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FAMILY OF SURFACES WITH A COMMON ALTERNATIVE PARTNER CURVE AS ASYMPTOTIC, GEODESIC AND CURVATURE LINE

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Abstract. In this study, the necessary and sufficient conditions are provided to form a surface family on which the alternative partner of any given curve lies as isogeodesic, isoasymptotic and curvature line by using the vector elements of alternative frame. The conditions for these surfaces to be minimal and developable are also examined by calculating the first and the second fundamental forms, and the mean and Gaussian curvatures. Moreover, the idea is discussed for the family of ruled surfaces defined with the alternative frame, as well. Finally a set of examples are given with the illustrations of corresponding surfaces.

1 Introduction

In Euclidean space, geodesics are special curves which describe the shortest paths between two points on a given surface. In other words, they take the role of straight lines on planes. According to the general relativity theory going back to the beginning of 20th century, Einstein integrated time as an another dimension with the three dimensional space and introduced the concept of space-time. The theory states that a freely falling particle bends the surrounding space proportional to its mass and it follows all along the geodesic paths [20]. Asymptotic curves, on the other hand, is another special type of curves used in astronomy, astrophysics, and architectural designs. Contopoulous [9] showed in his study that the motions of stars escaping from a galaxy can be found by the asymptotic curves of Lyapunov orbits, and he defined the orbit sets of escaping stars on asymptotic curves with some initial conditions. The asymptotic curves have also a great potential use of architectural designs, especially in modeling grid structures [19]. Another special characteristic for a curve is to be the curvature line on a given surface. The curvature lines have been of interest to researchers for many years, and they have especially been used for the process of extracting information from geometrical objects which is known in short as shape interrogation. Such characteristics of curves are determined based on an orthonormal basis settled

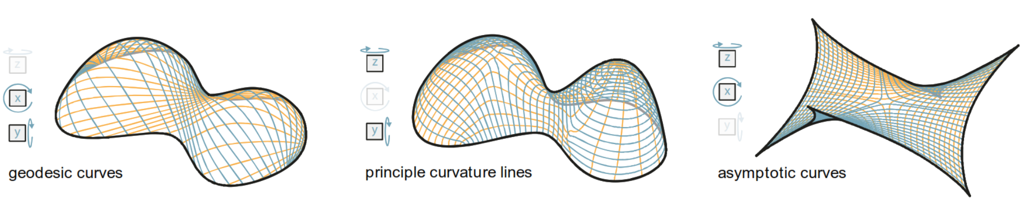


Figure 1. The view of geodesic curves, asymptotic curves and curvature lines on surfaces (adapted from [19])

at a point and moving all along the curve which is known as Frenet frame. To feature new aspects on curves, researchers defined other frames by following the same idea. One of those is known

to be the alternative frame which was first introduced by Uzunoğlu et al. [25]. They denoted the elements of the new frame as $\{N,\ C,\ W\}$ where N is the normal vector, W is the Darboux vector and C is defined to be the vector product of N and W. In [17], Özdoğan, established the transitional relations between Frenet frame and alternative frame, and identified new curves depending on some associations among the vector elements of this alternative frame. Similarly, Kaya and Önder [12] studied the alternative frame to generate new special curves. They introduced CN^* partner curves whereas Yılmaz and Has [27] defined WC^* partner curves. Both studies provided some characteristics for these new partner curves. Moreover, Şenyurt [21] defined the Darboux vector as the instantaneous rotational axis for the alternative frame.
Apart from these, another important subject of the differential geometry is to characterize curves on a given surface. However, characterizing a curve to be as geodesic, asymptotic or a curvature line on the surface was first questioned in a reverse approach by Wang et al. [26]. They defined the ways to parameterize surfaces which possess a given curve as geodesic. Providing such an insight into, Li et al. [13] gave the parametric representations for the surfaces that have a given curve as a curvature line. Likewise, Bayram et al. [7] constructed the surface family with a common asymptotic curve. With the help of the procedure on generating such new surface families, researchers began to examine the methods to construct new surfaces with some specific curves lying on the surface as geodesic, asymptotic or curvature line ([1–6, 8, 22–24]). By this study, we construct the parametric representations of the family of surfaces with a common alternative partner curve as of each geodesic, asymptotic and curvature line. We first recall some basic concepts regarding to alternative frame in Section 2. Then, in Section 3, by referring the alternative frame, the necessary and sufficient conditions are provided to form such surfaces, which is the main objective of this paper. The fundamental forms and the curvatures of the constructed surfaces are also examined in this section. Section 4 is comprised of the same process for the ruled surfaces. The necessary and sufficient conditions for these ruled surfaces to be minimal and developable are provided, as well. In addition, some examples of surfaces with their corresponding graphs are given in section 4. Finally, we summarize the paper and touch upon the potentials of generated surfaces, briefly, in Section 5.

