Integral bases of some families of quartic number fields.

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Abstract

Many authors have proposed Integral Basis of Biquadratic Number Fields, in 1984 Funakura was interested in Quartic Number Fields. D. Marcus in his book 1977, using the Dirichlet theorem proposed a theoretical method for Integral Basis of Number Field of degree n. In this paper using an other method, we will show the integral Basis of Quartic Number Field \( K = \mathbb{Q}(\sqrt[4]{p}) \), where \( p \) is any prime number.

1 Introduction

Let \( K \) be a number field of degree \( n \) and let \( R \) be the ring of integers of \( K \). \( R \) is a free \( \mathbb{Z} \)-module of rank \( n \). We call an integral basis of \( K \) any basis of the \( \mathbb{Z} \)-module \( R \). It is known how to determine explicitly an integral basis of \( K \), in the following cases:

- \( K = \mathbb{Q}(\sqrt{m}) \) \( m \in \mathbb{Z} \) without quadratic factor,
- \( K = \mathbb{Q}(\sqrt[3]{m}) \) \( m \) natural number without cubic factor,
- \( K \) cyclotomic field...

For the case \( n = 4 \), several authors have determined an integral basis of \( K \) \([1], [2], [5]...,\) For the general case, Daniel A. Marcus \([3]\) gives a theoretical method for determining an integral basis of \( K \). Using the Marcus method, we propose to determine explicitly an integral basis of \( K = \mathbb{Q}(\sqrt[4]{p}) \) where \( p \) is a prime number.

Theorem 1.1. \([3]\)

Let \( K = \mathbb{Q}(\alpha) \) be a number field of degree \( n \) where \( \alpha \) is an algebraic integer of \( K \). Then there is an integral basis \( (1, f_1(\alpha), \ldots, f_{n-1}(\alpha)) \) where \( d_i \) are in \( \mathbb{Z} \) and satisfying \( d_1 \mid d_2 \mid \ldots \mid d_{n-1} \), the \( f_i \) are monic polynomials over \( \mathbb{Z} \) and \( f_i \) has degree \( i \), the \( d_i \) are uniquely determined.

Proposition 1.2.

With the same notations of the theorem 1.1, we have

1) \( \text{disc}(\alpha) = (d_1d_2\ldots d_{n-1})^2\text{disc}(R) \), \( R \) is the integer ring of \( K \).

2) if \( i + j \leq n \) then \( d_i \mid d_{i+j} \)

3) for any \( i \leq n \) we have : \( d_1 \mid d_i \)

4) \( d_1^{n(n-1)} \mid \text{disc}(\alpha) \)

5) The \( f_i \) can be replaced by any other monic polynomials \( g_i \in \mathbb{Z}[X] \) such that \( g_i \) has degree \( i \) and all \( \frac{g_i(\alpha)}{d_i} \) are algebraic integers.

Lemma 1.3.

Using the same notations of the theorem 1.1, put for every \( i \) \( (1 \leq i \leq n - 1) \) \( f_i = X^i + \sum_{j=0}^{i-1} a_{ij}X^j \) For any \( i \) \( (1 \leq i \leq n - 1) \) and any \( j \) \( (0 \leq j \leq i - 1) \) let \( q_{ji} \) and \( r_{ji} \) respectively be the quotient
and the remainder of the euclidean division of \(a_{ji}\) by \(d_i\) and let \(g_i = X^i + \sum_{j=0}^{i-1} r_{ji} X^j\). Then \(\frac{f_i(\alpha)}{d_i}\) is an algebraic integer if and only if \(\frac{a_{(\alpha)}}{d_i}\) is an algebraic integer. Let \(t\) \((0 \leq t \leq i - 1)\) and \(h_i = X^i + \sum_{j=0}^{i-1} b_j X^j\) for where any \(j \neq t\), \(b_j = a_{ji}\) and \(b_t = a_{ti} + \delta d_i\) and \(\delta = \pm 1\). Then \(\frac{f_i(\alpha)}{d_i}\) is an algebraic integer if and only if \(\frac{b_i}{d_i}\) is an algebraic integer.

We announce the main result :

**Theorem 1.4.**

Let \(p\) be a prime number, \(\omega = \sqrt{p}\) and \(K = \mathbb{Q}(\omega) :\)

1. If \(p \equiv 2 \mod 4\) then \(B = [1, \omega, \omega^2, \omega^3]\) is an integral basis of \(K\).
2. If \(p \equiv 5 \mod 8\) then \(B = [1, \omega, \frac{1}{2}(\omega^2 - 1), \frac{1}{2}(\omega^3 - \omega)]\) is an integral basis of \(K\).
3. If \(p \equiv 1 \mod 8\) then \(B = [1, \omega, \frac{1}{2}(\omega^2 - 1), \frac{1}{2}(\omega^3 - \omega^2 + \omega - 1)]\) is an integral basis of \(K\).

**Proof** Using the theorem 1.1, \(K\) have an integral basis with the form

\[\left[1, \frac{f_i(\omega)}{d_i}, \frac{f_2(\omega)}{d_2}, \frac{f_3(\omega)}{d_3}\right]\]

We will determine explicitly \(d_1, d_2, d_3, f_1, f_2\) and \(f_3\), we will use the previous lemma. We have : \(\text{disc}(\omega) = -2^6 p^3\).

By proposition we have \(d_1^2\) divides \(\text{disc}(\omega)\), we conclude that \(d_1 = 1\) for any \(p\).

As \(\omega\) is an algebraic integer, we take \(f_1(\omega) = \omega\) for any \(p\).

Then we determine the possible values of \(d_i\), and we will start by the case : \(p \geq 3\).

