Palestine Journal of Mathematics

Vol. 12(2)(2023) , 475–486

Integral bases of some families of quartic number fields.

M. TALJAOUI and M. BOUHAMZA

Communicated by Ayman Badawi

MSC 2010 Classifications: 11R04, 11R16, 11Y40.

Keywords and phrases: Integral Basis; Quartic Number Fields..

The authors like to thank the referee for his Advice that ameliorat this manuscript

Abstract
Many autors have proposed Integral Basis of Biquadratic Number Fields, in 1984 Funakura was interested in Quartic Number Fields. D. Marcus in his book 1977, using the Dirichlet theorem proposed a theoretical method for Integral Basis of Number Field of degree n. In this paper using an other method, we will show the integral Basis of Quartic Number $K = Q(\sqrt[4]{p})$. *where* p *is any prime number.*

1 Introduction

Let K *be a number field of degree* n *and let* R *be the ring of integers of* K. R *is a free* $\mathbb{Z}$*-module of rank* n.*[4]. We call an integral basis of* K *any basis of the* $\mathbb{Z}$*-module* R. *It is known how to determine explicitly an integral basis of* K, *in the following cases :* $K = \mathbb{Q}(\sqrt{m})$ $m \in \mathbb{Z}$ *without quadratic factor),* $K = \mathbb{Q}(\sqrt[3]{m})$ (m *natural number without cubic factor),* K *cyclotomic field... For the case* $n = 4$, *several authors have determined an integral basis of* K *([1], [2], [5] ...). For the general case, Daniel A.Marcus[3] gives a theoretical method for determining an integral basis of* K. *Using the Marcus method, we propose to determine explicitly an integral basis of* $K = \mathbb{Q}(\sqrt[4]{p})$ *where* p *is a prime number.*

.

Theorem 1.1. *[3]*

Let $K = \mathbb{Q}(\alpha)$ *be a number field of degree* n *where* α *is an algebraic integer of* K. *Then there is an integral basis* $(1, \frac{f_1(\alpha)}{d_1}, ..., \frac{f_{n-1}(\alpha)}{d_{n-1}})$ *where* d_i *are in* $\mathbb{Z}$ *and satisfying* $d_1 \mid d_2 \mid ... \mid d_{n-1}$, *the* f_i *are monic polynomials over* $\mathbb{Z}$ *and* f_i *has degree* i, *the* d_i *are uniquely determined.*

Proposition 1.2.

*With the same notations of the theorem*1.1, *we have*

1) $disc(\alpha) = (d_1 d_2 ... d_{n-1})^2 disc(R)$, R *is the integer ring of* K.

2) *if* $i + j \leqslant n$ *then* $d_i d_j \mid d_{i+j}$

3) *for any* $i \leqslant n$ *we have :* $d_1^i \mid d_i$

4) $d_1^{n(n-1)} \mid disc(\alpha)$

5) *The* f_i *can be replaced by any other monic polynomials* $g_i \in \mathbb{Z}[X]$ *such that* g_i *has degree* i *and all* $\frac{g_i(\alpha)}{d_i}$ *are algebraic integers.*

Lemma 1.3.

*Using the same notations of the theorem*1.1, *put for every* i $(1 \leqslant i \leqslant n-1)$ $f_i = X^i + \sum_{j=0}^{i-1} a_{ji} X^j$
For any i $(1 \leqslant i \leqslant n-1)$ *and any* j $(0 \leqslant j \leqslant i-1)$ *let* q_{ji} *and* r_{ji} *respectively be the quotient*

and the remainder of the euclidean division of a_{ji} *by* d_i *and let* $g_i = X^i + \sum_{j=0}^{i-1} r_{ji}X^j$. *Then* $\frac{f_i(\alpha)}{d_i}$ *is an algebraic integer if and only if* $\frac{g_i(\alpha)}{d_i}$ *is an algebraic integer, Let* t $(0 \leqslant t \leqslant i-1)$ *and* $h_i = X^i + \sum_{j=0}^{i-1} b_j X^j$ *for where any* $j \neq t$, $b_j = a_{ji}$ *and* $b_t = a_{ti} + \delta d_i$ *and* $\delta = \pm 1$. *Then* $\frac{f_i(\alpha)}{d_i}$ *is an algebraic integer if and only if* $\frac{h_i(\alpha)}{d_i}$ *is an algebraic integer.*

We annonce the main result :

Theorem 1.4.

Let p *be a prime number,* $\omega = \sqrt[4]{p}$ *and* $K = \mathbb{Q}(\omega)$ *:*

1 *If* $p = 2$ *or* $p \equiv 3 \mod 4$ *then* $B = [1, \omega, \omega^2, \omega^3]$ *is an integral basis of* K.

2 *If* $p \equiv 5 \mod 8$ *then* $B = [1, \omega, \frac{1}{2}(\omega^2 - 1), \frac{1}{2}(\omega^3 - \omega)]$ *is an integral basis of* K.

3 *If* $p \equiv 1 \mod 8$ *then* $B = [1, \omega, \frac{1}{2}(\omega^2 - 1), \frac{1}{4}(\omega^3 - \omega^2 + \omega - 1)]$ *is an integral basis of* K.

Proof *Using the theorem*1.1, K *have an integral basis with the form*

$[1; \frac{f_1(\omega)}{d_1}; \frac{f_2(\omega)}{d_2}; \frac{f_3(\omega)}{d_3}]$

We will determine explicitly d_1, d_2 , d_3 , f_1, f_2 *and* f_3 , *we will use the previous lemma.*

we have : $disc(\omega) = -2^8 p^3$.

By proposition we have d_1^{12} *divides* $disc(\omega)$, *we conclude that* $d_1 = 1$ *for any* p.

As ω *is an algebraic integer, we take* $f_1(\omega) = \omega$ *for any* p.

Then we determine the possible values of d_i. *and we will start by the case :* $p \geqslant 3$.

2 Determination of d_2

From the proposition we have ${d_2}^4$ *divides* $2^8 p^3$

We conclude that $d_2 \in \{1, 2, 4\}$

We suppose $\gamma = \frac{\omega^2 + a\omega + b}{d_2}$ *with* a *and* b *are integer numbers satisfying :* $0 \leqslant a, b < d_2$.

