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A new class of ordinary generating functions of binary products of Mersenne Lucas numbers with several numbers

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Abstract In this paper, we provide some new results by using the symmetrizing endomorphism operator denoted by $\delta^{2-l}_{b_1 b_2}$, for which we can construct some new generating functions for the products of Mersenne Lucas numbers at positive and negative indices with several other numbers, such as Gaussian (p, q)-Jacobsthal numbers, Gaussian (p, q)-Jacobsthal Lucas numbers, etc.

1 Introduction

The first concept of Mersenne Lucas numbers is introduced by Saba et al. in [8]. They gave some results, such as generating function and Binet's formula of these numbers. Also, they presented some identities, including Catalan's identity, Cassini's identity and d'Ocagne's identity of them. Note that the Mersenne Lucas numbers are given either by the recurrence relation, for $n \geq 2$,

$$m_0 = 2,\ m_1 = 3 \text{ and } m_n = 3m_{n-1} - 2m_{n-2},$$

or by the explicit formula (see [9]):

$$m_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} 3^{n-2k} 2^k.$$

They are also given by the complete homogeneous symmetric function (see [8]):

$$m_n = 2h_n(b_1, [-b_2]) - 3h_{n-1}(b_1, [-b_2]), \text{ with } b_1 = 2 \text{ and } b_2 = 1. \tag{1.1}$$

In Binet's formula, they are also given by (see [8]):

$$m_n = 2^n + 1,$$

replacing (n) by $(-n)$ in the Binet's formula we get the negative extension of Mersenne Lucas numbers as follows:

$$m_{-n} = \frac{m_n}{2^n}. \tag{1.2}$$

In [7], Saba and Boussayoud defined and studied Gaussian (p, q)-Jacobsthal $\{GJ_{p,q,n}\}_{n\in\mathbb{N}}$ and Gaussian (p, q)-Jacobsthal Lucas $\{Gj_{p,q,n}\}_{n\in\mathbb{N}}$ numbers and by using these notions they discussed some intersting results. The recurrence relations of these numbers are respectively given by:

$$\begin{cases} GJ_{p,q,n} = pGJ_{p,q,n-1} + 2qGJ_{p,q,n-2},\ \forall n \geq 2 \\ GJ_{p,q,0} = \frac{i}{2}, GJ_{p,q,1} = 1 \end{cases},$$

$$\begin{cases} Gj_{p,q,n} = pGj_{p,q,n-1} + 2qGj_{p,q,n-2},\ \forall n \geq 2 \\ Gj_{p,q,0} = 2 - \frac{ip}{2}, Gj_{p,q,1} = p + 2iq \end{cases}.$$

Simillarly, Karaaslan and Yagmur in [6] introduced the Gaussian (p, q)-Pell $\{GP_{p,q,n}\}_{n\in\mathbb{N}}$ and Gaussian (p, q)-Pell-Lucas $\{GQ_{p,q,n}\}_{n\in\mathbb{N}}$ numbers under the following recurrence relations:

$$GP_{p,q,n} = 2pGP_{p,q,n-1} + qGP_{p,q,n-2}, \text{ for } n \geq 2,$$

with initial values $GP_{p,q,0} = i$ and $GP_{p,q,1} = 1$ and

$$GQ_{p,q,n} = 2pGQ_{p,q,n-1} + qGQ_{p,q,n-2}, \text{ for } n \geq 2,$$

with initial values $GQ_{p,q,0} = 2 - 2ip$ and $GQ_{p,q,1} = 2p + 2iq$.

On the other hand, The Gaussian (p, q)-Fibonacci numbers denoted by $\{GF_{p,q,n}\}_{n\in\mathbb{N}}$ are defined recursively by the relation:

$$\begin{cases} GF_{p,q,0} = i, \ GF_{p,q,1} = 1 \\ GF_{p,q,n} = pGF_{p,q,n-1} + qGF_{p,q,n-2} \ (n \geq 2) \end{cases}.$$

Moreover, the Gaussian (p, q)-Lucas numbers denoted by $\{GL_{p,q,n}\}_{n\in\mathbb{N}}$, are defined as:

$$\begin{cases} GL_{p,q,0} = 2 - ip, \ GL_{p,q,1} = p + 2iq \\ GL_{p,q,n} = pGL_{p,q,n-1} + qGL_{p,q,n-2} \ (n \geq 2) \end{cases}.$$

The $GJ_n = GJ_{1,1,n}$, $Gj_n = Gj_{1,1,n}$, $GP_n = GP_{1,1,n}$, $GQ_n = GQ_{1,1,n}$, $GF_n = GF_{1,1,n}$ and $GL_n = GL_{1,1,n}$, for $n \geq 0$, are called the Gaussian Jacobsthal numbers, Gaussian Jacobsthal Lucas numbers, Gaussian Pell numbers, Gaussian Pell Lucas numbers, Gaussian Fibonacci numbers and Gaussian Lucas numbers respectively, so that they are given and studied in [3, 4, 11, 16] by:

$$GJ_0 = \frac{i}{2}, \ GJ_1 = 1 \text{ and } GJ_n = GJ_{n-1} + 2GJ_{n-2},$$

$$Gj_0 = 2 - \frac{i}{2}, \ Gj_1 = 1 + 2i \text{ and } Gj_n = Gj_{n-1} + 2Gj_{n-2},$$

$$GP_0 = i, \ GP_1 = 1 \text{ and } GP_n = 2GP_{n-1} + GP_{n-2},$$

$$GQ_0 = 2 - 2i, \ GQ_1 = 2 + 2i \text{ and } GQ_n = 2GQ_{n-1} + GQ_{n-2},$$

$$GF_0 = i, \ GF_1 = 1 \text{ and } GF_n = GF_{n-1} + GF_{n-2},$$

$$GL_0 = 2 - i, \ GL_1 = 1 + 2i \text{ and } GL_n = GL_{n-1} + GL_{n-2}.$$

2 Some preliminary properties on the symmetric functions

Here we are going to recall some elementary facts on the symmetric functions and to state our main results on this paper. For more details on symmetric functions, we let the reader to [1, 2, 5, 13, 14].

Definition 2.1. [5] Let k and n be two positive integers and $\{a_1, a_2, ..., a_n\}$ an alphabet. Then, the k^{th} elementary symmetric function $e_k (a_1, a_2, ..., a_n)$ is defined by:

$$e_k^{(n)} = e_k (a_1, a_2, ..., a_n) = \sum_{i_1+i_2+...+i_n=k} a_1^{i_1} a_2^{i_2} ... a_n^{i_n} \qquad (0 \leq k \leq n),$$

with $i_1, i_2, ..., i_n = 0$ or 1.