2 Preliminaries

Let us recall some basic concepts which will be used throughout the paper. Let $\alpha(s) : I \subset \mathbb{R} \to \mathbb{R}^3,\ s \in I$ be a curve at least twice differentiable in the three dimensional Euclidean space $\mathbb{E}^3$. The general forms of Frenet vectors and formulae are given as

$$T(s) = \frac{\alpha'(s)}{\|\alpha'(s)\|}, \quad B(s) = \frac{\alpha'(s) \times \alpha''(s)}{\|\alpha'(s) \times \alpha''(s)\|}, \quad N(s) = B(s) \times T(s), \tag{2.1}$$

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}, \qquad \tau = \frac{\langle \alpha' \times \alpha'', \alpha''' \rangle}{\|\alpha' \times \alpha''\|^2}, \tag{2.2}$$

$$T' = \kappa\nu N, \quad N' = -\kappa\nu T + \tau\nu B, \quad B' = -\tau\nu N, \tag{2.3}$$

where $\nu = \|\alpha'\|$, κ is the curvature and τ is the torsion of the curve [16]. At every point s on the curve α the Frenet vectors has an instantaneous rotation of axis on the direction of a vector. This vector is called as Darboux vector and formulated by the following

$$\tilde{W} = \tau T + \kappa B. \tag{2.4}$$

If $\theta = \angle(\tilde{W}, B)$, then the unit Darboux vector can be written as

$$W = \frac{\tau T + \kappa B}{\sqrt{\kappa^2 + \tau^2}} = \sin\theta T + \cos\theta B, \tag{2.5}$$

where $\kappa = \|\tilde{W}\| \cos\theta,\ \ \tau = \|\tilde{W}\| \sin\theta$ [10]. By using the vector product of N and W, the unit vector C can be given as

$$C = \frac{-\kappa T + \tau B}{\sqrt{\kappa^2 + \tau^2}} = -\cos\theta T + \sin\theta B. \tag{2.6}$$

Such an orthonormal frame with the elements of $\{N, C, W\}$ is called as the alternative frame [25]. The transitions between the vectors of alternative frame and their derivatives are as follows:

$$N' = fC, \quad C' = -fN + gW, \quad W' = -gC, \tag{2.7}$$

where

$$f = \kappa\sqrt{1+h^2}, \quad h = \frac{\tau}{\kappa}, \quad g = \sigma f, \quad \sigma = \frac{h'}{\kappa(1+h^2)^{3/2}}. \tag{2.8}$$

If h is constant, then the curve α is a helix and if σ is a constant then α is known to be a slant helix [11]. Let us take α and α^* as two regular unit speed space curves with the corresponding Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$, respectively.
Denote by $\{N, C, W\}, f, g$, and $\{N^*, C^*, W^*\}, f^*, g^*$, as the alternative moving frame and the alternative curvatures of the given curves in respective order. Then, the curves α and α^* are named as WC^* -partner curves if the vector W of α coincides with the vector of C^* of α^* [27]. Thus the relations between two curves can be established by

$$\alpha^*(s) = \alpha(s) + \lambda(s)W(s),$$

where $\lambda(s)$ is known to be the distance function and defined to be as $\lambda(s) = \dfrac{\kappa(s)}{f(s)g(s)}$. Since W and C^* are assigned to be equal, if $\omega = \omega(s)$ is taken to be the angel between the vectors N and W^* (i.e. $\omega = \angle(N, W^*)$), the corresponding relations of alternative frames are given by

$$\begin{pmatrix} N^* \\ C^* \\ W^* \end{pmatrix} = \begin{pmatrix} \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \\ -\cos\omega & sin\omega & 0 \end{pmatrix} \begin{pmatrix} N \\ C \\ W \end{pmatrix},$$

where $\omega = \displaystyle\int_0^s f(s)ds$.
Now let us consider the surface $\psi = \psi(s, v)$ on which the curve α^* lies. If the normal curvature is zero in the direction of the velocity vector $\alpha^{*\prime}(s)$, then α^* is called to be as asymptotic curve on the surface. On the other hand, the curve α^* is named as geodesic if the acceleration vector $\alpha^{*\prime\prime}$ is parallel to the normal of surface [15]. Another specific curve on a surface is the line of curvature. A curve is said to be the line of curvature on a surface if the surface normals along that curve form a developable surface [14]. Moreover, if $\psi(s, v_0) = \alpha^*(s)$ where v_0 is constant, then α^* is named as isoparametric curve, and if α^* is both isoparametric and asymptotic (or geodesic), then it is called as isoasymptotic (or isogeodesic), respectively [7, 13, 26].
The first and second fundemental forms of the surface $\psi = \psi(s, v)$ are

$$\begin{aligned} I &= Eds^2 + 2Fdsdv + Gdv^2, \\ II &= Lds^2 + 2Mdsdv + Ndv^2, \end{aligned} \tag{2.9}$$

where the coefficients are calculated by

$$\begin{aligned} E &= \langle \psi_s, \psi_s \rangle, \quad F = \langle \psi_s, \psi_v \rangle, \quad G = \langle \psi_v, \psi_v \rangle, \\ L &= \langle \psi_{ss}, n \rangle, \quad M = \langle \psi_{sv}, n \rangle, \quad N = \langle \psi_{vv}, n \rangle. \end{aligned} \tag{2.10}$$

The Gauss and the mean curvatures of the surface are given as:

$$K = \frac{LN - M^2}{EG - F^2}, \qquad H = \frac{EN - 2FM + GL}{2(EG - F^2)}. \tag{2.11}$$

Moreover, a surface is called as ruled if it is formed with the locus of straight lines $x(s)$ moving along the curve $\alpha^*(s)$. The parametric representation is as follows:

$$\Delta(s, v) = \alpha^*(s) + vx(s). \tag{2.12}$$

In order for a ruled surface to be developable, the following relation must hold [18]:

$$\det(\alpha^{*\prime}, x, x') = 0. \tag{2.13}$$

3 Family of Surfaces with a Common Alternative Partner Curve as Asymptotic, Geodesic and Curvature Line

Let us denote the set of vectors $\{N^*, C^*, W^*\}$ as the alternative frame of $\alpha^*(s)$ which lies on the surface $\psi(s,v)$. It is of interest for us to determine the conditions such that α^* is geodesic, asymptotic and curvature line on ψ expressed by the alternative frame. Therefore we first parameterize the surface with the components of alternative frame as below:

$$\psi(s,v) = \alpha^*(s) + x(s,v)N^*(s) + y(s,v)C^*(s) + z(s,v)W^*(s). \tag{3.1}$$

Here, $x(s,v),\ y(s,v)$ and $z(s,v)$ are known to be differentiable marching scale functions.