γ *is an algebraic integer of* K *if and only if there are rational integers* $x, y, z,$ *and* t *satisfying :* $\gamma^4 + x\gamma^3 + y\gamma^2 + z\gamma + t = 0$. (E_1)

By replacing γ *in* (E_1), *reducing to the same denominator and using that* $(1, \omega, \omega^2, \omega^3)$ *is* $\mathbb{Q}$*-libre. We obtain the system :*

$$(S)\quad \begin{cases} a(a^2+6b)d_2x + 2ad_2^2y + 12ab^2 + 4a^3b + 4pa = 0 & (1) \\ (3a^2b + 3b^2 + p)d_2x + (a^2+2b)d_2^2y + d_2^3z + 4b^3 \\ +6a^2b^2 + 6a^2p + 4bp = 0 & (2) \\ (3ab^2 + 3ap)d_2x + 2abd_2^2y + ad_2^3z + 4ab^3 + 4a^3p \\ +12abp = 0 & (3) \\ (b^3 + 3a^2p + 3bp)d_2x + (b^2+p)d_2^2y + bd_2^3z + d_2^4t \\ +b^4 + 12a^2bp + a^4p + 6b^2p + p^2 = 0 & (4) \end{cases}$$

- If $a \neq 0$ *the system (S) becomes :*

$$(S_1)\quad \begin{cases} (a^2+6b)d_2x + 2d_2^2y + 12b^2 + 4a^2b + 4p = 0 & (1) \\ (3a^2b + 3b^2 + p)d_2x + (a^2+2b)d_2^2y + d_2^3z \\ +4b^3 + 6a^2b^2 + 6a^2p + 4bp = 0 & (2) \\ (3b^2 + 3p)d_2x + 2bd_2^2y + d_2^3z + 4b^3 + 4a^2p + 12bp = 0 & (3) \\ (b^3 + 3a^2p + 3bp)d_2x + (b^2+p)d_2^2y + bd_2^3z + d_2^4t + b^4 \\ +12a^2bp + a^4p + 6b^2p + p^2 = 0 & (4) \end{cases}$$

2.1 Case $d_2 = 4$

i) *If* $a = 0$

We suppose $\beta = \frac{\omega^2+b}{4} = \frac{\sqrt{p}+b}{4}$ *with* $b \in \{0,1,2,3\}$.

β *is an algebraic integer if and only if there are* $u, v \in \mathbb{Z}$ *such that :* $\beta^2 + u\beta + v = 0$.

Using $\{1, \sqrt{p}\}$ *is* $\mathbb{Q}$*-libre, we obtain* $u = \frac{-b}{2}$ *and* $v = \frac{b^2-p}{16}$. *As* $u, v \in \mathbb{Z}$ *we have* 2 *divides* b *and* 4 *divides* p*, which is impossible because* p *is a prime number. Then* $\beta = \frac{\omega^2+b}{4}$ *is not an algebraic integer.*

ii) *If* $a \neq 0$.

In this case the system S_1 *becomes :*

$$\begin{cases} (a^2+6b)x + 8y + p + 3b^2 + a^2b = 0 & (1)\\ (3a^2b+3b^2+p)2x + (a^2+2b)8y + 32z + 2b^3 + 3a^2b^2 + 3a^2p + 2bp = 0 & (2)\\ (3b^2+3p)x + 8by + ad_2^3y + b^3 + a^2p + 3bp = 0 & (3)\\ (b^3+3a^2p+3bp)4x + (b^2+p)16y + b4^3z + 4^4t + b^4 + 12a^2bp + a^4p + 6b^2p + p^2 = 0 & (4) \end{cases}$$

By solving this system we obtain :

$$\begin{cases} x = -b \\ y = \frac{3b^2-p}{8} \\ z = \frac{bp-b^3-a^2p}{16} \\ t = \frac{b^4+p^2+4a^2bp-a^4p-2b^2p}{256} \end{cases}$$

We give in the table Tab_1 *the values of* x, y, z *and* t *for different values of* $a \in \{1,2,3\}$ *and* $b \in \{0,1,2,3\}$.

Tab_1

(a, b)	x	y	z	t
(1,0)	0	$\frac{-p}{8}$	$\frac{-p}{16}$	$\frac{p(p-1)}{256}$
(1,1)	-1	$\frac{3-p}{8}$	$\frac{-1}{16}$	$\frac{p^2+p+1}{256}$
(1,2)	-2	$\frac{12-p}{8}$	$\frac{p-8}{16}$	$\frac{p^2-p+16}{256}$
(1,3)	-3	$\frac{27-p}{8}$	$\frac{2p-27}{16}$	$\frac{p^2-7p+81}{256}$
(2,0)	0	$\frac{-p}{8}$	$\frac{-p}{4}$	$\frac{p^2-16p}{256}$
(2,1)	-1	$\frac{3-p}{8}$	$\frac{-3p-1}{16}$	$\frac{(p-1)^2}{256}$
(2,2)	-2	$\frac{12-p}{8}$	$\frac{-p-4}{8}$	$\frac{(p+4)^2}{256}$
(2,3)	-3	$\frac{27-p}{8}$	$\frac{-p-27}{16}$	$\frac{p^2+14p+81}{256}$
(3,0)	0	$\frac{-p}{8}$	$\frac{-9p}{16}$	$\frac{p(p-81)}{256}$
(3,1)	-1	$\frac{3-p}{8}$	$\frac{-8p-1}{16}$	$\frac{p^2-47p+1}{256}$
(3,2)	-2	$\frac{12-p}{8}$	$\frac{-7p-8}{16}$	$\frac{(p-4)^2}{256}$
(3,3)	-3	$\frac{27-p}{8}$	$\frac{-6p-27}{16}$	$\frac{p^2-9p+81}{256}$

None of the 12 *cases of* (x, y, z, t) *is possible. We conclude that* $d_2 \neq 4$.