## 2 Determination of \(d_2\)

From the proposition we have \(d_2^4\) divides \(2^8 p^3\)

We conclude that \(d_2 \in \{1, 2, 4\}\).

We suppose \(\gamma = \frac{a^2 + \omega b}{d_2}\) with \(a\) and \(b\) are integer numbers satisfying : \(0 \leq a, b < d_2\).

\(\gamma\) is an algebraic integer of \(K\) if and only if there are rational integers \(x, y, z,\) and \(t\) satisfying :

\[\gamma^4 + x \gamma^3 + y \gamma^2 + z \gamma + t = 0.\]  \((E_1)\)

By replacing \(\gamma\) in \((E_1)\), reducing to the same denominator and using that \((1, \omega, \omega^2, \omega^3)\) is \(\mathbb{Q}\)-libre. We obtain the system :

\[
\begin{align*}
(1) & \quad a(a^2 + 6b)d_2x + 2ad_2^2y + 12ab^2 + 4a^3b + 4pa = 0 \\
(2) & \quad (3a^2b + 3b^2 + p)d_2x + (a^2 + 2b)d_2^2y + \alpha d_2z + 4b^3 + 6a^2b^2 + 6a^3p + 4bp = 0 \\
(3) & \quad (3ab^2 + 3ap)d_2x + 2abd_2^2y + ad_2z + 4ab^3 + 4a^3p + 12abp = 0 \\
(4) & \quad (b^3 + 3a^2b + 3bp)d_2x + (b^2 + p)d_2^2y + bd_2z + d_2^2t + b^4 + 12a^2bp + a^4p + 6b^2p + p^2 = 0 \\
\end{align*}
\]

- If \(a = 0\) the system \((S)\) becomes :

\[
\begin{align*}
(1) & \quad (a^2 + 6b)d_2x + 2d_2^2y + 12b^2 + 4a^2b + 4p = 0 \\
(2) & \quad (3a^2b + 3b^2 + p)d_2x + (a^2 + 2b)d_2^2y + d_2^2z + 4b^3 + 6a^2b^2 + 6a^3p + 4bp = 0 \\
(3) & \quad (b^3 + 3a^2b + 3bp)d_2x + (b^2 + p)d_2^2y + bd_2z + d_2^2t + b^4 + 12a^2bp + a^4p + 6b^2p + p^2 = 0 \\
\end{align*}
\]

2.1 Case \(d_2 = 4\)

i) If \(a = 0\)
We suppose $\beta = \frac{\omega^2 + b}{4} = \frac{\sqrt{p} + b}{4}$ with $b \in \{0, 1, 2, 3\}$.

$\beta$ is an algebraic integer if and only if there are $u, v \in \mathbb{Z}$ such that: $\beta^2 + u\beta + v = 0$.

Using $\{1, \sqrt{p}\}$ is $\mathbb{Q}$-libre, we obtain $u = \frac{-b}{2}$ and $v = \frac{b^2 - p}{16}$. As $u, v \in \mathbb{Z}$ we have 2 divides $b$ and 4 divides $p$, which is impossible because $p$ is a prime number. Then $\beta = \frac{\omega^2 + b}{4}$ is not an algebraic integer.

ii) If $a \neq 0$.

In this case the system $S_1$ becomes:

\[
\begin{align*}
(a^2 + 6b)x + 8y + p + 3b^2 + a^2b &= 0 \quad (1) \\
(3a^2b + 3b^2 + p)x + (a^2 + 2b)8y + 32z + 2b^3 + 3a^2b^2 + 3a^2p + 2bp &= 0 \quad (2) \\
(3b^2 + 3p)x + 8by + ad_2y + b^3 + a^2p + 3bp &= 0 \quad (3) \\
(b^3 + 3a^2p + 3bp)4x + ((b^2 + p)16y + b4^3z + 4^4t + b^4 + 12a^2bp + a^4p + 6b^2p + p^2 &= 0(4) \\
\end{align*}
\]

By solving this system we obtain:

\[
\begin{align*}
&\begin{align*}
&x = -b \\
y &= \frac{3p - p}{8} \\
z &= \frac{bp - b^2 - ap}{p} \\
t &= \frac{8^4 + 4p^2 - 4p - 2a^2p}{256}
&\end{align*}
\end{align*}
\]

We give in the table $\text{Tab}1$ the values of $x, y, z$ and $t$ for different values of $a \in \{1, 2, 3\}$ and $b \in \{0, 1, 2, 3\}$.