Theorem 3.1. *The necessary and sufficient conditions for α^* to be isogeodesic and isoasymptotic on ψ are*

$$\begin{aligned} &x(s,v_0) = y(s,v_0) = z(s,v_0) = \frac{\partial x(s,v_0)}{\partial v} = 0, \\ &\frac{\partial y(s,v_0)}{\partial v} = \beta(s)\sin\theta^*, \qquad \frac{\partial z(s,v_0)}{\partial v} = \beta(s)\cos\theta^*, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &x(s,v_0) = y(s,v_0) = z(s,v_0) = 0, \\ &\frac{\partial x(s,v_0)}{\partial v} \neq 0, \\ &\frac{\partial y(s,v_0)}{\partial v} = -\beta(s)\cos\theta^*, \qquad \frac{\partial z(s,v_0)}{\partial v} = \beta(s)\sin\theta^*, \end{aligned} \tag{3.3}$$

respectively.

Proof. From the definition of isoparametric curve and by referring the relation (3.1), we have $x(s,v_0) = y(s,v_0) = z(s,v_0) = 0$, for a constant $v = v_0$. On the other hand, the normal to ψ denoted by n can be calculated as

$$\begin{aligned} \vec{n}(s,v_0) &= \frac{\partial \psi(s,v)}{\partial s} \times \frac{\partial \psi(s,v)}{\partial v} \\ &= \left(-\cos\theta^* \frac{\partial z(s,v_0)}{\partial v} - \sin\theta^* \frac{\partial y(s,v_0)}{\partial v}\right) N^* \\ &\quad + \sin\theta^* \frac{\partial x(s,v_0)}{\partial v} C^* + \cos\theta^* \frac{\partial x(s,v_0)}{\partial v} W^*. \end{aligned}$$

In order for α^* to be geodesic on ψ, by definition, the normal vector N of α^* must be a scalar multiple of n. Therefore we can write

$$-\cos\theta^* \frac{\partial z(s,v_0)}{\partial v} - \sin\theta^* \frac{\partial y(s,v_0)}{\partial v} \neq 0$$

and

$$\sin\theta^* \frac{\partial x(s,v_0)}{\partial v} = \cos\theta^* \frac{\partial x(s,v_0)}{\partial v} = 0.$$

By using trigonometric identities and referring an arbitrary function $\beta(s) \neq 0$, we may simply rearrange the formulations given above as following:

$$\frac{\partial x(s,v_0)}{\partial v} = 0, \quad \frac{\partial y(s,v_0)}{\partial v} = \beta(s)\sin\theta^*, \quad \frac{\partial z(s,v_0)}{\partial v} = \beta(s)\cos\theta^*.$$

Now, by recalling the asymptoticity condition, it is clear to state that the binormal vector of α^* is parallel to the normal vector of ψ. By using the relationships between the Frenet frame and alternative frame given in [17], we first write

$$B^* = \sin\theta^* C^* + \cos\theta^* W^*. \tag{3.4}$$

Then, since $B^* \parallel n$, we have

$$\cos\theta^* \frac{\partial z(s,v_0)}{\partial v} + \sin\theta^* \frac{\partial y(s,v_0)}{\partial v} = 0,$$
$$\frac{\partial x(s,v_0)}{\partial v} \neq 0.$$

Similarly, we may reform the above relations with $\beta(s) \neq 0$ as

$$\frac{\partial x(s,v_0)}{\partial v} \neq 0, \quad \frac{\partial y(s,v_0)}{\partial v} = -\beta(s)\cos\theta^*, \quad \frac{\partial z(s,v_0)}{\partial v} = \beta(s)\sin\theta^*.$$

This completes the proof. □

Theorem 3.2. *The necessary and sufficient conditions for α^* to be a curvature line on the surface ψ are*

$$\begin{aligned}
&x(s,v_0) = y(s,v_0) = z(s,v_0) = 0, \quad \gamma = -\int f^*(s)\sin\theta^* ds, \quad \mu(s) \neq 0,\\
&\cos\gamma = -\mu(s)\left(\cos\theta^* \frac{\partial z(s,v_0)}{\partial v} + \sin\theta^* \frac{\partial y(s,v_0)}{\partial v}\right),\\
&\sin\gamma = \mu(s)\frac{\partial x(s,v_0)}{\partial v}.
\end{aligned}$$

Proof. Let us denote $\eta(s)$ as the orthogonal vector field of the surface ψ embedded with the curve α^*. By the definition of curvature line, $\eta(s) \in Sp\{N^*, B^*\}$. If $\gamma = \measuredangle(\eta(s), N^*)$, then

$$\eta(s) = \cos\gamma N^*(s) + \sin\gamma B^*(s).$$

Substituting the relation (3.4) into the last expression results

$$\eta(s) = \cos\gamma N^*(s) + \sin\gamma\sin\theta^* C^*(s) + \sin\gamma\cos\theta^* W^*(s).$$