Remark 2.2. Set $e_0 (a_1, a_2, ..., a_n) = 1$, by usual convention. For $k < 0$ or $k > n$, we set $e_k (a_1, a_2, ..., a_n) = 0$.

Example 2.3. The following lists the n elementary symmetric function for the first three positive values of n :
For $n = 1$

$$e_1 (a_1) = a_1.$$

For $n=2$

$$\begin{aligned} e_1(a_1,a_2) &= a_1+a_2, \\ e_2(a_1,a_2) &= a_1a_2. \end{aligned}$$

For $n=3$

$$e_1(a_1,a_2,a_3)=a_1+a_2+a_3,$$
$$e_2(a_1,a_2,a_3)=a_1a_2+a_1a_3+a_2a_3,$$
$$e_3(a_1,a_2,a_3)=a_1a_2a_3.$$

Proposition 2.4. *[15] Given an alphabet $A=\{a_1,a_2,...,a_n\}$, the generating function for the elementary symmetric function given as:*

$$\sum_{k=0}^{\infty} e_k(a_1,a_2,...,a_n)z^k = \prod_{a\in A}(1+az).$$

Definition 2.5. [5] Let k and n be two positive integers and $\{a_1,a_2,...,a_n\}$ an alphabet. Then, the k^{th} complete homogeneous symmetric function $h_k(a_1,a_2,...,a_n)$ is defined by:

$$h_k^{(n)} = h_k(a_1,a_2,...,a_n) = \sum_{i_1+i_2+...+i_n=k} a_1^{i_1}a_2^{i_2}...a_n^{i_n} \qquad (k\geq 0), \tag{2.1}$$

with $i_1,i_2,...,i_n\geq 0$.

Remark 2.6. Set $h_0(a_1,a_2,...,a_n)=1$, by usual convention. For $k<0$, we set $h_k(a_1,a_2,...,a_n)=0$.
If $n=2$, the k^{th} complete homogeneous symmetric function (2.1) gives us:

$$h_k^{(2)} = h_k(a_1,a_2) = \frac{a_1^{k+1}-a_2^{k+1}}{a_1-a_2}, \text{ for all } k\in\mathbb{N}_0,$$

with

$$\begin{aligned} h_0(a_1,a_2) &= 1, \\ h_1(a_1,a_2) &= a_1+a_2, \\ h_2(a_1,a_2) &= a_1^2+a_1a_2+a_2^2, \\ &\vdots \end{aligned}$$

Proposition 2.7. *[15] Given an alphabet $A=\{a_1,a_2,...,a_n\}$, the generating function for the complete homogeneous symmetric function given as:*

$$\sum_{k=0}^{\infty} h_k(a_1,a_2,...,a_n)z^k = \frac{1}{\prod_{a\in A}(1-az)}.$$

There is a fundamental relation between the elementary symmetric functions and the complete homogeneous ones:

$$\sum_{j=0}^{k}(-1)^j e_j(a_1,a_2,...,a_n)h_{k-j}(a_1,a_2,...,a_n)=0,$$

which is valid for all $k>0$.

Definition 2.8. [1] Given an alphabet $A=\{a_1,a_2\}$, the symmetrizing operator $\delta_{a_1a_2}^k$ is defined by:

$$\delta_{a_1a_2}^k f(a_1) = \frac{a_1^k f(a_1)-a_2^k f(a_2)}{a_1-a_2}, \text{ for all } k\in\mathbb{N}_0.$$

The following Theorem is one of key tools of the proof of our main result. It has been proved in [12] by applying the operator $\delta_{b_1 b_2}^{2-l}$ to the series

$$\sum_{n=0}^{\infty} h_n\left(a_1, a_2, ..., a_k\right) b_1^n z^n = \frac{1}{\sum\limits_{n=0}^{\infty} \left(-1\right)^n e_n\left(a_1, a_2, ..., a_k\right) b_1^n z^n}.$$

Theorem 2.9. *Let A and B be two alphabets, respectively, $\{a_1, a_2, ..., a_k\}$ and $\{b_1, b_2\}$, then we have:*

$$\sum_{n=0}^{\infty} h_n\left(a_1, a_2, ..., a_k\right) h_{n+1-l}\left(b_1, b_2\right) z^n =$$

$$\frac{h_{1-l}(b_1, b_2) - e_1\left(a_1, a_2, ..., a_k\right) b_1 b_2 h_{-l}(b_1, b_2) z}{\left(\sum\limits_{n=0}^{\infty} \left(-1\right)^n e_n\left(a_1, a_2, ..., a_k\right) b_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} \left(-1\right)^n e_n\left(a_1, a_2, ..., a_k\right) b_2^n z^n\right)}$$

$$-\frac{b_1^{2-l} b_2^{2-l} z^{3-l} \sum\limits_{n=0}^{\infty} \left(-1\right)^{n-l+3} e_{n-l+3}\left(a_1, a_2, ..., a_k\right) h_n(b_1, b_2) z^n}{\left(\sum\limits_{n=0}^{\infty} \left(-1\right)^n e_n\left(a_1, a_2, ..., a_k\right) b_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} \left(-1\right)^n e_n\left(a_1, a_2, ..., a_k\right) b_2^n z^n\right)},$$

for all $n, k \in \mathbb{N}_0$ and $l \in \{0, 1, 2\}$.

3 Symmetric and generating functions of binary products of Gaussian (p, q)-numbers with Mersenne Lucas numbers

In this part, we are now in a position to provide new Theorems. Also, we derive the new generating functions for the products of Gaussian (p, q)-Fibonacci numbers, Gaussian (p, q)-Lucas numbers, Gaussian (p, q)-Pell numbers, Gaussian (p, q)-Pell Lucas numbers, Gaussian (p, q)-Jacobsthal numbers and Gaussian (p, q)-Jacobsthal Lucas numbers with Mersenne Lucas numbers at positive and negative indices.

We consider $A = \{a_1, -a_2\}$, $B = \{b_1, -b_1\}$ and $l \in \{0, 1, 2\}$ in the Theorem 2.9, we deduce the following Lemmas.