<table>
<thead>
<tr>
<th>$(a, b)$</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 0)$</td>
<td>0</td>
<td>$-\frac{p}{8}$</td>
<td>$-\frac{p}{16}$</td>
<td>$\frac{p(p - 1)}{256}$</td>
</tr>
<tr>
<td>$(1, 1)$</td>
<td>$-1$</td>
<td>$\frac{3p}{8}$</td>
<td>$\frac{p}{16}$</td>
<td>$\frac{p^2 + p + 1}{256}$</td>
</tr>
<tr>
<td>$(1, 2)$</td>
<td>$-2$</td>
<td>$\frac{12 - p}{8}$</td>
<td>$\frac{p - 8}{16}$</td>
<td>$\frac{p^2 - p + 16}{256}$</td>
</tr>
<tr>
<td>$(1, 3)$</td>
<td>$-3$</td>
<td>$\frac{22 - p}{8}$</td>
<td>$\frac{2a - 27}{16}$</td>
<td>$\frac{p^2 - 7p + 81}{256}$</td>
</tr>
<tr>
<td>$(2, 0)$</td>
<td>0</td>
<td>$-\frac{p}{8}$</td>
<td>$-\frac{p}{16}$</td>
<td>$\frac{p^2 - 16p}{256}$</td>
</tr>
<tr>
<td>$(2, 1)$</td>
<td>$-1$</td>
<td>$\frac{3p}{8}$</td>
<td>$\frac{3p - 1}{16}$</td>
<td>$\frac{(p - 1)^2}{256}$</td>
</tr>
<tr>
<td>$(2, 2)$</td>
<td>$-2$</td>
<td>$\frac{12 - p}{8}$</td>
<td>$\frac{p - 4}{16}$</td>
<td>$\frac{p^2 + 4p + 81}{256}$</td>
</tr>
<tr>
<td>$(2, 3)$</td>
<td>$-3$</td>
<td>$\frac{27 - p}{8}$</td>
<td>$\frac{p - 27}{16}$</td>
<td>$\frac{p^2 + 14p + 81}{256}$</td>
</tr>
<tr>
<td>$(3, 0)$</td>
<td>0</td>
<td>$-\frac{p}{8}$</td>
<td>$\frac{7p - 8}{16}$</td>
<td>$\frac{p^2 - 9p + 81}{256}$</td>
</tr>
<tr>
<td>$(3, 1)$</td>
<td>$-1$</td>
<td>$\frac{3p}{8}$</td>
<td>$\frac{3p - 1}{16}$</td>
<td>$\frac{p^2 - 47p + 81}{256}$</td>
</tr>
<tr>
<td>$(3, 2)$</td>
<td>$-2$</td>
<td>$\frac{12 - p}{8}$</td>
<td>$\frac{p - 8}{16}$</td>
<td>$\frac{(p - 4)^2}{256}$</td>
</tr>
<tr>
<td>$(3, 3)$</td>
<td>$-3$</td>
<td>$\frac{27 - p}{8}$</td>
<td>$\frac{p - 27}{16}$</td>
<td>$\frac{p^2 - 9p + 81}{256}$</td>
</tr>
</tbody>
</table>

None of the 12 cases of $(x, y, z, t)$ is possible. We conclude that $d_2 \neq 4$.

2.2 Case $d_2 = 2$

- If $d_2 = 2$ the system $S_1$ becomes:

\[
\begin{align*}
&(a^3 + 6ab)x + 4ay + 6a^2 + 2a^3b + 2ap = 0 \quad (1) \\
&(3a^2b + 3b^2 + p)x + (a^2 + 2b)y + 4z + 2b^3 + 3a^2b^2 + 3a^2p + 2bp = 0 \quad (2) \\
&(3ab^2 + 3ap)x + 4aby + 4az + 2ab^3 + 2a^3p + 6abp = 0 \quad (3) \\
&(2b^3 + 3a^2b + 3bp)x + 4(b^2 + p)y + 8bz + 16t + b^4 + 12a^2bp + a^4p + 6b^2p + p^2 = 0(4) \\
\end{align*}
\]

with $a, b \in \{0, 1\}$.
i) \( a = b = 0. \)

\((E_1)\) becomes : 
\[
\begin{cases}
px + 4z = 0 \quad (1) \\
4py + 16t + p^2 = 0 \quad (2)
\end{cases}
\]

from equation (2), 4 divides \( p \), which is impossible.

ii) \( a = 0 \) and \( b = 1. \) We have \( \alpha = \frac{\omega^2 + 1}{2} = \frac{\sqrt{p^2 + 1}}{2}. \) \( \alpha \) is an algebraic integer if and only if \( p \equiv 1 \mod 4. \)

iii) \( a = 1 \) and \( b = 0. \)

The resolution of the system \((S_1)\) gives :
\[
\begin{cases}
x = 0 \\
y = -\frac{p}{2} \\
z = -\frac{p}{2} \\
t = \frac{p^2 - p}{16}
\end{cases}
\]

But this solution is not in \( \mathbb{Z}^4 \), so \( \alpha = \frac{\omega^2 + \omega}{2} \) is not an algebraic integer.

iv) \( a = b = 1. \)

By resolving \((S_3)\) we obtain :
\[
\begin{cases}
x = -2 \\
y = \frac{3-p}{2} \\
z = \frac{3-p}{4} \\
t = \frac{p^2 + p + 1}{16}
\end{cases}
\]

Observing that \( x = -2, y = \frac{3-p}{2}, z = \frac{3-p}{4}, t = \frac{p^2 + p + 1}{16} \notin \mathbb{Z}^4 \), we can deduce that \( \alpha = \frac{\omega^2 + \omega + 1}{2} \) is not an algebraic integer.

**Conclusion 1:** If \( p \equiv 1 \mod 4 \) we have \( d_2 = 2. \)

2.3 Case \( d_2 = 1. \)

From the previous paragraph, if \( p \equiv 3 \mod 4 \) we have \( d_2 = 1. \)

3 Determination of \( d_3 \)

We suppose \( \gamma = \omega^3 + a \omega^2 + b \omega + c. \) and \( \alpha = \frac{\omega^3 + a \omega^2 + b \omega + c}{d_3} \), \( \alpha \) is an algebraic integer if and only if the rational integers \( x, y, z \) and \( t \) exist such that :
\[
\alpha^4 + x \alpha^3 + y \alpha^2 + z \alpha + t = 0
\]

\[
\frac{\alpha^4}{d_3^4} + x \frac{\alpha^3}{d_3^4} + y \frac{\alpha^2}{d_3^4} + z \frac{\alpha}{d_3} + t = 0
\]

We reduce to the same denominator and simplify by \( \frac{1}{d_3^4} \), we obtain :
\[
\gamma^4 + x d_3^3 \gamma^3 + y d_3^3 \gamma^2 + z d_3 \gamma + t d_3^3 = 0
\]