Now, by recalling the two fold conditions for a curve to be a curvature line, we consider first $\eta(s) \parallel n(s,v_0)$. This results the following relations

$$-\frac{\cos\gamma}{\cos\theta^* \frac{\partial z(s,v_0)}{\partial v} + \sin\theta^* \frac{\partial y(s,v_0)}{\partial v}} = \frac{\sin\gamma\sin\theta^*}{\sin\theta^* \frac{\partial x(s,v_0)}{\partial v}} = \frac{\sin\gamma\cos\theta^*}{\cos\theta^* \frac{\partial x(s,v_0)}{\partial v}} = \mu(s).$$

With some algebraic manipulations, we have

$$\begin{aligned}
&\cos\gamma = -\mu(s)\left(\cos\theta^* \frac{\partial z(s,v_0)}{\partial v} + \sin\theta^* \frac{\partial y(s,v_0)}{\partial v}\right),\\
&\sin\gamma = \mu(s)\frac{\partial x(s,v_0)}{\partial v}.
\end{aligned}$$

Second, the surface defined as $\Phi(s,t) = \alpha^*(s) + t\eta(s)$ must be developable that is to say that $\det(\alpha^{*\prime}, \eta, \eta') = 0$. We calculate each component as

$$\begin{aligned}
\alpha^{*\prime} &= -\cos\theta^* C^* + \sin\theta^* W^*,\\
\eta(s) &= \cos\gamma N^* + \sin\gamma\sin\theta^* C^* + \sin\gamma\cos\theta^* W^*,\\
\eta'(s) &= \left(-\gamma'\sin\gamma - f^*\sin\gamma\sin\theta^*\right) N^*\\
&\quad + \left(f^*\cos\gamma + \gamma'\cos\gamma\sin\theta^* + \kappa^*\sin\gamma\cos\theta^* - g^*\sin\gamma\cos\theta^*\right) C^*\\
&\quad + \left(\gamma'\cos\gamma\cos\theta^* - \kappa^*\sin\gamma\sin\theta^* + g^*\sin\gamma\sin\theta^*\right) W^*.
\end{aligned}$$

Finally, the result of the determinant is given as:

$$\begin{aligned}
\det(\alpha^{*\prime}, \eta, \eta') = 0 &\Rightarrow \gamma' + f^*\sin\theta^* = 0\\
&\Rightarrow \gamma = -\int f^*\sin\theta^* ds,
\end{aligned}$$

which completes the proof. □

Theorem 3.3. *The first and the second fundamental forms together with the Gauss and mean curvatures of $\psi = \psi(s, v)$ embedded with the curve α^* are*

$$I = ds^2 + 2\left(-\frac{\partial y}{\partial v}\cos\theta^* + \frac{\partial z}{\partial v}\sin\theta^*\right) dsdv + \left(\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right) dv^2,$$

$$II = -\kappa^*\mathbf{A}ds^2 + \left(\mathbf{A}\left(f^*\frac{\partial y}{\partial v} - \frac{\partial^2 x}{\partial s\partial v}\right) + (\mathbf{B}+\mathbf{C})\frac{\partial x}{\partial v}\right) dsdv$$
$$+ \left(-\frac{\partial^2 x}{\partial^2 v}\mathbf{A} + \frac{\partial x}{\partial v}\left(\sin\theta^*\frac{\partial^2 y}{\partial^2 v} + \cos\theta^*\frac{\partial^2 z}{\partial^2 v}\right)\right) dv^2,$$

$$K = \frac{\kappa^*\mathbf{A}^2\frac{\partial^2 x}{\partial^2 v} - \kappa^*\mathbf{A}\left(\sin\theta^*\frac{\partial^2 y}{\partial^2 v} + \cos\theta^*\frac{\partial^2 z}{\partial^2 v}\right) - \left(\mathbf{A}\left(f^*\frac{\partial y}{\partial v} - \frac{\partial^2 x}{\partial s\partial v}\right) + (\mathbf{B}+\mathbf{C})\frac{\partial x}{\partial v}\right)^2}{\left(\frac{\partial x}{\partial v}\right)^2 + \mathbf{A}^2},$$

$$H = \frac{\begin{array}{c} -\frac{\partial^2 x}{\partial^2 v}\mathbf{A} + \frac{\partial x}{\partial v}\left(\sin\theta^*\frac{\partial^2 y}{\partial^2 v} + \cos\theta^*\frac{\partial^2 z}{\partial^2 v}\right) - \kappa^*\mathbf{A}\left(\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right) \\ -2\left(-\frac{\partial y}{\partial v}\cos\theta^* + \frac{\partial z}{\partial v}\sin\theta^*\right)\left(\mathbf{A}\left(f^*\frac{\partial y}{\partial v} - \frac{\partial^2 x}{\partial s\partial v}\right) + (\mathbf{B}+\mathbf{C})\frac{\partial x}{\partial v}\right)\end{array}}{2\left(\left(\frac{\partial x}{\partial v}\right)^2 + \mathbf{A}^2\right)},$$

where $\mathbf{A}$, $\mathbf{B}$ *and* $\mathbf{C}$ *are*

$$\mathbf{A} = \cos\theta^*\frac{\partial z}{\partial v} + \sin\theta^*\frac{\partial y}{\partial v}, \quad \mathbf{B} = \cos\theta^*\frac{\partial y}{\partial v} - \sin\theta^*\frac{\partial z}{\partial v},$$
$$\mathbf{C} = \left(f^*\frac{\partial x}{\partial v} + \frac{\partial^2 y}{\partial s\partial v}\right)\sin\theta^* + \frac{\partial^2 z}{\partial s\partial v}\cos\theta^*.$$