Lemma 3.1. *Given two alphabets $A = \{a_1, -a_2\}$ and $B = \{b_1, -b_2\}$, then we have:*

$$\sum_{n=0}^{\infty} h_n(a_1, [-a_2]) h_{n+1}(b_1, [-b_2]) z^n = \frac{b_1 - b_2 + b_1 b_2 (a_1 - a_2) z}{(1 - a_1 b_1 z)(1 + a_2 b_1 z)(1 + a_1 b_2 z)(1 - a_2 b_2 z)}.$$

The previous Lemma gives the following result as particular example:

$$\sum_{n=0}^{\infty} h_{n-1}(a_1, [-a_2]) h_n(b_1, [-b_2]) z^n = \frac{(b_1 - b_2) z + b_1 b_2 (a_1 - a_2) z^2}{(1 - a_1 b_1 z)(1 + a_2 b_1 z)(1 + a_1 b_2 z)(1 - a_2 b_2 z)}. \quad (3.1)$$

Lemma 3.2. *Given two alphabets $A = \{a_1, -a_2\}$ and $B = \{b_1, -b_2\}$, then we have:*

$$\sum_{n=0}^{\infty} h_n(a_1, [-a_2]) h_n(b_1, [-b_2]) z^n = \frac{1 - a_1 a_2 b_1 b_2 z^2}{(1 - a_1 b_1 z)(1 + a_2 b_1 z)(1 + a_1 b_2 z)(1 - a_2 b_2 z)}. \quad (3.2)$$

Note that, based on the relationship (3.2), we get:

$$\sum_{n=0}^{\infty} h_{n-1}(a_1, [-a_2]) h_{n-1}(b_1, [-b_2]) z^n = \frac{z - a_1 a_2 b_1 b_2 z^3}{(1 - a_1 b_1 z)(1 + a_2 b_1 z)(1 + a_1 b_2 z)(1 - a_2 b_2 z)}. \quad (3.3)$$

Lemma 3.3. *Given two alphabets $A = \{a_1, -a_2\}$ and $B = \{b_1, -b_2\}$, then we have:*

$$\sum_{n=0}^{\infty} h_n(a_1, [-a_2]) h_{n-1}(b_1, [-b_2]) z^n = \frac{(a_1 - a_2) z + a_1 a_2 (b_1 - b_2) z^2}{(1 - a_1 b_1 z)(1 + a_2 b_1 z)(1 + a_1 b_2 z)(1 - a_2 b_2 z)}. \quad (3.4)$$

This part consists of three cases.

Case 1. For $\left\{ \begin{array}{c} a_1 - a_2 = p \\ a_1 a_2 = q \end{array} \right.$ and $\left\{ \begin{array}{c} b_1 - b_2 = 3 \\ b_1 b_2 = -2 \end{array} \right.$, in the relationships (3.1), (3.2), (3.3) and (3.4) we get:

$$\sum_{n=0}^{\infty} h_{n-1}\left(a_1, [-a_2]\right) h_n(b_1, [-b_2]) z^n = \frac{3z - 2pz^2}{D_1}, \tag{3.5}$$

$$\sum_{n=0}^{\infty} h_n\left(a_1, [-a_2]\right) h_n(b_1, [-b_2]) z^n = \frac{1 + 2qz^2}{D_1}, \tag{3.6}$$

$$\sum_{n=0}^{\infty} h_{n-1}\left(a_1, [-a_2]\right) h_{n-1}(b_1, [-b_2]) z^n = \frac{z + 2qz^3}{D_1}, \tag{3.7}$$

$$\sum_{n=0}^{\infty} h_n\left(a_1, [-a_2]\right) h_{n-1}(b_1, [-b_2]) z^n = \frac{pz + 3qz^2}{D_1}, \tag{3.8}$$

with

$$D_1 = 1 - 3pz - \left(5q - 2p^2\right) z^2 + 6pqz^3 + 4q^2 z^4,$$

and we deduce the following theorems.

Theorem 3.4. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian (p, q)-Fibonacci numbers with Mersenne Lucas numbers is given by:*

$$\sum_{n=0}^{\infty} GF_{p,q,n} m_n z^n = \frac{2i + 3(1 - 2ip) z + \left(i\left(4p^2 - 5q\right) - 4p\right) z^2 + 6q(ip - 1) z^3}{1 - 3pz - (5q - 2p^2) z^2 + 6pqz^3 + 4q^2 z^4}. \tag{3.9}$$

Proof. Recall that, we have

$$GF_{p,q,n} = ih_n\left(a_1, [-a_2]\right) + (1 - ip) h_{n-1}\left(a_1, [-a_2]\right),$$

(see [10]) and according to the relationship (1.1). We see that:

$$\sum_{n=0}^{\infty} GF_{p,q,n} m_n z^n = \sum_{n=0}^{\infty} \left(\begin{array}{c} \left(ih_n\left(a_1, [-a_2]\right) + (1 - ip) h_{n-1}\left(a_1, [-a_2]\right)\right) \\ \times \left(2h_n(b_1, [-b_2]) - 3h_{n-1}(b_1, [-b_2])\right) \end{array} \right) z^n$$

$$= 2i \sum_{n=0}^{\infty} h_n\left(a_1, [-a_2]\right) h_n(b_1, [-b_2]) z^n$$

$$-3i \sum_{n=0}^{\infty} h_n\left(a_1, [-a_2]\right) h_{n-1}(b_1, [-b_2]) z^n$$

$$+2(1 - ip) \sum_{n=0}^{\infty} h_{n-1}\left(a_1, [-a_2]\right) h_n(b_1, [-b_2]) z^n$$

$$-3(1 - ip) \sum_{n=0}^{\infty} h_{n-1}\left(a_1, [-a_2]\right) h_{n-1}(b_1, [-b_2]) z^n,$$

by using the relationships (3.5), (3.6), (3.7) and (3.8), we obtain:

$$\sum_{n=0}^{\infty} GF_{p,q,n} m_n z^n = \frac{2i\left(1 + 2qz^2\right)}{D_1} - \frac{3i\left(pz + 3qz^2\right)}{D_1}$$

$$+\frac{2(1 - ip)\left(3z - 2pz^2\right)}{D_1} - \frac{3(1 - ip)\left(z + 2qz^3\right)}{D_1}$$

$$= \frac{2i + 3(1 - 2ip) z + \left(i\left(4p^2 - 5q\right) - 4p\right) z^2 + 6q(ip - 1) z^3}{1 - 3pz - (5q - 2p^2) z^2 + 6pqz^3 + 4q^2 z^4}.$$