2.2 Case $d_2 = 2$

- If $d_2 = 2$ *the system* S_1 *becomes :*

$$\begin{cases} (a^3+6ab)x + 4ay + 6ab^2 + 2a^3b + 2ap = 0 & (1)\\ (3a^2b+3b^2+p)x + (a^2+2b)y + 4z + 2b^3 + 3a^2b^2 + 3a^2p + 2bp = 0 & (2)\\ (3ab^2+3ap)x + 4aby + 4az + 2ab^3 + 2a^3p + 6abp = 0 & (3)\\ 2(b^3+3a^2p+3bp)x + 4(b^2+p)y + 8bz + 16t + b^4 + 12a^2bp + a^4p + 6b^2p + p^2 = 0 & (4) \end{cases}$$

with $a, b \in \{0,1\}$

i) $a = b = 0$.

(E_1) becomes : $\begin{cases} px + 4z = 0 & (1) \\ \\ 4py + 16t + p^2 = 0(2) \end{cases}$

from equation (2), 4 divides p, which is impossible.

ii) $a = 0$ *and* $b = 1$. *We have* $\alpha = \frac{\omega^2+1}{2} = \frac{\sqrt{p}+1}{2}$. α *is an algebraic integer if and only if* $p \equiv 1 mod 4$.

iii) $a = 1$ *and* $b = 0$.

The resolution of the system (S_1) gives : $\begin{cases} x = 0 \\ y = \frac{-p}{2} \\ z = \frac{-p}{2} \\ t = \frac{p^2-p}{16} \end{cases}$ *But this solution is not in* $\mathbb{Z}^4$, *so*

$\alpha = \frac{\omega^2+\omega}{2}$ *is not an algebraic integer.*

iv) $a = b = 1$.

By resolving (S_3) we obtain : $\begin{cases} x = -2 \\ y = \frac{3-p}{2} \\ z = \frac{-1}{2} \\ t = \frac{p^2+p+1}{16} \end{cases}$

Observing that $x = -2$, $y = \frac{3-p}{2}$, $z = \frac{-1}{2}$, $t = \frac{p^2+p+1}{16} \notin \mathbb{Z}^4$, *we can deduce that* $\alpha = \frac{\omega^2+\omega+1}{2}$ *is not an algebraic integer.*

Conclusion 1: *If* $p \equiv 1 mod 4$ *we have* $d_2 = 2$.

2.3 Case $d_2 = 1$.

From the previous paragraph, if $p \equiv 3 mod 4$ *we have* $d_2 = 1$.

3 Determination of d_3

We suppose $\gamma = \omega^3 + a\omega^2 + b\omega + c$. *and* $\alpha = \frac{\omega^3+a\omega^2+b\omega+c}{d_3}$, α *is an algebraic integer if and only if the rational integers* x, y, z *and* t *exist*

such that : $\alpha^4 + x\alpha^3 + y\alpha^2 + z\alpha + t = 0$

$\frac{\gamma^4}{d_3^4} + x\frac{\gamma^3}{d_3^3} + y\frac{\gamma^2}{d_3^2} + \frac{\gamma}{d_3}\alpha + t = 0$

We reduce to the same denominator and simplify by $\frac{1}{d_3^4}$, *we obtain :*

$$\gamma^4 + xd_3\gamma^3 + yd_3^2\gamma^2 + zd_3^3\gamma + td_3^4 = 0$$

We replace in this equation γ *by its expression and using that* $\{1, \omega, \omega^2, \omega^3\}$ *is* $\mathbb{Q}$*-libre, we obtain the following system (S)*

$$\begin{cases} (3a^2p+3bp+6abc+b^3+3c^2)d_3x+(2bc+2c)d_3^2y+d_3^3z+4ap^2+12bcp \\ +12a^2cp+12ab^2p+4c^3+4a^3bp+12abc^2+4b^3c=0 \quad (1) \\ (6abp+3cp+a^3p+3b^2c+3ac^2)d_3x+(b^2+2ac+p)d_3^2y+ad_3^3z+6a^2p^2 \\ +6a^2b^2+4bp^2+24abcp+4b^3p+6c^2p+6a^2b^2p+4a^3cp+4ac^3+6b^2c^2=0 \quad (2) \\ (3b^2p+6acp+3a^2bp+3bc^2+p^2)d_3x+(2ap+2bc)d_3^2y+bd_3^3z+12abp^2 \\ +4cp^2+12b^2cp+4a^3p^2+12ac^2p+12a^2bcp+4ab^3p+4bc^3=0 \quad (3) \\ (3ap^2+6bcp+3ab^2p+3a^2cp+c^3)d_3x+(a^2p+c^2+2bp)d_3^2y+cd_3^3z+td_3^4 \\ +6b^2p^2+12acp^2+12a^2bp^2+12bc^2p+p^3+a^4p^2+12ab^2cp+6a^2c^2p+b^4p+c^4=0 (4) \end{cases}$$

with $0 \leqslant a,b,c < d_3$.
Using 1) in the proposition, we have : d_3^2 *divides* $disc(\omega) = -2^8p^3$.
it follows that $d_3 \in \{1,2,4,8,16,p,2p,4p,8p,16p\}$
So we have 10 *cases to distinguish.*

3.1 Case $d_3 = 16p$

Let $\alpha = \frac{\omega^3+a\omega^2+b\omega+c}{16p}$. *If* α *is an algebraic integer then* $\alpha\omega = \frac{a\omega^3+b\omega^2+c\omega+p}{16p}$ *is also an algebraic integer and his trace* $T(\alpha\omega) = \frac{1}{4}$ *will be a rational integer.*

So α *is not an algebraic integer and* $d_3 \neq 16p$.

3.2 Case $d_3 = 8p$

Let $\alpha = \frac{\omega^3+a\omega^2+b\omega+c}{8p}$

We have $T(\alpha\omega) = \frac{1}{2}$ *then it follows that* $\alpha\omega$ *is not an algebraic integer and also for* α.

Then we conclude that $d_3 \neq 8p$.