Proof. We recall the coefficients given in (2.10) and calculate them as follows:

$$E = \langle\psi_s, \psi_s\rangle = 1, \qquad F = \langle\psi_s, \psi_v\rangle = -\frac{\partial y}{\partial v}cos\theta^* + \frac{\partial z}{\partial v}sin\theta^*,$$
$$G = \langle\psi_v, \psi_v\rangle = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2,$$
$$L = \langle\psi_{ss}, \psi_s \times \psi_v\rangle = -\kappa^*\mathbf{A},$$
$$N = \langle\psi_{vv}, \psi_s \times \psi_v\rangle = -\frac{\partial^2 x}{\partial^2 v}\mathbf{A} + \frac{\partial x}{\partial v}\left(\sin\theta^*\frac{\partial^2 y}{\partial^2 v} + \cos\theta^*\frac{\partial^2 z}{\partial^2 v}\right),$$
$$M = \langle\psi_{sv}, \psi_s \times \psi_v\rangle = \mathbf{A}\left(f^*\frac{\partial y}{\partial v} - \frac{\partial^2 x}{\partial s\partial v}\right) + (\mathbf{B}+\mathbf{C})\frac{\partial x}{\partial v}.$$

By substituting these relations into both (2.9) and (2.11), we complete the proof. □

Corollary 3.4. *If* α^* *is isogeodesic on* ψ *then its Gaussian and mean curvatures are given by*

$$K = (f^*\beta(s)\sin\theta^*)^2, \quad H = -\frac{1}{2}\kappa^*\beta(s),$$

respectively.

Corollary 3.5. *If* α^* *is isoasymptotic on* ψ *then its Gaussian and mean curvatures are given by*

$$K = -\left(f^*\frac{\partial x}{\partial v}\sin\theta^* + \beta(s)\theta^{*\prime} - \beta(s)\right)^2,$$
$$H = -\beta(s)\left(f^*\frac{\partial x}{\partial v}\sin\theta^* + \beta(s)\theta^{*\prime} - \beta(s)\right)\left(\frac{\partial x}{\partial v}\right)^{-1},$$

respectively.

4 Ruled Surfaces with a Common Isogeodesic and Isoasymptotic Curve by Alternative Frame

The parametric form of the ruled surface accepting α as the base curve can be written

$$\psi(s,v)=\alpha^*(s)+(v-v_0)R(s). \tag{4.1}$$

By considering the relations (3.1) and (4.1), we have

$$x(s,v)N^*(s)+y(s,v)C^*(s)+z(s,v)W^*(s)=(v-v_0)R(s). \tag{4.2}$$

The inner production of the above by N, C and W results

$$\begin{aligned} x(s,v)&=(v-v_0)\langle R,N^*\rangle, \\ y(s,v)&=(v-v_0)\langle R,C^*\rangle, \\ z(s,v)&=(v-v_0)\langle R,W^*\rangle. \end{aligned} \tag{4.3}$$

Now, consider the conditions given in Theorem 3.1. First, by using the relations (3.2), the family of ruled surfaces with a common isogeodesic curve can be given as

$$\psi_{igr}(s,v)=\alpha^*(s)+(v-v_0)\beta(s)\left(\sin\theta^*C^*+\cos\theta^*W^*\right), \qquad \beta(s)\neq 0. \tag{4.4}$$

Second, when we refer the relations given at (3.3), we obtain the parametrization for the family of ruled surfaces with a common isoasymptotic curve as

$$\psi_{iar}(s,v)=\alpha^*(s)+(v-v_0)\left(x(s)N^*-\beta(s)cos\theta^*C^*+\beta(s)sin\theta^*W^*\right), \tag{4.5}$$

where $x(s),\ \beta(s)\neq 0$.

Corollary 4.1. *The necessary and sufficient condition for the ruled surface with a common isogeodesic curve to be developable is that $f^*sin\theta^*=0$. However, since $f^*,\theta^*\neq 0$, there is no a developable ruled surface with a common isogeodesic curve.*

Corollary 4.2. *The necessary and sufficient condition for the ruled surface with a common isoasymptotic curve to be developable is*

$$g^*\beta(s)+\theta^{*\prime}\beta(s)\sin^2\theta^*=\theta^{*\prime}\beta(s)+f^*x(s)\sin\theta^*.$$

Example 4.3. Let us consider the slant helix curve given by the following parametrization

$$\alpha(s)=\left(-\frac{\cos(3s)}{12}-\frac{3\cos(s)}{4},-\frac{\sin(3s)}{12}-\frac{3\sin(s)}{4},-\frac{\sqrt{3}}{2}\cos(s)\right).$$

The corresponding curvatures of α can be easily computed as

$$\kappa(s)=\sqrt{3}\cos(s),\quad \tau(s)=\sqrt{3}\sin(s),\quad f=\sqrt{3},\quad g=1.$$

We also note that $\theta=\arctan\left(\frac{\tau}{\kappa}\right)=s$.