Which completes the proof. □

Theorem 3.5. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian (p,q)-Fibonacci numbers with Mersenne Lucas numbers at negative indice is given by:*

$$\sum_{n=0}^{\infty} GF_{p,q,n} m_{-n} z^n = \frac{8i + 6(1-2ip)z + \left(i\left(4p^2-5q\right)-4p\right)z^2 + 3q(ip-1)z^3}{4 - 6pz - (5q-2p^2)z^2 + 3pqz^3 + q^2z^4}. \tag{3.10}$$

Proof. We use the change of variable $z = \frac{z}{2}$ in (3.9) and according to relationship (1.2), we get:

$$\sum_{n=0}^{\infty} GF_{p,q,n} m_{-n} z^n = \sum_{n=0}^{\infty} GF_{p,q,n} m_n \left(\frac{z}{2}\right)^n$$

$$= \frac{2i + 3(1-2ip)\left(\frac{z}{2}\right) + \left(i\left(4p^2-5q\right)-4p\right)\left(\frac{z}{2}\right)^2 + 6q(ip-1)\left(\frac{z}{2}\right)^3}{1 - 3p\left(\frac{z}{2}\right) - (5q-2p^2)\left(\frac{z}{2}\right)^2 + 6pq\left(\frac{z}{2}\right)^3 + 4q^2\left(\frac{z}{2}\right)^4}$$

$$= \frac{8i + 6(1-2ip)z + \left(i\left(4p^2-5q\right)-4p\right)z^2 + 3q(ip-1)z^3}{4 - 6pz - (5q-2p^2)z^2 + 3pqz^3 + q^2z^4}.$$

Thus, this completes the proof. □

Theorem 3.6. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian (p,q)-Lucas numbers with Mersenne Lucas numbers is given by:*

$$\sum_{n=0}^{\infty} GL_{p,q,n} m_n z^n = \tag{3.11}$$

$$\frac{2(2-ip) + 3\left(2i\left(p^2+q\right)-3p\right)z + \left(2\left(2p^2-5q\right) - ip\left(4p^2+3q\right)\right)z^2 + 6q\left(p - i\left(p^2+2q\right)\right)z^3}{1 - 3pz - (5q-2p^2)z^2 + 6pqz^3 + 4q^2z^4}.$$

Proof. By referred to [10], we have:

$$GL_{p,q,n} = (2-ip)\,h_n(a_1,[-a_2]) + \left(i\left(p^2+2q\right)-p\right)h_{n-1}(a_1,[-a_2]),$$

and according to the relationship (1.1) we can easily see that:

$$\sum_{n=0}^{\infty} GL_{p,q,n} m_n z^n = \sum_{n=0}^{\infty} \left(\begin{array}{c} \left((2-ip)\,h_n(a_1,[-a_2]) + \left(i\left(p^2+2q\right)-p\right)h_{n-1}(a_1,[-a_2])\right) \\ \times\left(2h_n(b_1,[-b_2]) - 3h_{n-1}(b_1,[-b_2])\right) \end{array} \right) z^n$$

$$= 2(2-ip)\sum_{n=0}^{\infty} h_n(a_1,[-a_2])\,h_n(b_1,[-b_2])z^n$$

$$-3(2-ip)\sum_{n=0}^{\infty} h_n(a_1,[-a_2])\,h_{n-1}(b_1,[-b_2])z^n$$

$$+2\left(i\left(p^2+2q\right)-p\right)\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])\,h_n(b_1,[-b_2])z^n$$

$$-3\left(i\left(p^2+2q\right)-p\right)\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])\,h_{n-1}(b_1,[-b_2])z^n.$$

Then, according to the relationships (3.5), (3.6), (3.7) and (3.8), we obtain:

$$\sum_{n=0}^{\infty} GL_{p,q,n} m_n z^n = \frac{2(2-ip)\left(1+2qz^2\right)}{D_1} - \frac{3(2-ip)\left(pz+3qz^2\right)}{D_1}$$

$$+\frac{2\left(i\left(p^2+2q\right)-p\right)\left(3z-2pz^2\right)}{D_1} - \frac{3\left(i\left(p^2+2q\right)-p\right)\left(z+2qz^3\right)}{D_1}$$

$$= \frac{2(2-ip) + 3\left(2i\left(p^2+q\right)-3p\right)z + \left(2\left(2p^2-5q\right) - ip\left(4p^2+3q\right)\right)z^2 + 6q\left(p - i\left(p^2+2q\right)\right)z^3}{1 - 3pz - (5q-2p^2)z^2 + 6pqz^3 + 4q^2z^4}.$$

Hence, we obtain the desired result. □

Theorem 3.7. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian (p,q)-Lucas numbers with Mersenne Lucas numbers at negative indice is given by:*

$$\sum_{n=0}^{\infty} GL_{p,q,n} m_{-n} z^n = \tag{3.12}$$

$$\frac{8(2-ip)+6\left(2i\left(p^2+q\right)-3p\right)z+\left(2\left(2p^2-5q\right)-ip\left(4p^2+3q\right)\right)z^2+3q\left(p-i\left(p^2+2q\right)\right)z^3}{4-6pz-\left(5q-2p^2\right)z^2+3pqz^3+q^2z^4}.$$

Proof. We use the change of variable $z = \frac{z}{2}$ in (3.11) and according to relationship (1.2), we get:

$$\sum_{n=0}^{\infty} GL_{p,q,n} m_{-n} z^n = \sum_{n=0}^{\infty} GL_{p,q,n} m_n \left(\frac{z}{2}\right)^n$$

$$= \frac{2(2-ip)+3\left(2i\left(p^2+q\right)-3p\right)\left(\frac{z}{2}\right)+\left(2\left(2p^2-5q\right)-ip\left(4p^2+3q\right)\right)\left(\frac{z}{2}\right)^2+6q\left(p-i\left(p^2+2q\right)\right)\left(\frac{z}{2}\right)^3}{1-3p\left(\frac{z}{2}\right)-\left(5q-2p^2\right)\left(\frac{z}{2}\right)^2+6pq\left(\frac{z}{2}\right)^3+4q^2\left(\frac{z}{2}\right)^4}$$

$$= \frac{8(2-ip)+6\left(2i\left(p^2+q\right)-3p\right)z+\left(2\left(2p^2-5q\right)-ip\left(4p^2+3q\right)\right)z^2+3q\left(p-i\left(p^2+2q\right)\right)z^3}{4-6pz-\left(5q-2p^2\right)z^2+3pqz^3+q^2z^4}.$$