3.3 Case $d_3 = 4p$

The resolution of the system (S) with $d_3 = 4p$ *gives :*

$$\begin{cases} x = \frac{-c}{p} \\ y = \frac{3c^2-2bp-a^2p}{8p^2} \\ z = \frac{a^2cp+2bcp-c^3-ab^2p-ap^2}{16p^3} \\ t = \frac{2b^2p^2+a^4p^2+4ab^2cp+c^4+4acp^2-4a^2bp^2-4bc^2p-p^3-2a^2c^2p-b^4p}{256p^4} \end{cases}$$

We suppose that $x, y, z,$ *and* t *are the rational integers.*

Knowing that $x \in \mathbb{Z}$ *we have* p *divides* c *so* $c \in \{0, p, 2p, 3p\}$ *We do the same for* $y \in \mathbb{Z}$ *and* p *divides* c*, we have* p *divides* a^2+2b *We have* $z \in \mathbb{Z}$ *and* p *divides* c*, so we have* p *divides* ab^3*, then* p *divides* a *or* p *divides* b*. So* p *divides* a *or* p *divides* b *and* p *divides* a^2+2b*. It follows that* p *divides* a *and* p *divides* b*.*

In the conclusion, we have p *divides* a*,* p *divides* b *and* p *divides* c*, So we deduce that* a, b *and* $c \in \{0, p, 2p, 3p\}$.

We suppose $a = ps$*,* $b = pu$*,* $c = p = v$ *, and we replace in the expression of* t*, we obtain :*
$t = \frac{p(2u^2+p2s^4+4psu^2v+v^4+4sv-4ps^2u-4uv^2-2ps^2v^2-pu^4)-1}{256p}$

Knowing that $t \in \mathbb{Z}$ *we deduce that* p *divides* 1*, what is a contradicton.*

Then $d_3 \neq 4p$.

3.4 Case $d_3 = 2p$

We replace d_3 *by* $2p$ *in the system (S) and we solve it :*

$$\begin{cases} x = \frac{-2c}{p} \\ y = \frac{3c^2-2bp-a^2p}{2p^2} \\ z = \frac{a^2cp+2bcp-c^3-ab^2p-ap^2}{2p^3} \\ t = \frac{2b^2p^2+a^4p^2+4ab^2cp+c^4+4acp^2-4a^2bp^2-4bc^2p-p^3-2a^2c^2p-b^4p}{16p^4} \end{cases}$$

We suppose that $x, y, z,$ and t are the rational integers.

We have $x \in \mathbb{Z}$ and $p \neq 2$, so p divides c. It follows that $c \in \{0, p\}$ As the same $y \in \mathbb{Z}$ and p divides c we have p divides $a^2 + 2b$,

knowing that $z = \frac{(a^2+2b)cp-c^3-ab^2p-ap^2}{2p^3} \in \mathbb{Z}$, p divides c and p divides $a^2 + 2b$, so we have p divides a^2b. It follows that p divides a or b . Then p divides $a^2 + 2b$ and (p divides a or b). We deduce that p divides a and b as the following a, b, $c \in \{ 0, p\}$.

We suppose $a = ps$, $b = pu$, $c = pv$ and we replace theses values in the expression of t, we have : $t = \frac{2p^4u^2+p^6s^4+4p^5su^2v+p^4v^4+4p^4sv-4p^5s^2u-4p^4uv^2-p^5u^4-2p^5s^2v^2-p^3}{16p^4}$

knowing that $t \in \mathbb{Z}$ we deduce that p^4 divides p^3, so this is a contradiction.

Then $d_3 \neq 2p$.

3.5 Case $d_3 = p$

Let $\alpha = \frac{\omega^3+a\omega^2+b\omega+c}{p}$ with $a, b, c \in \{0, 1, ..., p-1\}$.

If α is an algebraic integer then its trace $T(\alpha) = \frac{4c}{p} \in \mathbb{Z}$, It follows that p divides c so $c = 0$.

We replace d_3 by p and c by 0, the system (S) becomes :

$$\begin{cases} (3a^2p + 3bp + b^3)x + 2abpy + p^2z + 12ab^2 + 4a^3b + 4ap = 0 & (1) \\ (6ab + a^3)px + (b^2 + p)py + ap^2z + 4b^3 + 6a^2b^2 + 6a^2p + 4bp = 0 & (2) \\ (3b^2 + 3a^2bp + p)px + 2ap^2y + bp^2z + 4ab^3 + 4a^3p + 12abp = 0 & (3) \\ (3ap + 3ab^2)px + (a^2 + 2p)p^2y + p^3t + b^4 + 12a^2bp + a^4p + 6b^2p + p^2 = 0 & (4) \end{cases}$$

From equation (4) we deduce that p divides b, then $b = 0$.

So the equation (4) becomes : $3apx + a^2py + p^2t + p + a^4 = 0$

We deduce that p divides a, then $a = 0$, So $a = b = c = 0$. The system (S) becomes :

$$\begin{cases} p^2z = 0 \\ p^2y = 0 \\ p^2x = 0 \\ p^3t = 0 \end{cases}$$

It follows that $x = y = z = t = 0$. So $\alpha^4 = 0$

We deduce that $0 = (\frac{\omega^3}{p})^4 = \frac{p^3}{p^4} = \frac{1}{p}$ is impossible .

Then $d_3 \neq p$.

3.6 Case $d_3 = 16$

Let $\alpha = \frac{\omega^3+a\omega^2+b\omega+c}{16}$

We suppose that α is an algebraic integer, we have also $\alpha\omega$ is an algebraic integer, We deduce that his trace $T(\alpha\omega) = \frac{p}{4}$ is a rational integer,

Then 4 divides p, is a contradiction .

Then α is not an algebraic integer and we deduce that $d_3 \neq 16$.

3.7 Case $d_3 = 8$

Let $\alpha = \frac{\omega^3+a\omega^2+b\omega+c}{8}$, We suppose that α is an algebraic integer, then $\alpha\omega = \frac{p+a\omega^3+b\omega^2+c\omega}{8}$ will be also an algebraic integer,

We deduce that its trace $T(\alpha\omega) = \frac{4p}{8} = \frac{p}{2}$ is in $\mathbb{Z}$, so we have a contradiction. Then $d_3 \neq 8$.