We replace in this equation \( \gamma \) by its expression and using that \( \{1, \omega, \omega^2, \omega^3\} \) is \( \mathbb{Q} \)-libre, we obtain the following system \((S)\).
\begin{equation}
\begin{pmatrix}
(3a^2p + 3bp + 6abc + b^3 + 3c^2)d_3x + (2bc + 2c)d_2^3y + d_3^2z + 4ap^2 + 12bcp \\
+ 12a^2cp + 12ab^2p + 4e^3 + 4a^3bp + 12abc^2 + 4b^3c = 0
\end{pmatrix}
\end{equation}

\begin{equation}
\begin{pmatrix}
(6abp + 3cp + a_3p + 3b^2c + 3ac^2)d_3x + (b^2 + 2ac + p)d_3^2y + ad_3^3z + 6a^2p^2 \\
+ 6a^2b^2 + 4b^2p + 24abc + 4b^3p + 6c_2p + 6a^2b^2p + 4a^3cp + 4c_2d + 6b^2c = 0
\end{pmatrix}
\end{equation}

\begin{equation}
\begin{pmatrix}
(3b^2p + 6acp + 3a_2p + b^2p^2 + p^3)d_3x + (2ap + 2bc)d_2^3y + bd_3^2z + 12abp^2 \\
+ 4cp^2 + 12b^3p + 4a^3cp + 12ac^2p + 2ab_3cp + 4ab^3p + 4bc^3 = 0
\end{pmatrix}
\end{equation}

\begin{equation}
\begin{pmatrix}
(3a^2p^2 + 6bp + 3abc + c_3)d_3x + (a^2p + c^2 + 2bp)d_3^2y + c_3d_3^2z + t_3d
\\
+ 6b_2p + 12ac^2p + 12a_2bp^2 + 12bc^2p + p^3 + a^2p^2 + 12ab_2cp + 6a_3c^2p + b^2p + c^4 = 0
\end{pmatrix}
\end{equation}

with \(0 \leq a, b, c < d_3\).

Using (1) in the proposition, we have: \(d_3^2 \text{ divides } \text{disc}(\omega) = -2^8p^3\).

It follows that \(d_3 \in \{1, 2, 4, 8, 16, p, 2p, 4p, 8p, 16p\}\)

So we have 10 cases to distinguish.

### 3.1 Case \(d_3 = 16p\)

Let \(\alpha = \frac{\omega^3 + \omega^2a^2 + b^2 + c}{16p}\). If \(\alpha\) is an algebraic integer then \(\alpha \omega = \frac{\omega^3 + b^2a^2 + c}{16p}\) is also an algebraic integer and his trace \(T(\alpha \omega) = \frac{1}{4}\) will be a rational integer.

So \(\alpha\) is not an algebraic integer and \(d_3 \neq 16p\).

### 3.2 Case \(d_3 = 8p\)

Let \(\alpha = \frac{\omega^3 + \omega^2a^2 + b^2 + c}{8p}\)

We have \(T(\alpha \omega) = \frac{1}{4}\) then it follows that \(\alpha \omega\) is not an algebraic integer and also for \(\omega\).

Then we conclude that \(d_3 \neq 8p\).

### 3.3 Case \(d_3 = 4p\)

The resolution of the system (S) with \(d_3 = 4p\) gives:

\begin{equation}
\begin{cases}
x = \frac{-c}{p} \\
y = \frac{3c^2 - 26b - a^2p}{8p} \\
z = \frac{a^2cp + 2bcp - b^2a^2p - ap^2}{16p} \\
t = \frac{2b^2p^2 + a^2p^3 + 4abc + a_3cp + 4a_3c^2p - 4b^4p - 4bcp - p^3 - 2a^2c^2p - b^2p}{256p^3}
\end{cases}
\end{equation}

We suppose that \(x, y, z, t\) are the rational integers.

Knowing that \(x \in \mathbb{Z}\) we have \(p\) divides \(c\) so \(c \in \{0, p, 2p, 3p\}\). We do the same for \(y \in \mathbb{Z}\) and \(p\) divides \(c\), we have \(p\) divides \(a^2 + 2b\) We have \(z \in \mathbb{Z}\) and \(p\) divides \(c\), so we have \(p\) divides \(ob^3\), then \(p\) divides \(a\) or \(p\) divides \(b\). So \(p\) divides \(a\) or \(p\) divides \(b\) and \(p\) divides \(a^2 + 2b\). It follows that \(p\) divides \(a\) and \(p\) divides \(b\).

In the conclusion, we have \(p\) divides \(a\), \(p\) divides \(b\) and \(p\) divides \(c\). So we deduce that \(a, b\) and \(c\) is \(\{0, p, 2p, 3p\}\).

We suppose \(a = ps, b = pu, c = p = v\), and we replace in the expression of \(t\), we obtain:

\(t = \frac{5(2u^2p + 2v^2 + 4uv + 1) - 4u^2 - 4uv - 2p^2(-u + v) - 4uv - 2uv}{256p^3}\)

Knowing that \(t \in \mathbb{Z}\) we deduce that \(p\) divides \(1\), what is a contradiction.

Then \(d_3 \neq 4p\).