As required. □

- By putting $p = q = 1$ in the relationships (3.9), (3.10), (3.11) and (3.12), we obtain the following table:

Coefficient of z^n	**Generating function**
$GF_n m_n$	$\frac{2i+3(1-2i)z-(i+4)z^2+6(i-1)z^3}{1-3z-3z^2+6z^3+4z^4}$
$GF_n m_{-n}$	$\frac{8i+6(1-2i)z-(i+4)z^2+3(i-1)z^3}{4-6z-3z^2+3z^3+z^4}$
$GL_n m_n$	$\frac{2(2-i)+3(4i-3)z-(6+7i)z^2+6(1-3i)z^3}{1-3z-3z^2+6z^3+4z^4}$
$GL_n m_{-n}$	$\frac{8(2-i)+6(4i-3)z-(6+7i)z^2+3(1-3i)z^3}{4-6z-3z^2+3z^3+z^4}$

Table 1. New generating functions of the products of Gaussian Fibonacci and Gaussian Lucas numbers with Mersenne Lucas numbers at positive and negative indices.

Case 2. For $\begin{cases} a_1 - a_2 = 2p \\ a_1 a_2 = q \end{cases}$ and $\begin{cases} b_1 - b_2 = 3 \\ b_1 b_2 = -2 \end{cases}$, in the relationships (3.1), (3.2), (3.3) and (3.4) we get:

$$\sum_{n=0}^{\infty} h_{n-1}\left(a_1, [-a_2]\right) h_n(b_1, [-b_2]) z^n = \frac{3z-4pz^2}{D_2}, \tag{3.13}$$

$$\sum_{n=0}^{\infty} h_n\left(a_1, [-a_2]\right) h_n(b_1, [-b_2]) z^n = \frac{1+2qz^2}{D_2}, \tag{3.14}$$

$$\sum_{n=0}^{\infty} h_{n-1}\left(a_1, [-a_2]\right) h_{n-1}(b_1, [-b_2]) z^n = \frac{z+2qz^3}{D_2}, \tag{3.15}$$

$$\sum_{n=0}^{\infty} h_n\left(a_1, [-a_2]\right) h_{n-1}(b_1, [-b_2]) z^n = \frac{2pz+3qz^2}{D_2}, \tag{3.16}$$

with

$$D_2 = 1-6pz-\left(5q-8p^2\right)z^2+12pqz^3+4q^2z^4,$$

and we deduce the following theorems.

Theorem 3.8. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian (p,q)-Pell numbers with Mersenne Lucas numbers is given by:*

$$\sum_{n=0}^{\infty} GP_{p,q,n} m_n z^n = \frac{2i + 3(1-4ip)z + \left(i\left(16p^2 - 5q\right) - 8p\right)z^2 + 6q(2ip-1)z^3}{1 - 6pz - (5q - 8p^2)z^2 + 12pqz^3 + 4q^2z^4}. \quad (3.17)$$

Proof. Recall that, we have:

$$GP_{p,q,n} = ih_n(a_1, [-a_2]) + (1-2ip)h_{n-1}(a_1, [-a_2]),$$

(see [10]) and according to the relationship (1.1). We see that:

$$\sum_{n=0}^{\infty} GP_{p,q,n} m_n z^n = \sum_{n=0}^{\infty} \left(\begin{array}{c} (ih_n(a_1, [-a_2]) + (1-2ip)h_{n-1}(a_1, [-a_2])) \\ \times (2h_n(b_1, [-b_2]) - 3h_{n-1}(b_1, [-b_2])) \end{array} \right) z^n$$

$$= 2i \sum_{n=0}^{\infty} h_n(a_1, [-a_2]) h_n(b_1, [-b_2]) z^n$$

$$-3i \sum_{n=0}^{\infty} h_n(a_1, [-a_2]) h_{n-1}(b_1, [-b_2]) z^n$$

$$+2(1-2ip) \sum_{n=0}^{\infty} h_{n-1}(a_1, [-a_2]) h_n(b_1, [-b_2]) z^n$$

$$-3(1-2ip) \sum_{n=0}^{\infty} h_{n-1}(a_1, [-a_2]) h_{n-1}(b_1, [-b_2]) z^n,$$

by using the relationships (3.13), (3.14), (3.15) and (3.16), we obtain:

$$\sum_{n=0}^{\infty} GP_{p,q,n} m_n z^n = \frac{2i\left(1 + 2qz^2\right)}{D_2} - \frac{3i\left(2pz + 3qz^2\right)}{D_2}$$

$$+ \frac{2(1-2ip)\left(3z - 4pz^2\right)}{D_2} - \frac{3(1-2ip)\left(z + 2qz^3\right)}{D_2}$$

$$= \frac{2i + 3(1-4ip)z + \left(i\left(16p^2 - 5q\right) - 8p\right)z^2 + 6q(2ip-1)z^3}{1 - 6pz - (5q - 8p^2)z^2 + 12pqz^3 + 4q^2z^4}.$$

Which completes the proof. □

Theorem 3.9. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian (p,q)-Pell numbers with Mersenne Lucas numbers at negative indice is given by:*

$$\sum_{n=0}^{\infty} GP_{p,q,n} m_{-n} z^n = \frac{8i + 6(1-4ip)z + \left(i\left(16p^2 - 5q\right) - 8p\right)z^2 + 3q(2ip-1)z^3}{4 - 12pz - (5q - 8p^2)z^2 + 6pqz^3 + q^2z^4}. \quad (3.18)$$

Proof. We use the change of variable $z = \frac{z}{2}$ in (3.17) and according to relationship (1.2), we get:

$$\sum_{n=0}^{\infty} GP_{p,q,n} m_{-n} z^n = \sum_{n=0}^{\infty} GP_{p,q,n} m_n \left(\frac{z}{2}\right)^n$$

$$= \frac{2i + 3(1-4ip)\left(\frac{z}{2}\right) + \left(i\left(16p^2 - 5q\right) - 8p\right)\left(\frac{z}{2}\right)^2 + 6q(2ip-1)\left(\frac{z}{2}\right)^3}{1 - 6p\left(\frac{z}{2}\right) - (5q - 8p^2)\left(\frac{z}{2}\right)^2 + 12pq\left(\frac{z}{2}\right)^3 + 4q^2\left(\frac{z}{2}\right)^4}$$

$$= \frac{8i + 6(1-4ip)z + \left(i\left(16p^2 - 5q\right) - 8p\right)z^2 + 3q(2ip-1)z^3}{4 - 12pz - (5q - 8p^2)z^2 + 6pqz^3 + q^2z^4}.$$

