B}, \overline{P}, \overline{Q_i}, \overline{W_i}, \overline{N}, \overline{T}, \overline{Z_j}$$

instead of

$$A, A_{d_n}, A_{h_n}, B, P, Q_i, W_i, N, T, Z_j, (i = 1, ..., 6; j = 1, ..., 8; n = 1, 2).$$

Pre- and post-multiplying (2.1) by $diag\{\underbrace{H^T, H^T, ..., H^T}_{15-H^T}\}$ and $diag\{\underbrace{H, H, ..., H}_{15-H}\}$ respectively, and using Lemma 1.3 and Lemma 1.4, we have:

$$\dot{V}(t,\eta) + 2\alpha V(t,\eta) \le 0.$$

Hence

$$V(t,\eta) \le e^{-2\alpha t}V(t,\phi).$$

According to the discussion above, we have

$$\lambda_1||\eta_1(t)||^2 \leq V(t,\eta) \leq e^{-2\alpha t}V(t,\phi) \leq \lambda_2||\phi||_\delta^2 e^{-2\alpha t}, \tag{2.20}$$

where $\lambda_1 = \lambda_{min}(P_1) > 0$ and

$$\begin{aligned}\lambda_2 =&\lambda_{max}(P_{11}) + \sum_{n=1}^{2}\{d_{n1}\lambda_{max}(\overline{Q_n}) + d_{n2}\lambda_{max}(\overline{Q_{n+2}}) + d_{n2}\lambda_{max}(\overline{Q_{n+4}})\\ &+ h_{n1}\lambda_{max}(\overline{W_n}) + h_{n2}\lambda_{max}(\overline{W_{n+2}}) + h_{n2}\lambda_{max}(\overline{W_{n+4}})\\ &+ \frac{1}{2}d_{n1}^3\lambda_{max}(\overline{Z_n}) + \frac{1}{2}(d_{n2}-d_{n1})^2(d_{n2}+d_{n1})\lambda_{max}(\overline{Z_{n+2}})\\ &+ \frac{1}{2}h_{n1}^3\lambda_{max}(\overline{Z_{n+4}}) + \frac{1}{2}(h_{n2}-h_{n1})^2(h_{n2}+h_{n1})\lambda_{max}(\overline{Z_{n+6}})\}\\ &+ \frac{1}{2}\tau_1^3\lambda_{max}(\overline{N}) + \frac{1}{2}(\tau_2-\tau_1)^2(\tau_2+\tau_1)\lambda_{max}(\overline{T}).\end{aligned}$$

From (2.20), we get

$$||\eta_1(t)|| \leq \sqrt{\frac{\lambda_2}{\lambda_1}}||\phi||_\delta e^{-\alpha t} = \varpi_1||\phi||_\delta e^{-\alpha t}. \tag{2.21}$$

The inequality (2.21) verifies that the subsystem (2.7) is the ES.

Now, we prove that the algebraic subsystem (2.7) is the ES. Let

$$\begin{aligned}p(t) =&A_3\eta_1(t) + \sum_{n=1}^{2}A_{d_{n3}}\eta_1(t-d_n(t))\\ &+ \sum_{n=1}^{2}A_{h_{n3}}\eta_1(t-h_n(t)) + B_3\int_{t-\tau(t)}^{t}\eta_1(s)ds.\end{aligned} \tag{2.22}$$

Observe that $t - d_1(t) \geq 0, \forall t \geq 0$. Then,

$$||\eta_1(t-d_1(t))||^2 \leq \frac{\lambda_2}{\lambda_1}||\phi||_\delta^2 e^{-2\alpha(t-d_1(t))} \leq \frac{\lambda_2}{\lambda_1}||\phi||_\delta^2 e^{2\alpha d_{12}}e^{-2\alpha t}.$$

Therefore, we have

$$||\eta_1(t-d_1(t))|| \leq \varpi_1 e^{\alpha d_{12}}||\phi||_\delta e^{-\alpha t}.$$