### 3.4 Case \(d_3 = 2p\)

We replace \(d_3\) by \(2p\) in the system (S) and we solve it:
\[ x = \frac{2c}{p} \\
\quad y = \frac{3a^2 - 2bp - a^2 p}{2p^2} \\
\quad z = \frac{a^2 cp + 2bcp - c^2 - ab^2 p - a p^2}{4p} \\
\quad t = \frac{20p^3 + 4a^2 p + 4ab^2 cp + c^2 + 4acp^2 - 4a^2 bp - 4bc^2 p - p^3 - 2a^2 c^3 p - b^2 p}{16p^3} \]

We suppose that \( x, y, z, \) and \( t \) are the rational integers.
We have \( x \in \mathbb{Z} \) and \( p \neq 2 \), so \( p \) divides \( c \). It follows that \( c \in \{0, p\} \) As the same \( y \in \mathbb{Z} \) and \( p \) divides \( c \) we have \( p \) divides \( a^2 + 2b \),
knowing that \( z = \frac{(a^2 + 2b)cp - c^2 - ab^2 p - ap^2}{2p} \in \mathbb{Z} \), \( p \) divides \( c \) and \( p \) divides \( a^2 + 2b \), so we have \( p \) divides \( a^2 b \). It follows that \( p \) divides \( a \) or \( b \). Then \( p \) divides \( a^2 + 2b \) and \( \{ p \) divides \( a \) or \( b \) \). We deduce that \( p \) divides \( a \) and \( b \) as the following \( a, b, c \in \{0, p\} \).
We suppose \( a = ps, b = pu, c = pv \) and we replace these values in the expression of \( t \), we have:
\[ t = \frac{2p^4 u^2 + pu^5 + u^6 - 2p^3 s^5 - 2p^3 s^3 v^2 - p^2 s^3 v - p s^3 v^2 - 16p^3}{16p^3} \]
knowing that \( t \in \mathbb{Z} \) we deduce that \( p^2 \) divides \( p^3 \), so this is a contradiction.
Then \( d_3 \neq 2p \).

3.5 Case \( d_3 = p \)

Let \( \alpha = \omega^3 + a\omega^2 + b\omega + c \) with \( a, b, c \in \{0, 1, \ldots, p-1\} \).
If \( \alpha \) is an algebraic integer then its trace \( T(\alpha) = \frac{4c}{p} \in \mathbb{Z} \), It follows that \( p \) divides \( c \) so \( c = 0 \).
We replace \( d_3 \) by \( p \) and \( c \) by \( 0 \), the system \( S \) becomes:
\[
\begin{align*}
(3a^2 p + 3bp + b^3)x + 2abpy + p^2 z + 12ab^2 + 4a^2 b + 4ap = 0 \\
(6ab + a^3)px + (b^2 + p)py + ap^2 z + 4b^3 + 6a^2 b^2 + 6a^2 p + 4bp = 0 \\
(3b^2 + 3a^2 bp + p)px + 2ap^2 y + bp^2 z + 4ab^3 + 4a^3 p + 12abp = 0 \\
(3ap + 3a^2 bp + p^2 y + (a^2 + 2p)p^3 y + p^3 t + b^4 + 12a^2 bp + a^4 p + 6b^2 p + p^2 = 0(4)
\end{align*}
\]
From equation \( 4 \) we deduce that \( p \) divides \( b \), then \( b = 0 \).
So the equation \( 4 \) becomes:
\[
3apx + a^2 py + p^3 t + p + a^4 = 0
\]
We deduce that \( p \) divides \( a \), then \( a = 0 \), So \( a = b = c = 0 \). The system \( S \) becomes:
\[
\begin{align*}
p^2 z &= 0 \\
p^2 y &= 0 \\
p^2 x &= 0 \\
p^3 t &= 0
\end{align*}
\]
It follows that \( x = y = z = t = 0 \). So \( \alpha^4 = 0 \).
We deduce that \( 0 = (\frac{\alpha^2}{p})^4 = (\frac{\alpha}{p})^4 = \frac{1}{p} \) is impossible.
Then \( d_3 \neq p \).

3.6 Case \( d_3 = 16 \)

Let \( \alpha = \omega^3 + a\omega^2 + b\omega + c \) with \( a, b, c \in \{0, 1, \ldots, p-1\} \).
We suppose that \( \alpha \) is an algebraic integer, we have also \( \omega \) is an algebraic integer, We deduce that his trace \( T(\alpha\omega) = \frac{4c}{p} \) is a rational integer,
Then \( 4 \) divides \( p \), is a contradiction.
Then \( \alpha \) is not an algebraic integer and we deduce that \( d_3 \neq 16 \).

3.7 Case \( d_3 = 8 \)

Let \( \alpha = \omega^3 + a\omega^2 + b\omega + c \). We suppose that \( \alpha \) is an algebraic integer, then \( \omega = \frac{p + a\omega^3 + b\omega^2 + c\omega}{8} \) will be also an algebraic integer,
We deduce that its trace \( T(\alpha\omega) = \frac{4p}{8} = \frac{p}{2} \) is in \( \mathbb{Z} \), so we have a contradiction. Then \( d_3 \neq 8 \).
3.8 Case $d_3 = 4$

If $d_3 = 4$, the system (S) becomes:

\[
\begin{align*}
(3a^2p + 3bp + 6abc + b^3 + 3c^2)x + (2ab + 2bc)y + 16z + ap^2 + 3bcp \\
+ 3a^2c + 3ab^2p + c^3 + a^3bp + 3abc^2 + b^3c &= 0 \\
(6abp + 3c^2p + a^3p + 3b^2p + a^3c^2)p + [(b^2 + 2ac + p)8y + 32az + 3a^2p^2] \\
+ 3a^3bp^2 + 12abc + 2b^3p + 3c^2p + 3a^3b^2p + 2a^3c^2p + 2ac^3 + 3b^3c^2 &= 0 \\
(3b^2p + 6acp + 3a^2bp + 3bc^2 + p^2)x + (ap + bc)y + 16bz + 3abyp^2 + cp^2 \\
+ 3b^2c + a^3p^2 + 3ac^2p + 3a^3bc + ab^3p + bc^3 &= 0 \\
(3ap^2 + 6bcp + 3ab^2p + 3c^2p + c^3)4x + (a^2p + c^2 + 2bp)y + 16y + 64cz + 256t \\
+ 6b^2p^2 + 12acp^2 + 12a^2bp^2 + 12b^2p^2 + p^3 + a^4p^2 + 12ab^2p + 6a^2c^2p + b^4p + c^4 &= 0(4)
\end{align*}
\]