Thus, this completes the proof. □

Theorem 3.10. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian (p, q)-Pell Lucas numbers with Mersenne Lucas numbers is given by:*

$$\sum_{n=0}^{\infty} GQ_{p,q,n} m_n z^n = \tag{3.19}$$

$$\frac{4(1-ip)+6\left(i\left(4p^2+q\right)-3p\right)z+2\left(8p^2-5q-ip\left(16p^2+3q\right)\right)z^2+12q\left(p-i\left(2p^2+q\right)\right)z^3}{1-6pz-\left(5q-8p^2\right)z^2+12pqz^3+4q^2z^4}.$$

Proof. By referred to [10], we have

$$GQ_{p,q,n} = (2-2ip)\,h_n\left(a_1,[-a_2]\right)+\left(i\left(4p^2+2q\right)-2p\right)h_{n-1}\left(a_1,[-a_2]\right),$$

and according to the relationship (1.1) we can easily see that:

$$\sum_{n=0}^{\infty} GQ_{p,q,n} m_n z^n = \sum_{n=0}^{\infty}\left(\begin{array}{c}\left((2-2ip)\,h_n\left(a_1,[-a_2]\right)+\left(i\left(4p^2+2q\right)-2p\right)h_{n-1}\left(a_1,[-a_2]\right)\right)\\ \times\left(2h_n(b_1,[-b_2])-3h_{n-1}(b_1,[-b_2])\right)\end{array}\right)z^n$$

$$= 2(2-2ip)\sum_{n=0}^{\infty} h_n\left(a_1,[-a_2]\right)h_n(b_1,[-b_2])z^n$$

$$-3(2-2ip)\sum_{n=0}^{\infty} h_n\left(a_1,[-a_2]\right)h_{n-1}(b_1,[-b_2])z^n$$

$$+2\left(i\left(4p^2+2q\right)-2p\right)\sum_{n=0}^{\infty} h_{n-1}\left(a_1,[-a_2]\right)h_n(b_1,[-b_2])z^n$$

$$-3\left(i\left(4p^2+2q\right)-2p\right)\sum_{n=0}^{\infty} h_{n-1}\left(a_1,[-a_2]\right)h_{n-1}(b_1,[-b_2])z^n.$$

Then, according to the relationships (3.13), (3.14), (3.15) and (3.16), we obtain:

$$\sum_{n=0}^{\infty} Q_{p,q,n} m_n z^n = \frac{2(2-2ip)\left(1+2qz^2\right)}{D_2}-\frac{3(2-2ip)\left(2pz+3qz^2\right)}{D_2}$$

$$+\frac{2\left(i\left(4p^2+2q\right)-2p\right)\left(3z-4pz^2\right)}{D_2}-\frac{3\left(i\left(4p^2+2q\right)-2p\right)\left(z+2qz^3\right)}{D_2}$$

$$=\frac{4(1-ip)+6\left(i\left(4p^2+q\right)-3p\right)z+2\left(8p^2-5q-ip\left(16p^2+3q\right)\right)z^2+12q\left(p-i\left(2p^2+q\right)\right)z^3}{1-6pz-\left(5q-8p^2\right)z^2+12pqz^3+4q^2z^4}.$$

Hence, we obtain the desired result. □

Theorem 3.11. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian (p, q)-Pell Lucas numbers with Mersenne Lucas numbers at negative indice is given by:*

$$\sum_{n=0}^{\infty} GQ_{p,q,n} m_{-n} z^n = \tag{3.20}$$

$$\frac{16(1-ip)+12\left(i\left(4p^2+q\right)-3p\right)z+2\left(8p^2-5q-ip\left(16p^2+3q\right)\right)z^2+6q\left(p-i\left(2p^2+q\right)\right)z^3}{4-12pz-\left(5q-8p^2\right)z^2+6pqz^3+q^2z^4}.$$

Proof. We use the change of variable $z = \frac{z}{2}$ in (3.19) and according to relationship (1.2), we get:

$$\sum_{n=0}^{\infty} GQ_{p,q,n} m_{-n} z^n = \sum_{n=0}^{\infty} GQ_{p,q,n} m_n \left(\frac{z}{2}\right)^n$$

$$= \frac{4(1-ip)+6\left(i\left(4p^2+q\right)-3p\right)\left(\frac{z}{2}\right)+2\left(8p^2-5q-ip\left(16p^2+3q\right)\right)\left(\frac{z}{2}\right)^2+12q\left(i\left(2p^2+q\right)-p\right)\left(\frac{z}{2}\right)^3}{1-6p\left(\frac{z}{2}\right)-\left(5q-8p^2\right)\left(\frac{z}{2}\right)^2+12pq\left(\frac{z}{2}\right)^3+4q^2\left(\frac{z}{2}\right)^4}$$

$$= \frac{16(1-ip)+12\left(i\left(4p^2+q\right)-3p\right)z+2\left(8p^2-5q-ip\left(16p^2+3q\right)\right)z^2+6q\left(p-i\left(2p^2+q\right)\right)z^3}{4-12pz-\left(5q-8p^2\right)z^2+6pqz^3+q^2z^4}.$$

As required. □

- By setting $p=q=1$ in the relationships (3.17), (3.18), (3.19) and (3.20), we obtain the following table:

Coefficient of z^n	Generating function
$GP_n m_n$	$\frac{2i+3(1-4i)z+(11i-8)z^2+6(2i-1)z^3}{1-6z+3z^2+12z^3+4z^4}$
$GP_n m_{-n}$	$\frac{8i+6(1-4i)z+(11i-8)z^2+3(2i-1)z^3}{4-12z+3z^2+6z^3+z^4}$
$GQ_n m_n$	$\frac{4(1-i)+6(5i-3)z+2(3-19i)z^2+12(1-3i)z^3}{1-6z+3z^2+12z^3+4z^4}$
$GQ_n m_{-n}$	$\frac{16(1-i)+12(5i-3)z+2(3-19i)z^2+6(1-3i)z^3}{4-12z+3z^2+6z^3+z^4}$

Table 2. New generating functions of the products of Gaussian Pell and Gaussian Pell Lucas numbers with Mersenne Lucas numbers at positive and negative indices.