3.8 Case $d_3 = 4$

If $d_3 = 4$, *the system (S) becomes :*

$$\begin{cases} (3a^2p+3bp+6abc+b^3+3c^2)x+(2ab+2c)4y+16z+ap^2+3bcp \\ +3a^2cp+3ab^2p+c^3+a^3bp+3abc^2+b^3c=0 & (1) \\ (6abp+3cp+a^3p+3b^2c+3ac^2)2x+(b^2+2ac+p)8y+32az+3a^2p^2 \\ +3a^2b^2+2bp^2+12abcp+2b^3p+3c^2p+3a^2b^2p+2a^3cp+2ac^3+3b^2c^2=0 & (2) \\ (3b^2p+6acp+3a^2bp+3bc^2+p^2)x+(ap+bc)8y+16bz+3abp^2+cp^2 \\ +3b^2cp+a^3p^2+3ac^2p+3a^2bcp+ab^3p+bc^3=0 & (3) \\ (3ap^2+6bcp+3ab^2p+3a^2cp+c^3)4x+(a^2p+c^2+2bp)16y+64cz+256t \\ +6b^2p^2+12acp^2+12a^2bp^2+12bc^2p+p^3+a^4p^2+12ab^2cp+6a^2c^2p+b^4p+c^4=0 & (4) \end{cases}$$

After solving this system, we obtain :

$$\begin{cases} x=-c \\ y=\frac{3c^2-2bp-a^2p}{8} \\ z=\frac{a^2cp+2bcp-c^3-ab^2p-ap^2}{16} \\ t=\frac{2b^2p^2+a^4p^2+4ab^2cp+c^4+4acp^2-4a^2bp^2-4bc^2p-p^3-2a^2c^2p-b^4p}{256} \end{cases}$$

By using the lemma we will take a, b *and* c *in* $\{-1,0,1,2\}$
We suppose that x, y, z, *and* t *are the rational integers.*

i) *If c is even* ($c \in \{0,2\}$)

Using $y = \frac{3c^2-2bp-a^2p}{8} \in \mathbb{Z}$ *we have* 2 *divides* a^2p *and* $p > 2$ *so* 2 *divides* a *The same for* $t \in \mathbb{Z}$, *we deduce that* 2 *divides* $p(p^2+b^4)$ *so* b *is odd (* $b \in \{-1,1\}$*).*

- if $b=-1$ *we have* $y = \frac{3c^2+2p-a^2p}{8}$ *It follows that* 8 *divides* $p(2-a^2)$ *so* 8 *divides* $2-a^2$ *we deduce that* 4 *divides* 2, *so it's a contradiction.*

- if $b = 1$ *we have* $y = \frac{3c^2-2p-a^2p}{8}$. *we deduce that* 4 *divides* $2+a^2$ *so* 4 *divides* 2. *Then it's a contradiction. We conclude that* c *is odd.*

ii) *If c is odd (* $c \in \{-1,1\}$ *)*

Using $y = \frac{3c^2-2bp-a^2p}{8} \in \mathbb{Z}$, *we have* 2 *divides* $3c^2-a^2p$, *it follows that* a *is odd.*

as the same $z = \frac{a^2cp+2bcp-c^3-ab^2p-ap^2}{16} \in \mathbb{Z}$, *so we deduce that* b *is odd, Then* a, b *and* c *are odds.*

We give the values of x, y, z, t *in the table* Tab_2 *for* $a, b, c \in \{-1,1\}$.

Tab_2

(a,b,c)	x	y	z	t
(-1,-1,-1)	1	$\frac{3+p}{8}$	$\frac{(p+1)^2}{16}$	$\frac{-p^3+11p^2+5p+1}{256}$
(-1,-1,1)	-1	$\frac{3+p}{8}$	$\frac{p^2-1}{16}$	$\frac{-(p-1)^3}{256}$
(-1,1,-1)	1	$\frac{3-3p}{8}$	$\frac{(p-1)^2}{16}$	$\frac{-(p-1)^3}{256}$
(-1,1,1)	-1	$\frac{3-3p}{8}$	$\frac{p^2+4p-1}{16}$	$\frac{-p^3-5p^2-11p+1}{256}$
(1,-1,-1)	1	$\frac{3+p}{8}$	$\frac{1-p^2}{16}$	$\frac{-(p-1)^3}{256}$
(1,-1,1)	-1	$\frac{3+p}{8}$	$\frac{-(p+1)^2}{16}$	$\frac{-p^3+11p^2+5p+1}{256}$
(1,1,-1)	1	$\frac{3-3p}{8}$	$\frac{-p^2-4p+1}{16}$	$\frac{-p^3-5p^2-11p+1}{256}$
(1,1,1)	-1	$\frac{3-3p}{8}$	$\frac{-(p-1)^2}{16}$	$\frac{-(p-1)^3}{256}$

Conclusion2**:**

i) *If* $p \equiv 1 mod 8$, *we suppose* $p = 8k+1$, *then we have :*

$\frac{3-3p}{8} = -3k \in \mathbb{Z}$.

$\frac{(p-1)^2}{16} = 4k^2 \in \mathbb{Z}$.

$\frac{(p-1)^3}{256} = 2k^3 \in \mathbb{Z}$.

It follows for $(a,b,c) = (1,1,1)$ *that :* $x = -1$, $y = \frac{3-3p}{8}$, $z = \frac{(p-1)^2}{16}$ *and* $t = \frac{(p-1)^3}{256}$ *are rational integers. As the following* $\alpha = \frac{\omega^3+\omega^2+\omega+1}{4}$ *is an algebraic integer. Then if* $p \equiv 1 mod 8$, *we have* $d_3 = 4$.

If $p \equiv 3 mod 8$, *we suppose* $p = 8k+3$, *then we have :*

- for $(a,b,c) \in \{(-1,-1,-1),(-1,-1,1),(1,-1,-1),(1,-1,1)\}$

$y = \frac{3+p}{8} = k + \frac{3}{4} \notin \mathbb{Z}$.

- for $(a,b,c) \in \{(-1,1,-1),(-1,1,1),(1,1,-1),(1,1,1)\}$.

$y = \frac{3-3p}{8} = -3k - \frac{3}{4} \notin \mathbb{Z}$.

It follows that : for $p \equiv 3 mod 8$, $d_3 \neq 4$.

If $p \equiv 5 mod 8$, *we suppose* $p = 8k+5$, *then we have :*

$\frac{(p+1)^2}{16} = 4k^2 + 6k + \frac{9}{4} \notin \mathbb{Z}$. $\frac{p^2-1}{16} = 4k^2 + 5k + \frac{3}{2} \notin \mathbb{Z}$.