After solving this system, we obtain:

\[
\begin{align*}
x &= -c \\
y &= \frac{3c^2 - 2bp - a^2p}{8} \\
z &= \frac{a^3cp + 12bc + c^3 - ab^2p - ap^2}{16} \\
t &= \frac{2b^2p^2 + 4ab^2 + c^4 + 4acp^2 - 4a^2bp^2 - 4bc^3p^2 - 2ab^2c^2p - b^4p}{256}
\end{align*}
\]

By using the lemma we will take $a, b, c$ in $\{-1, 0, 1, 2\}$

We suppose that $x, y, z$, and $t$ are the rational integers.

i) If $c$ is even ($c \in \{0, 2\}$)

Using $y = \frac{3c^2 - 2bp - a^2p}{8} \in \mathbb{Z}$ we have $2$ divides $a^2p$ and $p > 2$ so $2$ divides $a$ The same for $t \in \mathbb{Z}$, we deduce that $2$ divides $p(p^2 + b^4)$ so $b$ is odd ($b \in \{-1, 1\}$).

- if $b = -1$ we have $y = \frac{3c^2 + 2p - a^2p}{8}$. It follows that $8$ divides $p(2 - a^2)$ so $8$ divides $2 - a^2$ we deduce that $4$ divides $2$, so it’s a contradiction.

- if $b = 1$ we have $y = \frac{3c^2 - 2p - a^2p}{8}$. we deduce that $4$ divides $2 + a^2$ so $4$ divides $2$. Then it’s a contradiction. We conclude that $c$ is odd.

ii) If $c$ is odd ($c \in \{-1, 1\}$)

Using $y = \frac{3c^2 - 2bp - a^2p}{8} \in \mathbb{Z}$, we have $2$ divides $3c^2 - a^2p$, it follows that $a$ is odd.

as the same $z = \frac{a^3cp + 12bc + c^3 - ab^2p - ap^2}{16} \in \mathbb{Z}$, so we deduce that $b$ is odd, Then $a, b$ and $c$ are odds.

We give the values of $x, y, z, t$ in the table $T_{ab}2$ for $a, b, c \in \{-1, 1\}$.

\[
\begin{array}{|c|c|c|c|}
\hline
(a,b,c) & x & y & z \\
\hline
(-1,-1,1) & 3 + p & \frac{[p+1]^2}{16} & -p' + 11p + 5p' + 1 \\
(-1,1,1) & 3 + 3p & \frac{[p+1]^2}{16} & -p' + 11p + 5p' + 1 \\
(-1,1,1) & 3 + 3p & \frac{[p+1]^2}{16} & -p' + 11p + 5p' + 1 \\
(-1,1,1) & 3 + 3p & \frac{[p+1]^2}{16} & -p' + 11p + 5p' + 1 \\
(1,1,1) & 3 + p & \frac{[p+1]^2}{16} & -p' + 11p + 5p' + 1 \\
(1,1,1) & 3 + p & \frac{[p+1]^2}{16} & -p' + 11p + 5p' + 1 \\
(1,1,1) & 3 + p & \frac{[p+1]^2}{16} & -p' + 11p + 5p' + 1 \\
(1,1,1) & 3 + p & \frac{[p+1]^2}{16} & -p' + 11p + 5p' + 1 \\
(1,1,1) & 3 + p & \frac{[p+1]^2}{16} & -p' + 11p + 5p' + 1 \\
\hline
\end{array}
\]

Conclusion 2:

i) If $p \equiv 1 \mod 8$, we suppose $p = 8k + 1$, then we have:
If \( p \equiv 3 \mod 8 \), we suppose \( p = 8k + 3 \), then we have:

- for \((a, b, c) \in \{(-1, -1, 1), (-1, 1, -1), (1, -1, 1), (1, 1, 1)\}\)

\[
y = \frac{3-3p}{8} = k + \frac{3}{4} \notin \mathbb{Z}.
\]

- for \((a, b, c) \in \{(-1, -1, 1), (-1, 1, 1), (1, 1, -1), (1, 1, 1)\}\)

\[
y = \frac{3-3p}{8} = -3k - \frac{3}{4} \notin \mathbb{Z}.
\]

It follows that: for \( p \equiv 3 \mod 8 \), \( d_3 \neq 4 \).

If \( p \equiv 5 \mod 8 \), we suppose \( p = 8k + 5 \), then we have:

\[
\frac{(p+1)^2}{16} = 4k^2 + 6k + \frac{9}{4} \notin \mathbb{Z}.
\]

\[
\frac{p^2+4p-1}{16} = 4k^2 + 7k + \frac{11}{4} \notin \mathbb{Z}.
\]

\[
\frac{p^2+4p-1}{16} \text{ is not a rational integer}.
\]

As the following \( \alpha = \frac{\omega^3 + \omega^2 + \omega + 1}{4} \) is not an algebraic integer. Then in this case \( \alpha = \frac{\omega^3 + \omega^2 + \omega + 1}{4} \) is not an algebraic integer. So for \( p \equiv 5 \mod 8 \), we have \( d_3 \neq 4 \).

iv) If \( p \equiv 7 \mod 8 \), we pose \( p = 8k + 7 \), then we have:

\[
\frac{3-3p}{8} = k + \frac{5}{4} \notin \mathbb{Z} \quad \text{and} \quad \frac{3-3p}{8} = -3k - \frac{3}{4} \notin \mathbb{Z}.
\]

It follows that: for any triplet \((a, b, c) \notin \mathbb{Z}\). Then \( \alpha = \frac{\omega^3 + \omega^2 + \omega + 1}{4} \) is not an algebraic integer.