Case 3. For $\begin{cases} a_1-a_2=p \\ a_1a_2=2q \end{cases}$ and $\begin{cases} b_1-b_2=3 \\ b_1b_2=-2 \end{cases}$, in the relationships (3.1), (3.2), (3.3) and (3.4) we get:

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])\, h_n(b_1,[-b_2])z^n = \frac{3z-2pz^2}{D_3}, \tag{3.21}$$

$$\sum_{n=0}^{\infty} h_n(a_1,[-a_2])\, h_n(b_1,[-b_2])z^n = \frac{1+4qz^2}{D_3}, \tag{3.22}$$

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])\, h_{n-1}(b_1,[-b_2])z^n = \frac{z+4qz^3}{D_3}, \tag{3.23}$$

$$\sum_{n=0}^{\infty} h_n(a_1,[-a_2])\, h_{n-1}(b_1,[-b_2])z^n = \frac{pz+6qz^2}{D_3}, \tag{3.24}$$

with

$$D_3 = 1-3pz-2\left(5q-p^2\right)z^2+12pqz^3+16q^2z^4,$$

and we deduce the following theorems.

Theorem 3.12. *For $n\in\mathbb{N}$, the new generating function of the product of Gaussian (p,q)-Jacobsthal numbers with Mersenne Lucas numbers is given by:*

$$\sum_{n=0}^{\infty} GJ_{p,q,n} m_n z^n = \frac{i+3(1-ip)z+\left(i\left(2p^2-5q\right)-4p\right)z^2+6q(ip-2)z^3}{1-3pz-2\left(5q-p^2\right)z^2+12pqz^3+16q^2z^4}. \tag{3.25}$$

Proof. Recall that, we have

$$GJ_{p,q,n} = \frac{i}{2}h_n(a_1,[-a_2]) + \left(1-\frac{ip}{2}\right)h_{n-1}(a_1,[-a_2]),$$

(see [7]) and according to the relationship (1.1). We see that:

$$\sum_{n=0}^{\infty} GJ_{p,q,n} m_n z^n = \sum_{n=0}^{\infty} \left(\begin{array}{c} \left(\frac{i}{2}h_n(a_1,[-a_2]) + \left(1-\frac{ip}{2}\right)h_{n-1}(a_1,[-a_2])\right) \\ \times\left(2h_n(b_1,[-b_2]) - 3h_{n-1}(b_1,[-b_2])\right) \end{array}\right) z^n$$

$$= i\sum_{n=0}^{\infty} h_n\left(a_1,[-a_2]\right) h_n(b_1,[-b_2])z^n$$

$$-\frac{3i}{2}\sum_{n=0}^{\infty} h_n\left(a_1,[-a_2]\right) h_{n-1}(b_1,[-b_2])z^n$$

$$+2\left(1-\frac{ip}{2}\right)\sum_{n=0}^{\infty} h_{n-1}\left(a_1,[-a_2]\right) h_n(b_1,[-b_2])z^n$$

$$-3\left(1-\frac{ip}{2}\right)\sum_{n=0}^{\infty} h_{n-1}\left(a_1,[-a_2]\right) h_{n-1}(b_1,[-b_2])z^n,$$

by using the relationships (3.21) , (3.22) , (3.23) and (3.24), we obtain:

$$\sum_{n=0}^{\infty} GJ_{p,q,n}m_n z^n = \frac{2i\left(1+4qz^2\right)}{2D_3} - \frac{3i\left(pz+6qz^2\right)}{2D_3}$$

$$+\frac{2\left(2-ip\right)\left(3z-2pz^2\right)}{2D_3} - \frac{3\left(2-ip\right)\left(z+4qz^3\right)}{2D_3}$$

$$= \frac{2i+6\left(1-ip\right)z+2\left(i\left(2p^2-5q\right)-4p\right)z^2+12q\left(ip-2\right)z^3}{2-6pz-4\left(5q-p^2\right)z^2+24pqz^3+32q^2z^4}$$

$$= \frac{i+3\left(1-ip\right)z+\left(i\left(2p^2-5q\right)-4p\right)z^2+6q\left(ip-2\right)z^3}{1-3pz-2\left(5q-p^2\right)z^2+12pqz^3+16q^2z^4}.$$

Which completes the proof. □

Theorem 3.13. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian (p,q)-Jacobsthal numbers with Mersenne Lucas numbers at negative indice is given by:*

$$\sum_{n=0}^{\infty} GJ_{p,q,n}m_{-n} z^n = \frac{4i+6\left(1-ip\right)z+\left(i\left(2p^2-5q\right)-4p\right)z^2+3q\left(ip-2\right)z^3}{4-6pz-2\left(5q-p^2\right)z^2+6pqz^3+4q^2z^4}. \tag{3.26}$$

Proof. We use the change of variable $z=\frac{z}{2}$ in Eq. (3.25) and according to relationship (1.2), we get:

$$\sum_{n=0}^{\infty} GJ_{p,q,n}m_{-n} z^n = \sum_{n=0}^{\infty} GJ_{p,q,n}m_n \left(\frac{z}{2}\right)^n$$

$$= \frac{i+3\left(1-ip\right)\left(\frac{z}{2}\right)+\left(i\left(2p^2-5q\right)-4p\right)\left(\frac{z}{2}\right)^2+6q\left(ip-2\right)\left(\frac{z}{2}\right)^3}{1-3p\left(\frac{z}{2}\right)-2\left(5q-p^2\right)\left(\frac{z}{2}\right)^2+12pq\left(\frac{z}{2}\right)^3+16q^2\left(\frac{z}{2}\right)^4}$$

$$= \frac{4i+6\left(1-ip\right)z+\left(i\left(2p^2-5q\right)-4p\right)z^2+3q\left(ip-2\right)z^3}{4-6pz-2\left(5q-p^2\right)z^2+6pqz^3+4q^2z^4}.$$

Thus, this completes the proof. □

Theorem 3.14. *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian (p,q)-Jacobsthal Lucas numbers with Mersenne Lucas numbers is given by:*

$$\sum_{n=0}^{\infty} Gj_{p,q,n}m_n z^n = \tag{3.27}$$

$$\frac{4-ip+3\left(i\left(p^2+2q\right)-3p\right)z+\left(4\left(p^2-5q\right)-ip\left(2p^2+3q\right)\right)z^2+6q\left(2p-i\left(p^2+4q\right)\right)z^3}{1-3pz-2\left(5q-p^2\right)z^2+12pqz^3+16q^2z^4}.$$

Proof. Recall that, we have

$$Gj_{p,q,n}=\left(2-\frac{ip}{2}\right)h_n\left(a_1,[-a_2]\right)+\left(i\left(\frac{p^2}{2}+2q\right)-p\right)h_{n-1}\left(a_1,[-a_2]\right),$$