Similarly, we obtain

$$\begin{aligned}&||\eta_1(t-d_2(t))|| \leq \varpi_1 e^{\alpha d_{22}}||\phi||_\delta e^{-\alpha t},\\ &||\eta_1(t-h_1(t))|| \leq \varpi_1 e^{\alpha h_{12}}||\phi||_\delta e^{-\alpha t},\\ &||\eta_1(t-h_2(t))|| \leq \varpi_1 e^{\alpha h_{22}}||\phi||_\delta e^{-\alpha t}\end{aligned}$$

and

$$||\eta_1(s)|| \leq \varpi_1||\phi||_\delta e^{-\alpha s}, \forall s \in [t-\tau(t), t], t \geq 0.$$

From (2.22), it follows that

$$\begin{aligned}||p(t)|| \leq&||A_3||||\eta_1(t)|| + \sum_{n=1}^{2}||A_{d_n3}||||\eta_1(t-d_n(t))||\\ &+ \sum_{n=1}^{2}||A_{h_n3}||||\eta_1(t-h_n(t))|| + ||B_3||\int_{t-\tau(t)}^{t}||\eta_1(s)||ds\\ \leq&\{k_1 + \sum_{n=1}^{2}k_{n+1}e^{\alpha d_{n2}} + \sum_{n=1}^{2}k_{n+3}e^{\alpha h_{n2}} + \frac{k_6}{\alpha}(e^{\alpha\tau_2}-1)\}\\ &\times \varpi_1||\phi||_\delta e^{-\alpha t} \leq \sigma e^{-\alpha t},\end{aligned} \tag{2.23}$$

where

$$k_1 = ||A_3||, k_{n+1} = ||A_{d_n3}||, k_{n+3} = ||A_{h_n3}||, k_6 = ||B_3||, (n = 1, 2)$$

and

$$\sigma = \{k_1 + \sum_{n=1}^{2} k_{n+1} e^{\alpha d_{n2}} + \sum_{n=1}^{2} k_{n+3} e^{\alpha h_{n2}} + \frac{k_6}{\alpha}(e^{\alpha \tau_2} - 1)\}\varpi_1 ||\phi||_\delta, \sigma > 0.$$

To verify the exponential stability of the algebraic subsystem (2.7), we define a function by

$$\begin{aligned} L(t) =& \sum_{n=1}^{2} \eta_2^T(t)[Q_{(n+2)4} + W_{(n+2)4}]\eta_2(t) + \sum_{n=1}^{2} \eta_2^T(t - d_n(t)) \\ & \times [-e^{-2\alpha d_{n2}}(1 - \mu_n)]Q_{(n+2)4}\eta_2(t - d_n(t)) \\ & + \sum_{n=1}^{2} \eta_2^T(t - h_n(t))[-e^{-2\alpha h_{n2}}(1 - \mu_{n+2})]W_{(n+2)4}\eta_2(t - h_n(t)) \\ & - e^{-2\alpha\tau_2}\tau(t) \int_{t-\tau(t)}^{t} \eta_2^T(s) T_4 \eta_2(s) ds. \end{aligned} \tag{2.24}$$

By pre-multiplying the second equation of (2.7) with $\eta_2^T(t)P_4^T$, we have

$$\begin{aligned} 0 =& \eta_2^T(t)P_4^T A_4 \eta_2(t) + \sum_{n=1}^{2} \eta_2^T(t)P_4^T A_{d_n4}\eta_2(t - d_n(t)) \\ & + \sum_{n=1}^{2} \eta_2^T(t)P_4^T A_{h_n4}\eta_2(t - h_n(t)) \\ & + \eta_2^T(t)P_4^T B_4 \int_{t-\tau(t)}^{t} \eta_2(s)ds + \eta_2^T(t)P_4^T p(t). \end{aligned}$$