We conclude that: if \( p \equiv 7 \mod 8 \), \( d_3 \neq 4 \).

3.9 Case \( d_3 = 2 \)

Let \( \alpha = \frac{\omega^3 + \omega^2 + \omega + 1}{16} \) with \( a, b, c \in \{0, 1\} \).

The \( \mathbb{Q} \)-isomorphisms of \( K = \mathbb{Q}(\omega) \) are defined by:

\[
\sigma_1(\omega) = \omega \quad \sigma_2(\omega) = -\omega \quad \sigma_3(\omega) = i\omega
\]

and \( \sigma_4(\omega) = -i\omega \).

Let \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) denote the conjugates of \( \alpha \) over \( \mathbb{Q} \), then we have:

\[
\alpha_1 = \sigma_1(\alpha) = \omega^3 + \omega^2 + \omega + 1
\]

\[
\alpha_2 = \sigma_2(\alpha) = -\left(\omega^3 + \omega^2 + \omega + 1\right)
\]

\[
\alpha_3 = \sigma_3(\alpha) = \frac{c-\omega^3-i(\omega^3+\omega+1)}{2}
\]

\[
\alpha_4 = \sigma_4(\alpha) = \frac{c-\omega^3+i(\omega^3+\omega+1)}{2}
\]

The minimal polynomial \( P \) of \( \alpha \) over \( \mathbb{Q} \) is:

\[
P = X^4 - S_1X^3 + S_2X^2 - SX + S_4
\]

where \( S_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \)

\( S_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4 \)

\( S_3 = \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4 \)

\( S_4 = \alpha_1\alpha_2\alpha_3\alpha_4 \)
After calculating, we have : \( S_1 = -2c, \quad S_2 = \frac{3c^2-2bp-a^3p}{2}, \quad S_3 = \frac{ap^3+c^2+ab^2p-a^3cp-2bp}{2} \), \
\[ S_4 = \frac{2b^2p+a^2p^2+4ab^2cp+c^2+4acp^2-4bc^2by^2-4c^2p^2-p^3-2a^2c^2p-b^2p}{16} \]
\( \alpha \) is an algebraic integer if and only if \( S_1, S_2, S_3 \) and \( S_4 \) are rational integers.

We give in the table \( \text{Tab}_3 \) the values of \( S_1, S_2, S_3 \) and \( S_4 \) when the values of \( a, b, c \) are in \{0, 1\}.

\[ \begin{array}{|c|c|c|c|}
\hline
\text{Tab}_3 & S_1 & S_2 & S_3 & S_4 \\
\hline
(0,0,0) & 0 & 0 & 0 & -p^2 \\
\hline
(0,0,1) & -2 & \frac{1}{2} & \frac{1}{2} & 1-p^2 \\
\hline
(0,1,0) & 0 & -p & 0 & -p(p-1)^2 \\
\hline
(0,1,1) & -2 & -\frac{3p}{2} & -\frac{3p}{2} & -p^2-2p+5p+1 \\
\hline
(1,0,0) & 0 & -p & p^3 & p(p-1)^2 \\
\hline
(1,0,1) & -2 & -\frac{p+1}{2} & -\frac{p+1}{2} & -p^2+5p^2-2p+1 \\
\hline
(1,1,0) & 0 & -3p & \frac{p(p+1)}{2} & -p^2-p \\
\hline
(1,1,1) & -2 & -\frac{3p}{2} & -\frac{(p-1)^2}{2} & -\frac{(p-1)^2}{16} \\
\hline
\end{array} \]

**Conclusion 3:**

i) If \( p \equiv 1 \pmod{4} \), we have : for the triplet \( (a, b, c) = (0, 1, 0) \).
\[ (S_1 = 0, \quad S_2 = -p, \quad S_3 = 0, \quad S_4 = \frac{p(p-1)^2}{16}) \in \mathbb{Z}^4. \]

It follows that : \( \alpha = \frac{\omega^2+c^2}{2} \) is an algebraic integer, so \( d_3 = 2 \).

ii) If \( p \equiv 3 \pmod{4} \), we have for any \( (a, b, c) \) \( (S_1, S_2, S_3, S_4) \notin \mathbb{Z}^4. \) Then for \( p \equiv 3 \pmod{4} \), we have \( d_3 \neq 2 \).

**4 Particular Case :** \( p = 2 \).

Let \( \omega = \sqrt{2} \).
We have disc(\( \omega \)) = -2^{11}.

i) we know that \( d_1^2 \) divides disc(\( \omega \)), It follows that \( d_1 = 1 \).

ii) \( d_2^2 \) divides disc(\( \omega \)) = -2^{11} . It follows that \( d_2 \in \{1, 2, 4\} \).

**4.1 Determination of \( d_2 \).**

1- Case \( d_2 = 4 \)

Let \( \gamma = \omega^2 + \omega + b \) and \( \alpha = \frac{\gamma}{2} \) with \( a, b \in \{0, 1, 2, 3\} \). \( \alpha \) is an algebraic integer if and only if there are four rational integers \( x, y, z \) and \( t \), such that : \( \alpha^4 + x\alpha^3 + y\omega^2 + z\omega + t = 0 \).