$\frac{p^2+4p-1}{16} = 4k^2 + 7k + \frac{11}{4} \notin \mathbb{Z}$. $frac3 - 3p8 = -3k - \frac{3}{2} \notin \mathbb{Z}$. *It follows that : - for* $(a,b,c) \in \{(-1,1,-1),(-1,1,1),(1,1,-1),(1,1,1)\}$

$y = \frac{3-3p}{8} \notin \mathbb{Z}$. *so* $\alpha = \frac{\omega^3+a\omega^2+b\omega+c}{4}$ *is not an algebraic integer.*

- for $(a,b,c) \in \{(-1,-1,1),(1,-1,-1)\}$.

$z = \frac{p^2-1}{16}$ *or* $z = -\frac{p^2-1}{16}$ *is not a rational integer.*

As the following $\alpha = \frac{\omega^3+a\omega^2+b\omega+c}{4}$ *is not an algebraic integer. - for* $(a,b,c) \in \{(-1,1,1),(1,1,-1)\}$. $y = \frac{p^2+4p-1}{16}$ *or* $y = -\frac{p^2+4p-1}{16}$ *is not a rational integer. Then in this case* $\alpha = \frac{\omega^3+a\omega^2+b\omega+c}{4}$ *is not an algebraic integer. So for* $p \equiv 5 mod 8$, *we have* $d_3 \neq 4$.

iv) *If* $p \equiv 7 mod 8$, *we pose* $p = 8k+7$, *then we have :* $\frac{3+p}{8} = k + \frac{5}{4} \notin \mathbb{Z}$. *et* $\frac{3-3p}{8} = -3k - \frac{9}{4} \notin \mathbb{Z}$.

It follows that : for any triplet (a,b,c) $y \notin \mathbb{Z}$. *Then* $\alpha = \frac{\omega^3+a\omega^2+b\omega+c}{4}$ *is not an algebraic integer.*

We conclude that : if $p \equiv 7 mod 8$, $d_3 \neq 4$.

3.9 Case $d_3 = 2$

Let $\alpha = \frac{\omega^3+a\omega^2+b\omega+c}{16}$ *with* $a,b,c \in \{0,1\}$

The $\mathbb{Q}$*-isomorphisms of* $K = \mathbb{Q}(\omega)$ *are defined by :*

$\sigma_1(\omega) = \omega$.

$\sigma_2(\omega) = -\omega$.

$\sigma_3(\omega) = i\omega$.

and $\sigma_4(\omega) = -i\omega$.

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ *denote the conjugates of* α *over* $\mathbb{Q}$, *then we have*

$\alpha_1 = \sigma_1(\alpha) = \frac{\omega^3+a\omega^2+b\omega+c}{2}$

$\alpha_2 = \sigma_2(\alpha) = \frac{-\omega^3+a\omega^2-b\omega+c}{2}$

$\alpha_3 = \sigma_3(\alpha) = \frac{c-a\omega^2-i(\omega^3-b\omega)}{2}$

$\alpha_4 = \sigma_4(\alpha) = \frac{c-a\omega^2+i(\omega^3-b\omega)}{2}$

The minimal polynomial P *of* α *over* $\mathbb{Q}$ *is :*

$P = X^4 - S_1X^3 + S_2X^2 - SX + S_4$.

where $S_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ $S_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4$

$S_3 = \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4$

$S_4 = \alpha_1\alpha_2\alpha_3\alpha_4$

After calculating, we have : $S_1 = -2c$, $S_2 = \frac{3c^2-2bp-a^2p}{2}$, $S_3 = \frac{ap^2+c^3+ab^2p-a^2cp-2bcp}{2}$, $S_4 = \frac{2b^2p^2+a^4p^2+4ab^2cp+c^4+4acp^2-4a^2bp^2-4bc^2p-p^3-2a^2c^2p-b^4p}{16}$

α is an algebraic integer if and only if S_1, S_2, S_3 *and* S_4 *are rational integers.*

We give in the table Tab_3 *the values of* S_1, S_2, S_3 *and* S_4 *when the values of* a, b, c *are in* $\{0, 1\}$.

Tab_3

(a,b,c)	S_1	S_2	S_3	S_4
(0,0,0)	0	0	0	$\frac{-p^3}{16}$
(0,0,1)	-2	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1-p^3}{16}$
(0,1,0)	0	$-p$	0	$-\frac{p(p-1)^2}{16}$
(0,1,1)	-2	$\frac{3-2p}{2}$	$\frac{1-2p}{2}$	$\frac{-p^3+2p^2-5p+1}{16}$
(1,0,0)	0	$\frac{-p}{2}$	$\frac{p^2}{2}$	$\frac{p^2(1-p)}{16}$
(1,0,1)	-2	$\frac{3-p}{2}$	$\frac{p^2+1-p}{2}$	$\frac{-p^3+5p^2-2p+1}{16}$
(1,1,0)	0	$\frac{-3p}{2}$	$\frac{p(p+1)}{2}$	$\frac{-p^3-p^2-p}{16}$
(1,1,1)	-2	$\frac{3-3p}{2}$	$\frac{(p-1)^2}{2}$	$\frac{-(p-1)^3}{16}$

Conclusion*3*:

i) *If* $p \equiv 1 mod 4$, *we have : for the triplet* $(a, b, c) = (0, 1, 0)$.

$(S_1 = 0,\ S_2 = -p,\ S_3 = 0,\ S_4 = \frac{p(p-1)^2}{16}) \in \mathbb{Z}^4$.

It follows that : $\alpha = \frac{\omega^3+\omega}{2}$ *is an algebaic integer, so* $d_3 = 2$.

ii) *If* $p \equiv 3 mod 4$, *we have for any* (a, b, c) $(S_1, S_2, S_3, S_4) \notin \mathbb{Z}^4$. *Then for* $p \equiv 3 mod 4$, *we have* $d_3 \neq 2$.

4 Particular Case : $p = 2$.

Let $\omega = \sqrt{2}$.