Adding this equality and its transpose to (2.24) and using Lemma 1.1, we obtain

$$\begin{aligned} L(t) =& \eta_2^T(t)[P_4^T A_4 + A_4^T P_4 + \sum_{n=1}^{2} Q_{(n+2)4} + \sum_{n=1}^{2} W_{(n+2)4} + e^{2\alpha\tau_2} P_4^T B_4 T_4^{-1} B_4^T P_4]\eta_2(t) \\ & + \sum_{n=1}^{2} \eta_2^T(t)P_4^T A_{d_n4}\eta_2(t - d_n(t)) + \sum_{n=1}^{2} \eta_2^T(t - d_n(t))A_{d_n4}^T P_4 \eta_2(t) \\ & + \sum_{n=1}^{2} \eta_2^T(t)P_4^T A_{h_n4}\eta_2(t - h_n(t)) + \sum_{n=1}^{2} \eta_2^T(t - h_n(t))A_{h_n4}^T P_4 \eta_2(t) \\ & + \sum_{n=1}^{2} \eta_2^T(t - d_n(t))[-e^{-2\alpha d_{n_2}}(1 - \mu_n)]Q_{(n+2)4}\eta_2(t - d_n(t)) \\ & + \sum_{n=1}^{2} \eta_2^T(t - h_n(t))[-e^{-2\alpha h_{n_2}}(1 - \mu_{n+2})]W_{(n+2)4}\eta_2(t - h_n(t)) \\ & + 2\eta_2^T(t)P_4^T p(t) \\ \leq & \xi^T(t)\Theta\xi(t) + \upsilon_1 \eta_2^T(t)\eta_2(t) + \frac{1}{\upsilon_1} p^T(t) P_4 P_4^T p(t), \end{aligned} \tag{2.25}$$

where

$$\xi(t) = \begin{bmatrix} \eta_2^T(t) & \eta_2^T(t - d_1(t)) & \eta_2^T(t - d_2(t)) & \eta_2^T(t - h_1(t)) & \eta_2^T(t - h_2(t)) \end{bmatrix},$$

υ_1 is a positive constant, and

$$\Theta = \begin{bmatrix} \Theta_{11} & P_4^T A_{d_1 4} & P_4^T A_{d_2 4} & P_4^T A_{h_1 4} & P_4^T A_{h_2 4} \\ * & \Theta_{22} & 0 & 0 & 0 \\ * & * & \Theta_{33} & 0 & 0 \\ * & * & * & \Theta_{44} & 0 \\ * & * & * & * & \Theta_{55} \end{bmatrix}$$

here

$$\begin{aligned} \Theta_{11} =& P_4^T A_4 + A_4^T P_4 + Q_{34} + Q_{44} + W_{34} + W_{44} + e^{2\alpha\tau_2} P_4^T B_4 T_4^{-1} B_4^T P_4, \\ \Theta_{22} =& [-e^{-2\alpha d_{12}}(1-\mu_1)]Q_{34}, \Theta_{33} = [-e^{-2\alpha d_{22}}(1-\mu_2)]Q_{44}, \\ \Theta_{44} =& [-e^{-2\alpha h_{12}}(1-\mu_3)]W_{34}, \Theta_{55} = [-e^{-2\alpha h_{22}}(1-\mu_4)]W_{44}, \end{aligned}$$

Using Lemma 1.3, i.e., the Schur's complement, we have

$$\begin{bmatrix} \Omega_{11} & P_4^T A_{d_1 4} & P_4^T A_{d_2 4} & P_4^T A_{h_1 4} & P_4^T A_{h_2 4} \\ * & \Omega_{22} & 0 & 0 & 0 \\ * & * & \Omega_{33} & 0 & 0 \\ * & * & * & \Omega_{44} & 0 \\ * & * & * & * & \Omega_{55} \end{bmatrix} < 0. \tag{2.26}$$

Since $S > 0$, pre- and post-multiplying the inequality (2.26) by $diag\{H^T, H^T, H^T, H^T, H^T\}$ and $diag\{H, H, H, H, H\}$ respectively, and by using Lemma 1.3, we have

$$\begin{bmatrix} \overline{\Theta}_{11} & P_4^T A_{d_1 4} & P_4^T A_{d_2 4} & P_4^T A_{h_1 4} & P_4^T A_{h_2 4} \\ * & \Theta_{22} & 0 & 0 & 0 \\ * & * & \Theta_{33} & 0 & 0 \\ * & * & * & \Theta_{44} & 0 \\ * & * & * & * & \Theta_{55} \end{bmatrix} < 0.$$

where

$$\begin{aligned} \overline{\Theta}_{11} =& P_4^T A_4 + A_4^T P_4 + Q_{14} + Q_{24} + Q_{34} + Q_{44} + Q_{54} + Q_{64} \\ &+ W_{14} + W_{24} + W_{34} + W_{44} + W_{54} + W_{64}, \end{aligned}$$