Now, recall the distance function $\lambda(s)=\dfrac{\kappa(s)}{f(s)g(s)}=-cos(s)$. Since α^* is the WC^* alternative partner curve of α, the parametrization can alternatively be given as

$$\alpha^*(s)=\alpha(s)-cos(s)W(s).$$

Thus, we reach the parametric equation for the alternative partner curve as

$$\alpha^*=\left(\frac{1}{3}\left(2\cos^2(s)-3\right)\cos(s),-\frac{2}{3}\sin^3(s),-\sqrt{3}\cos(s)\right),$$

and curvature functions as

$$\kappa^*(s) = \frac{1}{2sin(s)}, \quad \tau^*(s) = \frac{\sqrt{3}}{2\sin(s)}, \quad f^* = \frac{1}{sin(s)}, \quad g^* = 0.$$

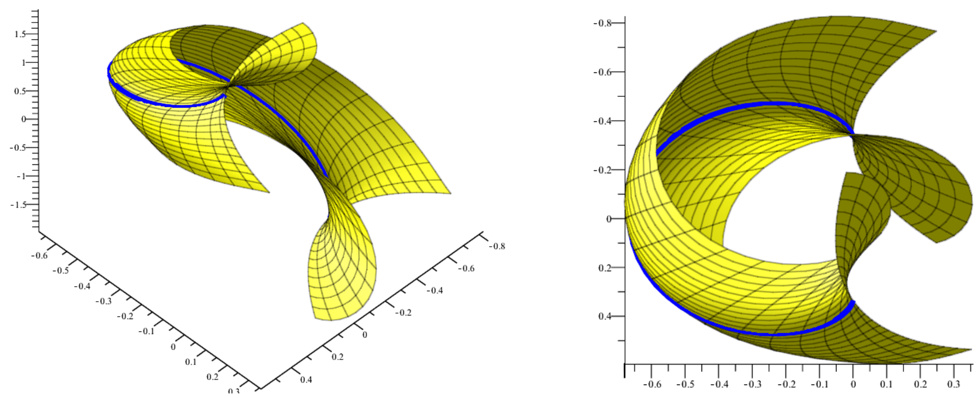
After this line, throughout the example section, we will choose specific marching scale functions that satisfy the corresponding conditions on the following parametric form for the surface $\psi(s, v)$ defined by

$$\psi(s, v) = \alpha^*(s) + x(s, v)N^*(s) + y(s, v)C^*(s) + z(s, v)W^*(s).$$

- First, in order for α^* to be geodesic on the surface $\psi(s, v)$, we choose the set of marching scale functions as

  $$x(s, v) = v^2, \quad y(s, v) = v\sin(\theta^*), \quad z(s, v) = v\cos(\theta^*) \quad \text{and} \quad v_0 = 0,$$

  where $\theta^* = arctan(\frac{\tau^*}{\kappa^*})$. Fig. 2 illustrates this surface ψ.



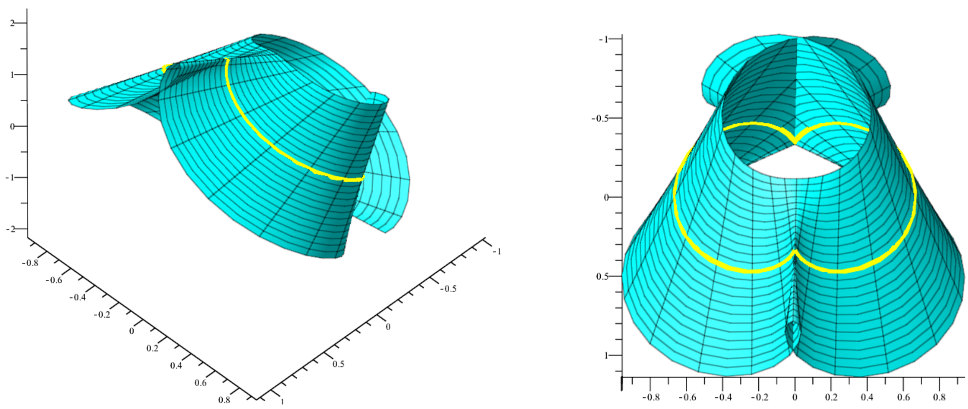
(a) default orientation (b) oriented to z-axis

Figure 2. The surface ψ with a common geodesic curve α^* from different angles

- Second, in order for α^* to be asymptotic on the surface $\zeta(s, v)$, we use the following set of marching scale functions

  $$x(s, v) = v, \quad y(s, v) = -v\cos(\theta^*), \quad z(s, v) = v\sin(\theta^*) \quad \text{and} \quad v_0 = 0.$$

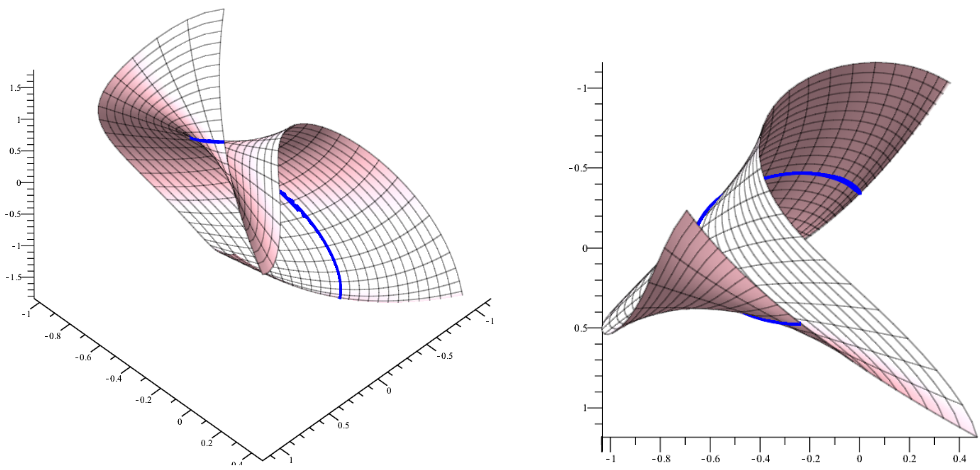
  The graph of the surface ζ is given in Fig. 3.