We have $disc(\omega) = -2^{11}$.

i) *we know that* d_1^{12} *divides* $disc(\omega)$, *It follows that* $d_1 = 1$.

ii) d_2^4 *divides* $disc(\omega) = -2^{11}$. *It follows that* $d_2 \in \{1, 2, 4\}$.

4.1 Determination of d_2.

1- *Case* $d_2 = 4$

Let $\gamma = \omega^2 + a\omega + b$ *and* $\alpha = \frac{\gamma}{4}$ *with* $a, b \in \{0, 1, 2, 3\}$. *α is an algebraic integer if and only if there are four rational integers* x, y, z *and* t, *such that :* $\alpha^4 + x\alpha^3 + y\alpha^2 + z\alpha + t = 0$.

By replacing α *by* $\frac{\gamma}{4}$ *we obtain the equation :*

$\gamma^4 + 4x\gamma^3 + 4^2y\gamma^2 + 4^3z\gamma + 4^4t = 0$ *By replacing* γ *by* $\omega^2 + a\omega + b$ *and using that* $\{1, \omega, \omega^2, \omega^3\}$ *is* $\mathbb{Q}$*-libre, we obtain the system :*

$$\begin{cases} (a^3+6ab)x + 8ay + 3ab^2 + a^3b + 2a = 0 & (1) \\ 2(3a^2b+3b^2+2)x + 8(a^2+2b)y + 32z + 2b^3 + 3a^2b^2 + 6a^2 + 4b = 0 & (2) \\ (3ab^2+6a)x + 8aby + 16az + ab^3 + 2a^3 + 6ab = 0 & (3) \\ 4(b^3+6a^2+6b)x + 4^2(b^2+2)y + 4^3bz + 4^4t + b^4 + 24a^2b + 2a^4 + 12b^2 + 4 = 0 & (4) \end{cases}$$

We suppose that x, y, z and t are in $\mathbb{Z}$.

(4) $\Longrightarrow$ *2 divides $b \Longrightarrow 2$ divides a.*

So $a, b \in \{0, 2\}$.

i) $a = b = 0$

(4) becomes $8y + 64t + 1 = 0$ it is a contradiction.

ii) $a = 0$ *and* $b = 2$

The equation (4) becomes :

$20x + 24y + 32z + 64t + 17 = 0$

it's also a contradiction.

iii) $a = 2$ *and* $b = 0$

The equation (4) becomes :

$24x + 8y + 64t + 9 = 0$ *we have also a contradiction.*

iv) $a = b = 2$

In this case the equation (4) becomes : $8x + 4y + 11 = 0$ so it'is a contradiction.

In conclusion $d_2 \neq 4$.

2- *Case $d_2 = 2$.*

Let $\gamma = \omega^2 + a\omega + b$ and $\alpha = \frac{\gamma}{2}$ with $a, b \in \{0, 1\}$. α is an algebraic integer if and only if there are four rational integers x, y, z and t so that : $\alpha^4 + x\alpha^3 + y\alpha^2 + z\alpha + t = 0$. $\frac{\gamma^4}{2^4} + x\frac{\gamma^3}{2^3} + y\frac{\gamma^2}{2^2} + z\frac{\gamma}{2} + t = 0$. $\gamma^4 + 2xy^3 + 4y\gamma^2 + 8z\gamma + 16t = 0$.

In the last equation we replace γ by its expression and using $\{1, \omega, \omega^2, \omega^3\}$ is $\mathbb{Q}$-libre we obtain the system :

$$\begin{cases} 2(3ab^2 + a^3 + 2a) + (a^3 + 6ab)x + 4ay + 4z = 0 & (1) \\ 2b^3 + 3a^2b^2 + 6a^2 + 4b + (3a^2b + 3b^2 + 2)x + 2(a^2 + 2b)y + 4z = 0 & (2) \\ 2ab^3 + 4a^3 + 12ab + (3ab^2 + 6a)x + 4aby + 4az = 0 & (1) \\ b^4 + 24a^2b + 2a^4 + 12b^2 + 4 + 2(b^3 + 6a^2 + 6b)x + 4(b^2 + 2)y + 8bz + 16t = 0 & (4) \end{cases}$$

we suppose that x, y, z and t are in $\mathbb{Z}$, with equation (4) we deduce that 2 divides b it follows that $b \in \{0, 1\}$ so $b = 0$.

Replacing b by 0, the equation (4) becomes : $a^4 + 2 + 6a^2x + 4y + 8t = 0$ It follows that 2 divides a and so $a = 0$.Then $a = b = 0$. and $\alpha = \frac{\sqrt{2}}{2}$.

As $\frac{\sqrt{2}}{2}$ is not an algebraic integer, we conclude that $d_2 \neq 2$.

***Conclusion*4:** *We have $d_2 = 1$.*

4.2 Determination of d_3.

We distinguish d_3 in $\{1, 2, 2^2, 2^3, 2^4, 2^5\}$

1- *Case $d_3 = 32$*

Let $\beta = \alpha\omega = \frac{a\omega^3 + b^2 + c\omega + 2}{32}$, if α is an algebraic integer, it's the same for β and his trace $T(\beta)$ is a rational integer. But $T(\beta) = \frac{1}{4}$ is a contradiction. Then $d_3 \neq 32$.

2- *Case $d_3 = 16$.*

Let $\beta = \alpha\omega = \frac{a\omega^3 + b\omega^2 + c\omega + 2}{16}$, if α is an algebraic integer, it's the same for β and his trace $T(\beta)$ is a rational integer. But $T(\beta) = \frac{1}{2}$. Then we have a contradiction. So $d_3 \neq 16$.