Additionally, from $\overline{\Omega} < 0$, we have

$$\begin{bmatrix} \overline{\Omega}_{1,1} & \overline{\Omega}_{1,2} & \overline{\Omega}_{1,3} & \overline{\Omega}_{1,4} & \overline{\Omega}_{1,5} & P_4^T B_4^T \\ * & \overline{\Omega}_{2,2} & 0 & 0 & 0 & 0 \\ * & * & \overline{\Omega}_{3,3} & 0 & 0 & 0 \\ * & * & * & \overline{\Omega}_{4,4} & 0 & 0 \\ * & * & * & * & \overline{\Omega}_{5,5} & 0 \\ * & * & * & * & * & \overline{\Omega}_{14,14} \end{bmatrix} < 0.$$

By using Lemma 2 given in [8], we can write

$$\begin{bmatrix} \hat{\Theta}_{1,1} & P_4^T A_{d_1 4} & P_4^T A_{d_2 4} & P_4^T A_{h_1 4} & P_4^T A_{h_2 4} & P_4^T B_4^T \\ * & \Theta_{2,2} & 0 & 0 & 0 & 0 \\ * & * & \Theta_{3,3} & 0 & 0 & 0 \\ * & * & * & \Theta_{4,4} & 0 & 0 \\ * & * & * & * & \Theta_{5,5} & 0 \\ * & * & * & * & * & \Theta_{14,14} \end{bmatrix} < 0$$

where

$$\begin{aligned}\hat{\Theta}_{1,1} =& P_4^T A_4 + A_4^T P_4 + Q_{14} + Q_{24} + Q_{34} + Q_{44} + Q_{54} + Q_{64} \\ &+ W_{14} + W_{24} + W_{34} + W_{44} + W_{54} + W_{64} + \tau_1^2 N_4 + (\tau_2 - \tau_1)^2 T_4, \\ \Theta_{14,14} =& - e^{-2\alpha\tau_2} T_4.\end{aligned}$$

Consequently, it follows that

$$\begin{bmatrix} \hat{\Theta}_{1,1} + e^{2\alpha\tau_2} P_4^T B_4 T_4^{-1} B_4^T P_4 & P_4^T A_{d_1 4} & P_4^T A_{d_2 4} & P_4^T A_{h_1 4} & P_4^T A_{h_2 4} \\ * & \Theta_{2,2} & 0 & 0 & 0 \\ * & * & \Theta_{3,3} & 0 & 0 \\ * & * & * & \Theta_{4,4} & 0 \\ * & * & * & * & \Theta_{5,5} \end{bmatrix} < 0. \quad (2.27)$$

Then, we have

$$\Theta \le diag\left\{ -\Psi \quad 0 \quad 0 \quad 0 \quad 0 \right\},$$

where

$$\begin{aligned}\Psi =& Q_{14} + Q_{24} + Q_{54} + Q_{64} + W_{14} + W_{24} + W_{54} + W_{64} \\ &+ \tau_1^2 N_4 + (\tau_2 - \tau_1)^2 T_4 + e^{2\alpha\tau_2} P_4^T B_4 T_4^{-1} B_4^T P_4.\end{aligned}$$