(see [7]) and according to the relationship (1.1). We see that:

$$\sum_{n=0}^{\infty}Gj_{p,q,n}m_nz^n=\sum_{n=0}^{\infty}\left(\begin{array}{c}\left(\left(2-\frac{ip}{2}\right)h_n\left(a_1,[-a_2]\right)+\left(i\left(\frac{p^2}{2}+2q\right)-p\right)h_{n-1}\left(a_1,[-a_2]\right)\right)\\ \times\left(2h_n(b_1,[-b_2])-3h_{n-1}(b_1,[-b_2])\right)\end{array}\right)z^n$$

$$=2\left(2-\frac{ip}{2}\right)\sum_{n=0}^{\infty}h_n\left(a_1,[-a_2]\right)h_n(b_1,[-b_2])z^n$$

$$-3\left(2-\frac{ip}{2}\right)\sum_{n=0}^{\infty}h_n\left(a_1,[-a_2]\right)h_{n-1}(b_1,[-b_2])z^n$$

$$+2\left(i\left(\frac{p^2}{2}+2q\right)-p\right)\sum_{n=0}^{\infty}h_{n-1}\left(a_1,[-a_2]\right)h_n(b_1,[-b_2])z^n$$

$$-3\left(i\left(\frac{p^2}{2}+2q\right)-p\right)\sum_{n=0}^{\infty}h_{n-1}\left(a_1,[-a_2]\right)h_{n-1}(b_1,[-b_2])z^n,$$

by using the relationships (3.21), (3.22), (3.23) and (3.24), we obtain:

$$\sum_{n=0}^{\infty}Gj_{p,q,n}m_nz^n=\frac{2\left(4-ip\right)\left(1+4qz^2\right)}{2D_3}-\frac{3\left(4-ip\right)\left(pz+6qz^2\right)}{2D_3}$$

$$+\frac{2\left(i\left(p^2+4q\right)-2p\right)\left(3z-2pz^2\right)}{2D_3}-\frac{3\left(i\left(p^2+4q\right)-2p\right)\left(z+4qz^3\right)}{2D_3}$$

$$=\frac{2\left(4-ip\right)+6\left(i\left(p^2+2q\right)-3p\right)z+2\left(4\left(p^2-5q\right)-ip\left(2p^2+3q\right)\right)z^2+12q\left(2p-i\left(p^2+4q\right)\right)z^3}{2-6pz-4\left(5q-p^2\right)z^2+24pqz^3+32q^2z^4}$$

$$=\frac{4-ip+3\left(i\left(p^2+2q\right)-3p\right)z+\left(4\left(p^2-5q\right)-ip\left(2p^2+3q\right)\right)z^2+6q\left(2p-i\left(p^2+4q\right)\right)z^3}{1-3pz-2\left(5q-p^2\right)z^2+12pqz^3+16q^2z^4}.$$

Which completes the proof. □

Theorem 3.15. *For $n\in\mathbb{N}$, the new generating function of the product of Gaussian (p,q)-Jacobsthal lucas numbers with Mersenne Lucas numbers at negative indice is given by:*

$$\sum_{n=0}^{\infty}Gj_{p,q,n}m_{-n}z^n= \tag{3.28}$$

$$\frac{4\left(4-ip\right)+6\left(i\left(p^2+2q\right)-3p\right)z+\left(4\left(p^2-5q\right)-ip\left(2p^2+3q\right)\right)z^2+3q\left(2p-i\left(p^2+4q\right)\right)z^3}{4-6pz-2\left(5q-p^2\right)z^2+6pqz^3+4q^2z^4}.$$

Proof. We use the change of variable $z=\frac{z}{2}$ in (3.27) and according to relationship (1.2), we get:

$$\sum_{n=0}^{\infty}Gj_{p,q,n}m_{-n}z^n=\sum_{n=0}^{\infty}Gj_{p,q,n}m_n\left(\frac{z}{2}\right)^n$$

$$=\frac{4-ip+3\left(i\left(p^2+2q\right)-3p\right)\left(\frac{z}{2}\right)+\left(4\left(p^2-5q\right)-ip\left(2p^2+3q\right)\right)\left(\frac{z}{2}\right)^2+6q\left(2p-i\left(p^2+4q\right)\right)\left(\frac{z}{2}\right)^3}{1-3p\left(\frac{z}{2}\right)-2\left(5q-p^2\right)\left(\frac{z}{2}\right)^2+12pq\left(\frac{z}{2}\right)^3+16q^2\left(\frac{z}{2}\right)^4}$$

$$=\frac{4\left(4-ip\right)+6\left(i\left(p^2+2q\right)-3p\right)z+\left(4\left(p^2-5q\right)-ip\left(2p^2+3q\right)\right)z^2+3q\left(2p-i\left(p^2+4q\right)\right)z^3}{4-6pz-2\left(5q-p^2\right)z^2+6pqz^3+4q^2z^4}.$$

As required. □

- By setting $p=q=1$ in the relationships (3.25) , (3.26) , (3.27) and (3.28), we obtain the following table:

Coefficient of z^n	Generating function
$GJ_n m_n$	$\frac{i+3(1-i)z-(3i+4)z^2+6(i-2)z^3}{1-3z-8z^2+12z^3+16z^4}$
$GJ_n m_{-n}$	$\frac{4i+6(1-i)z-(3i+4)z^2+3(i-2)z^3}{4-6z-8z^2+6z^3+4z^4}$
$Gj_n m_n$	$\frac{4-i+9(i-1)z-(5i+16)z^2+6(2-5i)z^3}{1-3z-8z^2+12z^3+16z^4}$
$Gj_n m_{-n}$	$\frac{4(4-i)+18(i-1)z-(5i+16)z^2+3(2-5i)z^3}{4-6z-8z^2+6z^3+4z^4}$

Table 3. New generating functions of the products of Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers with Mersenne Lucas numbers at positive and negative indices.

4 Conclusion

In this paper, we studied the Mersenne Lucas numbers and some Gaussian numbers. By using the symmetric functions we gave some new generating functions for the products of Gaussian (p,q)-Fibonacci numbers, Gaussian (p,q)-Lucas numbers, Gaussian (p,q)-Pell numbers, Gaussian (p,q)-Pell Lucas numbers, Gaussian (p,q)-Jacobsthal numbers and Gaussian (p,q)-Jacobsthal Lucas numbers with Mersenne Lucas numbers at positive and negative indices.

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