3- *Case* $d_3 = 8$

Let $\alpha = \frac{\omega^3+a\omega^2+b\omega+c}{8}$ *(Using the calculs of paragraph 3-7) with* $p = 2$ *we have the following expressions:*

$$\begin{cases} x = \frac{-c}{2} \\ y = \frac{3c^2-4b-2a^2}{32} \\ z = \frac{2a^2c+4bc-c^3-2ab^2-4a}{128} \\ t = \frac{8b^2+4a^4+8ab^2c+c^4+16ac-16a^2b-8bc^2-8-4a^2c^2-2b^4}{4096} \end{cases}$$

with $a, b, c \in \{0,1,2,3,4,5,6,7\}$. *Knowing* $x \in \mathbb{Z}$ *and* $x = T(\alpha) = -\frac{c}{2}$ *we have c is even,* y *is in* $\mathbb{Z}$ *and c is even, we deduce that* a *is even.* $t \in \mathbb{Z}$, a *and* c *are evens, we deduce that* b *is also even. We suppose that* $a = 2s$, $b = 2u$, $c = 2v$ *and we replace in* t, *we obtain :* $t = \frac{2(2u^2+4s^2+8su^2v+v^4+4sv-8s^2u-4uv^2-4s^2v^2-2u^4)-1}{512}$
What is a contradiction because the numerator of t *is odd. Then* $d_3 \neq 8$.

4) *Case* $d_3 = 4$

Let $\alpha = \frac{\omega^3+a\omega^2+b\omega+c}{4}$ *(Using the calculs of paragraph 3-8) with* $p = 2$ *we have the following expressions:*

$$\begin{cases} x = c \\ y = \frac{3c^2-4b-2a^2}{/}8 \\ z = \frac{2a^2c+4bc-c^3-2ab^2-4a}{16} \\ t = \frac{8b^2+4a^4+8ab^2c+c^4+16ac-16a^2b-8bc^2-8-4a^2c^2-2b^4}{256} \end{cases}$$

with $a, b, c \in \{0,1,2,3\}$, *If c is even, from* $y \in \mathbb{Z}$ *we deduce that* a *is even, and from* $t \in \mathbb{Z}$ *we obtain b is even. We suppose that*

$a = 2s$, $b = 2u$, $c = 2v$ *and replace this in* t, *we obtain :* $t = \frac{2(2u^2+4s^2+8su^2v+v^4+4sv-8s^2u-4uv^2-4s^2v^2-2u^4)-}{64}$
This is a contradiction because the numerator of t *is odd. If c is odd then* $y = \frac{3c^2-4b-2a^2}{8} \notin \mathbb{Z}$ *because the numerator of* y *is odd. We deduce that :* $d_3 \neq 4$.

5) *Case* $d_3 = 2$

Let $\alpha = \frac{\omega^3+a\omega^2+b\omega+c}{2}$ *(Using the calculs of paragraph 3-9) (with* $p = 2$*) we have the following solutions:*

$$\begin{cases} x = -2c \\ y = \frac{3c^2-4b-2a^2}{2} \\ z = \frac{2a^2c+4bc-c^3-2ab^2-4a}{2} \\ t = \frac{8b^2+4a^4+8ab^2c+c^4+16ac--16a^2b-8bc^2-8-4a^2c^2-2b^4}{16} \end{cases}$$

with $a, b, c \in \{0,1\}$, *If* $c = 0$ *then* $t = \frac{4b^2+2a^4--8a^2b-4-b^4}{8} \in \mathbb{Z}$ *this implies that b is even. So* $b = 0$, *we deduce that* $t = \frac{6a^4-1}{2}$. *So it's a contradiction. If* $c = 1$ *then* $y = \frac{3-4b-2a^2}{2} \notin \mathbb{Z}$ *because* $3 - 4b - 2a^2$ *is odd. Then* $d_3 \neq 2$. *In conclusion we have* $d_3 = 1$.

Corollary 4.1. *Let* p *be a prime number,* $\omega = \sqrt[4]{p}$ *and* $K = \mathbb{Q}(\omega)$ *then :*

1) *If* $p = 2$ *or* $p \equiv 3 \mod 4$ *we have :* $disc(K) = -2^8p^3$.

2) *If* $p \equiv 5 \mod 8$ *we have :* $disc(K) = -2^4p^3$.

3) *If* $p \equiv 1 \mod 8$ *we have :* $disc(K) = -2^2p^3$.

Remark 4.2. Let m be an integer number without quartic factors, $\omega = \sqrt[4]{m}$ and $K = \mathbb{Q}(\omega)$. By theorem1, K admits an integral basis of the form : $(1; \frac{f_1(\omega)}{d_1}; \frac{f_2(\omega)}{d_2}; \frac{f_3(\omega)}{d_3})$
Let's put $m = fg^2h^3$ where f, g and h are integers mutualy coprimes $f = a_1a_2...a_r$, $g = b_1b_2...b_s$ and $h = c_1c_2...c_t$ where $a_1, .., a_r$, $b_1, ..., b_s$, $c_1, ...c_t$ are prime numbers We have $disc(\omega) = -2^8m^3 = -2^8a_1^3...a_r^3b_1^6...b_s^6c_1^9...c_t^9$. Under the proposal we have d_2^4 divides $disc(\omega)$. It follows that $d_2 \in \{1, 2, 4, a_1, ..., a_r, b_1, ..., b_s, c_1, ..., c_s, ...\}$ Which proves the complexity of the determination of d_2. This is how limited ourselves to the case $m = p$ where p is a prime integer.

References

[1] *T. Funakura : On integral bases of pure quartic fields, Mathematical Journal of Okayama University. Volume 26. Issue 1 (1984).*

[2] *J.G.Huard, B.K.Spearman and K.S.Williams: Integral bases for quartic Fields with Quadratic Subfields, Journal of Number Theory 51. 87-102 (1995).*

[3] *D.A.Marcus : Number Fields Springer-Verlag, (1977).*

[4] *P.Samuel : Théorie algébrique des nombres. Hermann, (1967).*

[5] *K.S.Williams : Integer of biquadratic Fields, Canad. Math. Bull.13 N°4, 519-526 (1970).*

Mouhcine TALJAOUI and Mostapha BOUHAMZA
Department of Mathematics and Computing Science
Faculty of Science Ain Chock
Hassan 2 University BOP 5366
Casablanca 20100 Morocco
taljaoui@gmail.com (M. TALJAOUI)
mosbouhamza@yahoo.fr (M. BOUHAMZA)

Author information

M. TALJAOUI and M. BOUHAMZA, Laboratory of Fondamental Mathematic and Applications, FSAC, Hassan2 University, BP 5366 Casablanca 20100., Morocco..
E-mail: taljaoui@gmail.com (M. TALJAOUI)

Received: 2022-07-11
Accepted: 2023-04-16