Combining (2.25) and (2.27), we get

$$\begin{aligned}&\eta_2^T(t)[Q_{34} + Q_{44} + W_{34} + W_{44}]\eta_2(t) + [-e^{-2\alpha d_{12}}(1-\mu_1)]\eta_2^T(t - d_1(t)) \\ &\quad \times Q_{34}\eta_2(t - d_1(t)) + [-e^{-2\alpha d_{22}}(1-\mu_2)]\eta_2^T(t - d_2(t)) \\ &\quad \times Q_{44}\eta_2(t - d_2(t)) + [-e^{-2\alpha h_{12}}(1-\mu_3)]\eta_2^T(t - h_1(t)) \\ &\quad \times W_{34}\eta_2(t - h_1(t)) + [-e^{-2\alpha h_{22}}(1-\mu_4)]\eta_2^T(t - h_2(t)) \\ &\quad \times W_{44}\eta_2(t - h_2(t)) \\ \le& \eta_2^T(t)[-\Psi + \upsilon_1 I]\eta_2(t) + \frac{1}{\upsilon_1} p^T(t) P_4 P_4^T p(t)\end{aligned}$$

which can be arranged as

$$\begin{aligned}&\eta_2^T(t)[Q_{34} + Q_{44} + W_{34} + W_{44} + \Psi - \upsilon_1 I]\eta_2(t) \\ \le& [e^{-2\alpha d_{12}}(1-\mu_1)]\eta_2^T(t - d_1(t)) Q_{34}\eta_2(t - d_1(t)) \\ &+ [e^{-2\alpha d_{22}}(1-\mu_2)]\eta_2^T(t - d_2(t)) Q_{44}\eta_2(t - d_2(t)) \\ &+ [e^{-2\alpha h_{12}}(1-\mu_3)]\eta_2^T(t - h_1(t)) W_{34}\eta_2(t - h_1(t)) \\ &+ [e^{-2\alpha h_{22}}(1-\mu_4)]\eta_2^T(t - h_2(t)) W_{44}\eta_2(t - h_2(t)) \\ &+ \frac{1}{\upsilon_1} p^T(t) P_4 P_4^T p(t).\end{aligned} \quad (2.28)$$

Since υ_1 is an arbitrary positive scalar, we can choose a sufficiently small positive constant υ_1 such that

$$Q_{34} + Q_{44} + W_{34} + W_{44} + \Psi - \upsilon_1 I \ge W_{44} + \Psi - \upsilon_1 I \ge (1 + b) W_{44}, \quad (2.29)$$

where $b > 0, b \in R$ such that $\Psi > bW_{44}$.

Let

$$f(t) = \eta_2^T(t) W_{44} \eta_2(t), t \ge -\delta,$$

$$\upsilon_2 = \frac{e^{-2\alpha d_{12}}(1-\mu_1) + e^{-2\alpha d_{22}}(1-\mu_2) + e^{-2\alpha h_{12}}(1-\mu_3) + e^{-2\alpha h_{22}}(1-\mu_4)}{1 + b}$$

and

$$\upsilon_3 = \frac{\sigma||P_4^T P_4||}{\upsilon_1(1+b)}.$$

From (2.28) and (2.29), we can write

$$f(t) \le \upsilon_2 \sup_{t-\delta \le s \le t} f(s) + \upsilon_3 e^{-2\alpha t}.$$

Using Lemma 1.2, we derive that

$$f(t) \le \sup_{-\delta \le \theta \le 0} f(\theta) e^{-2\alpha t} + \frac{\upsilon_3 e^{-2\alpha t}}{1 - \upsilon_2 e^{2\alpha\delta}},$$

which means that

$$\begin{aligned}
&\lambda_{min}(W_{44})||\eta_2(t)||^2 \\
&\le [\{\frac{e^{-2\alpha d_{12}}(1-\mu_1) + e^{-2\alpha d_{22}}(1-\mu_2) + e^{-2\alpha h_{12}}(1-\mu_3) + e^{-2\alpha h_{22}}(1-\mu_4)}{1+b}\} \\
&\quad \times e^{2\alpha\delta}\lambda_{max}(W_{44}) + \lambda_{max}(P_4^T P_4)\sigma^2 \\
&\quad \times \frac{e^{-2\alpha\delta}}{\upsilon_1[1+b-\{e^{-2\alpha d_{12}}(1-\mu_1) + e^{-2\alpha d_{22}}(1-\mu_2) + e^{-2\alpha h_{12}}(1-\mu_3) + e^{-2\alpha h_{22}}(1-\mu_4)\}]}] \\
&\quad \times ||\phi||_\delta^2 e^{-2\alpha t}, t \ge 0.
\end{aligned}$$