(a) default orientation (b) oriented to z-axis

Figure 3. The surface ζ with a common asymptotic curve α^* from different angles

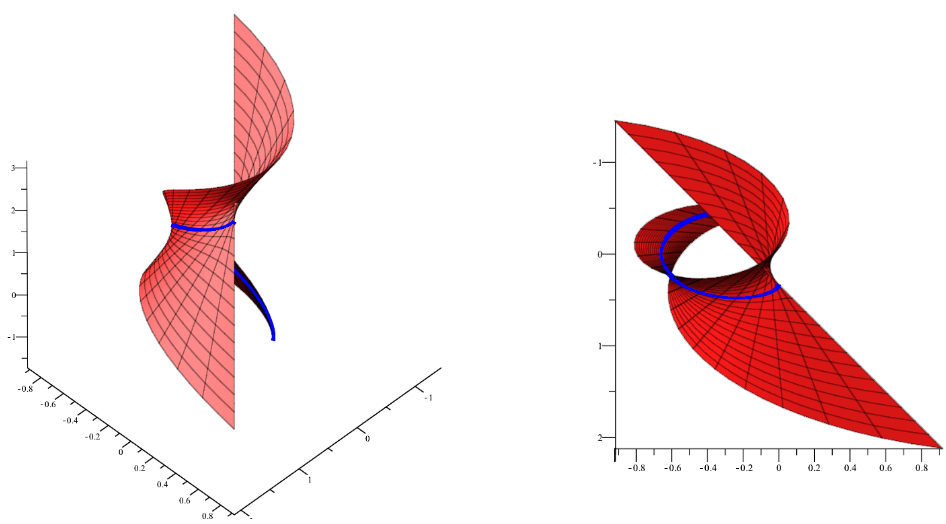
- Third, by choosing the scale functions as $x(s,v) = v\sin(\gamma(s))$, $y(s,v) = v^2$, $z(s,v) = -\dfrac{v\cos(\gamma(s))}{\cos(\theta^*)}$, $v_0 = 0$, $\mu(s) = 1$ where $\gamma(s) = -\int f^* cos(\theta^*)ds$, we can form the surface ϕ with a common alternative partner curve as a curvature line (see Fig. 4).



(a) default orientation (b) oriented to z-axis

Figure 4. The surface ϕ with a common curvature line α^* from different angles

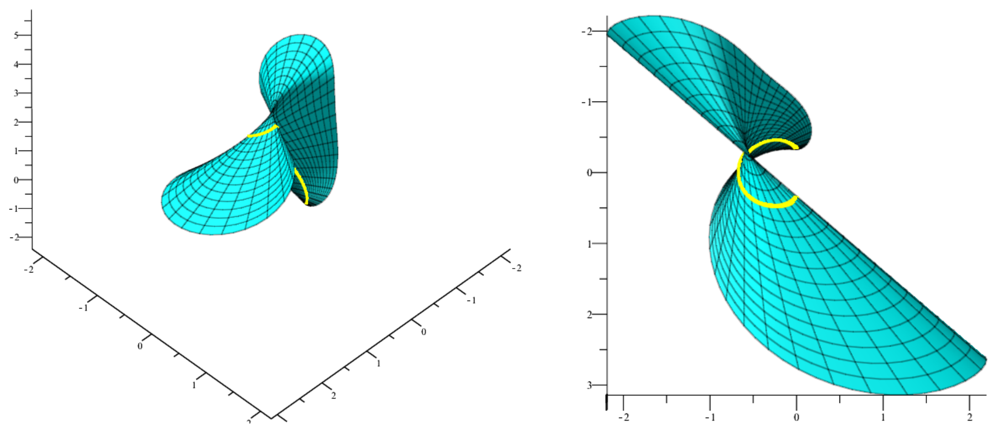
- Similar to the previous ones but for the ruled surface with a common isogeodesic partner curve α^*, we choose $x(s) = 0$, $\beta(s) = s^2$, $v_0 = 0$. Fig. 5 corresponds to the surface χ.



(a) default orientation (b) oriented to z-axis

Figure 5. The ruled surface χ with a common geodesic curve α^* from different angles

- Lastly, for the ruled surface with a common isoasymptotic partner curve α^*, we choose $x(s) = s$, $\beta(s) = s^2$, $v_0 = 0$. The graph of this surface Γ is given in Fig. 6.