Consequently,

$$||\eta_2(t)|| \le \varpi_2 ||\phi||_\delta e^{-\alpha t}, t \ge 0. \tag{2.30}$$

where

$$\begin{aligned}
\varpi_2 =& [\{\frac{e^{-2\alpha d_{12}}(1-\mu_1) + e^{-2\alpha d_{22}}(1-\mu_2) + e^{-2\alpha h_{12}}(1-\mu_3) + e^{-2\alpha h_{22}}(1-\mu_4)}{(1+b)\lambda_{min}(W_{44})}\} e^{2\alpha\delta}\lambda_{max}(W_{44}) \\
&+ \frac{\lambda_{max}(P_4^T P_4)\sigma^2 e^{-2\alpha\delta}\{\lambda_{min}(W_{44})\}^{-1}}{\upsilon_1(1+b-\{e^{-2\alpha d_{12}}(1-\mu_1) + e^{-2\alpha d_{22}}(1-\mu_2) + e^{-2\alpha h_{12}}(1-\mu_3) + e^{-2\alpha h_{22}}(1-\mu_4)\})}]^{\frac{1}{2}}
\end{aligned}$$

Combining (2.21) and (2.30), we obtain

$$||x(t)|| \le \varpi_3 ||\phi||_\delta e^{-\alpha t}, t \ge 0,$$

where

$$\varpi_3 = ||H||||H^{-1}||\sqrt{\varpi_1^2 + \varpi_2^2}.$$

This completes the proof.

3 Numerical examples

Example 3.1. For the particular case of system (1.1), we consider the following singular system with variable delays:

$$\begin{aligned}
\frac{d}{dt}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}\right) =& \begin{bmatrix} -5.2 & 0 \\ -4.5 & -10 \end{bmatrix} \times \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\
&+ \begin{bmatrix} 0.1 & 0.35 \\ 0 & -0.25 \end{bmatrix} \times \begin{bmatrix} x_1(t-(0.3+0.2sint)) \\ x_2(t-(0.3+0.2sint)) \end{bmatrix} \\
&+ \begin{bmatrix} -0.2 & -0.4 \\ 0 & 0.4 \end{bmatrix} \times \begin{bmatrix} x_1(t-(0.1+0.3sin^2t)) \\ x_2(t-(0.1+0.3sin^2t)) \end{bmatrix} \\
&+ \begin{bmatrix} 0 & -0.5 \\ -0.6 & -0.4 \end{bmatrix} \times \begin{bmatrix} \int_{t-\frac{2+sint}{20}}^t x_1(s)ds \\ \int_{t-\frac{2+sint}{20}}^t x_2(s)ds \end{bmatrix},
\end{aligned} \tag{3.1}$$

where

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -5.2 & 0 \\ -4.5 & -10 \end{bmatrix}, A_d = \begin{bmatrix} 0.1 & 0.35 \\ 0 & -0.25 \end{bmatrix},$$
$$A_h = \begin{bmatrix} -0.2 & -0.4 \\ 0 & 0.4 \end{bmatrix}, B = \begin{bmatrix} 0 & -0.5 \\ -0.6 & -0.4 \end{bmatrix},$$

and

$$\begin{aligned} d_1 =& 0.1 \le d(t) = 0.3 + 0.2sint \le 0.5 = d_2, \dot{d}(t) = 0.2cost \le 0.2 = \mu_1, \\ h_1 =& 0.1 \le h(t) = 0.1 + 0.3sin^2 t \le 0.4 = h_2, \dot{h}(t) = 0.3sin2t \le 0.3 = \mu_2, \\ \tau_1 =& 0.05 \le \tau(t) = \frac{2 + sint}{20} \le 0.15 = \tau_2, \alpha = 0.1, \end{aligned}$$

It is clear that the system (3.1) is regular and IF. Next, the LMIs (2.1)-(2.3) are feasible with

$$P = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}, Q_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, Q_2 = \begin{bmatrix} 0.5 & 0.2 \\ 0.2 & 0.6 \end{bmatrix},$$
$$Q_3 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.3 \end{bmatrix}, T = \begin{bmatrix} 5.6 & 0.6 \\ 0.6 & 4.4 \end{bmatrix}, W_1 = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix},$$
$$W_2 = \begin{bmatrix} 0.5 & 0.2 \\ 0.2 & 0.6 \end{bmatrix}, W_3 = \begin{bmatrix} 0.8 & 0.3 \\ 0.3 & 0.6 \end{bmatrix}, Z_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$
$$Z_2 = \begin{bmatrix} 0.8 & 0.4 \\ 0.4 & 0.6 \end{bmatrix}, Z_3 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, Z_4 = \begin{bmatrix} 0.8 & 0.5 \\ 0.5 & 0.7 \end{bmatrix},$$

and

$$N = \begin{bmatrix} 1 & 0.8 \\ 0.8 & 1.2 \end{bmatrix}.$$

Therefore, system (3.1) is exponentially admissible. There exist two regular matrices

$$G = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, H = \begin{bmatrix} 0.5 & 0 \\ -0.225 & -0.1 \end{bmatrix}.$$

such that

$$GEH = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, GAH = \begin{bmatrix} -5.2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Consequently, by Theorem 2.1, the solutions of the system (3.1) satisfy

$$||x(t, \phi)|| \le 51.3722||\phi||_\delta e^{-0.1t}, t \ge 0.$$

Below, by using MATLAB-Simulink, we draw the trajectories of solutions of system (3.1).

Example 3.2. As a special case of system (1.1), we consider the following singular delay system for $n = 3$:

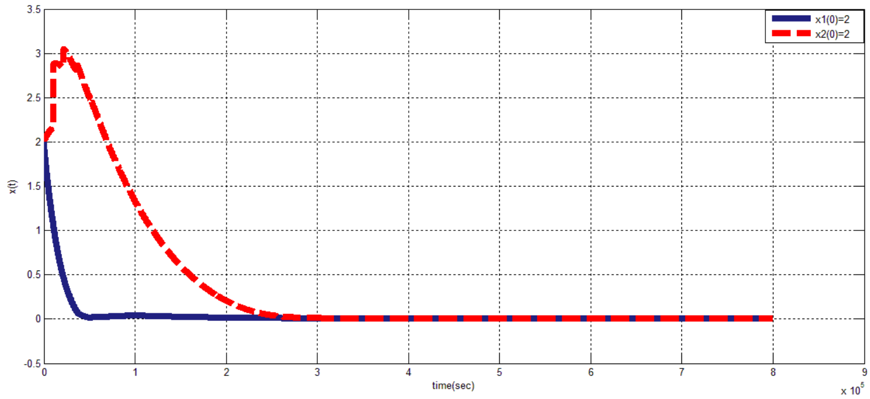

Figure 1. Trajectories of the solutions $x(t)$ of system (3.1), when $\delta = 0,5$.

$$\frac{d}{dt}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}\right) = \begin{bmatrix} -5.2 & 0 & 0 \\ -4.5 & -10 & 0 \\ -1 & -2 & -12 \end{bmatrix} \times \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$
$$+ \begin{bmatrix} 0.1 & 0.35 & 0.2 \\ 0 & -0.25 & -0.2 \\ 0.2 & -0.3 & 0.5 \end{bmatrix} \times \begin{bmatrix} x_1(t-(0.3+0.2sint)) \\ x_2(t-(0.3+0.2sint)) \\ x_3(t-(0.3+0.2sint)) \end{bmatrix}$$
$$+ \begin{bmatrix} -0.2 & -0.4 & -0.3 \\ 0 & 0.4 & 0.2 \\ 0 & 0.3 & 0.2 \end{bmatrix} \times \begin{bmatrix} x_1(t-(0.1+0.3sin^2t)) \\ x_2(t-(0.1+0.3sin^2t)) \\ x_3(t-(0.1+0.3sin^2t)) \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & -0.5 & -0.4 \\ -0.6 & -0.4 & -0.2 \\ -1 & -0.5 & 0 \end{bmatrix} \times \begin{bmatrix} \int_{t-\frac{2+sint}{20}}^{t} x_1(s)ds \\ \int_{t-\frac{2+sint}{20}}^{t} x_2(s)ds \\ \int_{t-\frac{2+sint}{20}}^{t} x_3(s)ds \end{bmatrix}, \tag{3.2}$$