(a) default orientation (b) oriented to z-axis

Figure 6. The ruled surface Γ with a common asymptotic curve α^* from different angles

5 Conclusion

Curves and surfaces have been the subject of many different fields, not only in terms of theoretical perspective but also in terms of their possible applications. For this reason, in this study, we have introduced a way of generating new surface family where the alternative partner curve of a given curve has a specific characteristic on the surface. Then, we have provided the required conditions for these surfaces to be developable and minimal, as well. It is for sure that the constructed surfaces may have the potential to be used in related fields, and also providing their mathematical language may enable researchers to easily adapt these to the computational area. Researchers may also benefit from this paper and apply other orthonormal frames rather than alternative frame to generate new kinds of surfaces with some other distinct marching scale functions satisfying some different conditions corresponding to that frame. Moreover, this study can be extended by considering different space forms.

References

[1] G. Ş. Atalay and E. Kasap, Surfaces family with common null asymptotic. *Applied Mathematics and Computation* **260**, 135–139 (2015).

[2] G. Ş. Atalay and E. Kasap, Surfaces family with common Smarandache asymptotic curve, *Boletim da Sociedade Paranaense de Matemática* **34**(1), 9–20 (2016).

[3] G. Ş. Atalay and E. Kasap, Surfaces family with common Smarandache geodesic curve, *Journal of Science and Arts* **17**(4), 651–664 (2017).

[4] G. Ş. Atalay, Surfaces Family with a Common Mannheim Geodesic Curve, *Journal of Applied Mathematics and Computation* **2**(4), 155–165 (2018).

[5] G. Ş. Atalay and K. H. Ayvacı, Surface family with a common Mannheim B-geodesic curve, *Balkan Journal of Geometry and Its Applications* **26**(2), 1–12 (2021).

[6] K. H. Ayvacı and G. Ş. Atalay, Surface Family With A Common Bertrand-B Isogeodesic Curve, *Journal of the Institute of Science and Technology* **10**(3), 1975–1983 (2020).

[7] E. Bayram, F. Güler and E. Kasap, Parametric representation of a surface pencil with a common asymptotic curve, *Computer-Aided Design* **44**(7), 637-–643 (2012).

[8] E. Bayram and M. Bilici, Surface family with a common involute asymptotic curve, *International Journal of Geometric Methods in Modern Physics* **13**(5), 1650062 (2016).

[9] G. Contopoulos, Asymptotic curves and escapes in Hamiltonian systems, *Astronomy and Astrophysics* **231**, 41–55 (1990).

[10] W. Fenchel, On The Differential Geometry of Closed Space Curves, *Bulletin of American Mathematical Society* **57**, 44–54 (1951).

[11] S. Izumiya and N. Takeuchi, New special curves and developable surfaces, *Turk. J. Math.* **28**, 153–163 (2004).

[12] O. Kaya and M. Önder, New Partner Curves in the Euclidean 3-Space, *International Journal of Geometry* **6**(2), 41–50 (2017).

[13] C. Y. Li, R. H. Wang and C. G. Zhu, Parametric representation of a surface pencil with a common line of curvature, *Comput. Aided Des.* **43**(9), 1110—1117 (2011).

[14] C. Y. Li, R. H. Wang and C. G. Zhu, An approach for designing a developable surface through a given line of curvature, *Computer-Aided Design* **45**(3), 621–627 (2013).

[15] R. S. Millman and G. D. Parker, *Elements of Differential Geometry*, Prentice-Hall, Englewood Clis, New Jersey (1977).

[16] B. O'Neill, *Elementary differential geometry*, New York, Academic Press Inc (1966).

[17] S. Özdoğan, O. Tuncer, İ. Gök and Y. Yaylı, Some new types of associated curves in Euclidean 3-space, *Analele Universitatii "Ovidius" Constanta-Seria Matematica* **26**(1), 205–221 (2018).

[18] B. Ravani and T. S. Ku, Bertrand Offsets of Ruled and Developable Surfaces, *Comput Aided Geom Design* **23(2)**, 145–152 (1991).

[19] E. Schling, D. Hitrec and R. Barthel, Designing grid structures using asymptotic curve networks, *In Humanizing Digital Reality Springer Singapore* 125–140 (2018).

[20] B. Schutz, *A first course in general relativity*, Cambridge University press (2009).

[21] S. Şenyurt, D-Smarandache Curves According to the Sabban Frame of the Spherical Indicatrix Curve, *Turk. J. Math. Comput. Sci.* **9**, 39–49 (2018).

[22] S. Şenyurt, K. H. Ayvacı and D. Canlı, On the Surfaces with Common Mannheim D- Isogeodesic Curve *Ordu Univ. J. Sci. Tech.* **10**(2), 105–116 (2020).

[23] S. Şenyurt, K. H. Ayvacı and D. Canlı, Family of Surfaces with A Common Mannheim D- Isoasymptotic Curve *Ordu Univ. J. Sci. Tech.* **10**(2), 143–153 (2020).

[24] S. Şenyurt, K. H. Ayvacı and D. Canlı, Family of Surfaces with a Common Special Involute and Evolute Curves, *International Electronic Journal of Geometry* **15**(1), 160–174 (2022).

[25] B. Uzunoğlu, İ. Gök and Y. Yaylı, A New Approach on Curves of Constant Precession, *Applied Mathematics and Computation* **275**, 317–323 (2016).

[26] G. J. Wang, K. Tang and C. L. Tai, Parametric representation of a surface pencil with a common spatial geodesic, *Comput. Aided Des.* **36**(5), 447–459 (2004).

[27] B. Yılmaz and A. Has, Alternative Partner Curves in the Euclidean 3-Space, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* **69**(1), 1–10 (2020).

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