where

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} A = \begin{bmatrix} -5.2 & 0 & 0 \\ -4.5 & -10 & 0 \\ -1 & -2 & -12 \end{bmatrix} A_d = \begin{bmatrix} 0.1 & 0.35 & 0.2 \\ 0 & -0.25 & -0.2 \\ 0.2 & -0.3 & 0.5 \end{bmatrix},$$
$$A_h = \begin{bmatrix} -0.2 & -0.4 & -0.3 \\ 0 & 0.4 & 0.2 \\ 0 & 0.3 & 0.2 \end{bmatrix}, B = \begin{bmatrix} 0 & -0.5 & -0.4 \\ -0.6 & -0.4 & -0.2 \\ -1 & -0.5 & 0 \end{bmatrix},$$

and

$$\begin{aligned} d_1 &= 0.1 \le d(t) = 0.3 + 0.2sint \le 0.5 = d_2, \dot{d}(t) = 0.2cost \le 0.2 = \mu_1, \\ h_1 &= 0.1 \le h(t) = 0.1 + 0.3sin^2t \le 0.4 = h_2, \dot{h}(t) = 0.3sin2t \le 0.3 = \mu_2, \\ \tau_1 &= 0.05 \le \tau(t) = \frac{2+sint}{20} \le 0.15 = \tau_2, \alpha = 0.1, \end{aligned}$$

It is clear that system (3.2) is regular and IF. The LMIs (2.1)–(2.3) are feasible with

$$P = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}, Q_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, Q_2 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.4 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}, T = \begin{bmatrix} 5.6 & 0 & 0 \\ 0 & 4.4 & 0 \\ 0 & 0 & 0.6 \end{bmatrix}, W_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$W_2 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}, W_3 = \begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}, Z_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$Z_2 = \begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, Z_3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, Z_4 = \begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 0.7 & 0 \\ 0 & 0 & 0.5 \end{bmatrix},$$

and

$$N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 0.8 \end{bmatrix}.$$

Therefore, by Theorem 2.1, system (3.2) is exponentially admissible.

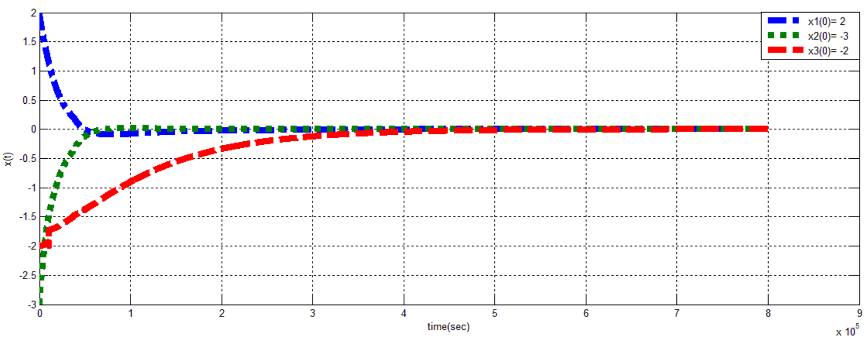

Figure 2. Trajectories of the solution of $x(t)$ of system (3.2) when $d(t) = 0.3 + 0.2sint$.

4 Conclusion

In this work, we consider a class of singular system with multiply variable delays. By using a new LKF, LMI and convex polyhedron method, we investigate exponential admissibility for the considered singular system. Finally, two numerical examples are given with their simulations to demonstrate the applicability of the proposed results. We believe that our results include and generalize some recent results in the literature.

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Received: 2022-06-30
Accepted: 2022-31-10