By replacing \( \alpha \) by \( \frac{\gamma}{2} \) we obtain the equation :
\[ \gamma^4 + 4x\gamma^3 + 4^2y\gamma^2 + 4^3z\gamma + 4^4t = 0 \]
By replacing \( \gamma \) by \( \omega^2 + \omega + b \) and using that \( \{1, \omega, \omega^2, \omega^3\} \) is \( \mathbb{Q} \)-libre, we obtain the system :
\[
\begin{align*}
(a^3 + 6ab)x + 8ay + 3ab^2 + a^3b + 2a &= 0 \\
2(3a^2b + 3b^2 + 2)x + 8(a^2 + 2b)y + 32z + 2b^3 + 3a^2b^2 + 6a^2 + 4b &= 0 \\
(3ab^2 + 6a)x + 8aby + 16az + ab^3 + 2a^3 + 6ab &= 0 \\
4(b^3 + 6a^2 + 6b)x + 4^2(b^2 + 2)y + 4^3bz + 4^4t + b^4 + 24a^2b + 2a^4 + 12b^2 + 4 &= 0(4)
\end{align*}
\]
We suppose that \( x, y, z \) and \( t \) are in \( \mathbb{Z} \).

(4) \( \Rightarrow \) 2 divides \( b \) \( \Rightarrow \) 2 divides \( a \).

So \( a, b \in \{0, 2\} \).

i) \( a = b = 0 \)

(4) becomes \( 8y + 64t + 1 = 0 \) it is a contradiction.

ii) \( a = 0 \) and \( b = 2 \)

The equation (4) becomes:
\[
20x + 24y + 32z + 64t + 17 = 0
\]

it’s also a contradiction.

iii) \( a = 2 \) and \( b = 0 \)

The equation (4) becomes:
\[
24x + 8y + 64t + 9 = 0 \quad \text{we have also a contradiction.}
\]

iv) \( a = b = 2 \)

In this case the equation (4) becomes: \( 8x + 4y + 11 = 0 \) so it’s a contradiction.

In conclusion \( d_2 \neq 4 \).

2- Case \( d_2 = 2 \).

Let \( \gamma = \omega^2 + aw + b \) and \( \alpha = \frac{1}{2} \) with \( a, b \in \{0, 1\} \). \( \alpha \) is an algebraic integer if and only if there are four rational integers \( x, y, z \) and \( t \) so that:
\[
\alpha^4 + x\alpha^3 + y\alpha^2 + z\alpha + t = 0.
\]

The last equation we replace \( \gamma \) by its expression and using \( \{1, \omega, \omega^2, \omega^3\} \) is \( \mathbb{Q} \)-libre we obtain the system:
\[
\begin{align*}
2(3ab^2 + a^3 + 2a) + (a^3 + 6ab)x + 4ay + 4z &= 0 \quad (1) \\
2b^3 + 3a^2b^2 + 6a^2 + 4b + (3a^2b + 3b^2 + 2)x + 2(a^2 + 2b)y + 4z &= 0 \quad (2) \\
2ab^3 + 4a^3 + 12ab + (3ab^2 + 6a)x + 4aby + 4az &= 0 \quad (1)
\end{align*}
\]

we suppose that \( x, y, z \) and \( t \) are in \( \mathbb{Z} \), with equation (4) we deduce that 2 divides \( b \) it follows that \( b \in \{0, 1\} \) so \( b = 0 \).

Replacing \( b \) by 0, the equation (4) becomes: \( a^4 + 2 + 6a^2x + 4y + 8z = 0 \). It follows that 2 divides \( a \) and so \( a = 0 \). Then \( a = b = 0 \) and \( \alpha = \frac{\sqrt{2}}{2} \).

As \( \frac{\sqrt{2}}{2} \) is not an algebraic integer, we conclude that \( d_2 \neq 2 \).

Conclusion: We have \( d_2 = 1 \).

4.2 Determination of \( d_3 \).

We distinguish \( d_3 \) in \( \{1, 2, 2^2, 2^3, 2^4, 2^5\} \)

1- Case \( d_3 = 32 \)

Let \( \beta = \omega \alpha = \frac{\omega^3 + b^2 + c\omega + 2}{16} \), if \( \alpha \) is an algebraic integer, it’s the same for \( \beta \) and his trace \( T(\beta) \) is a rational integer. But \( T(\beta) = \frac{1}{4} \) is a contradiction. Then \( d_3 \neq 32 \).

2- Case \( d_3 = 16 \)

Let \( \beta = \omega \alpha = \frac{\omega^3 + b^2 + c\omega + 2}{16} \), if \( \alpha \) is an algebraic integer, it’s the same for \( \beta \) and his trace \( T(\beta) \) is a rational integer. But \( T(\beta) = \frac{1}{4} \). Then we have a contradiction. So \( d_3 \neq 16 \).
3. Case $d_3 = 8$

Let $\alpha = \omega^3 + a\omega^2 + b\omega + c$ (Using the calculus of paragraph 3-7) with $p = 2$ we have the following expressions:

\[
\begin{cases}
  x = \frac{3c - 4b - 2a^2}{2} \\
y = \frac{3c - 4b - 2a^2}{2} \\
z = \frac{3c - 4b - 2a^2}{2} \\
t = \frac{3c - 4b - 2a^2}{2}
\end{cases}
\]

with $a, b, c \in \{0, 1, 2, 3, 4, 5, 6, 7\}$. Knowing $x \in \mathbb{Z}$ and $x = T(\omega) = -\frac{\omega}{2}$ we have $c$ is even, $y$ is in $\mathbb{Z}$ and $c$ is even, we deduce that $a$ is even. $t \in \mathbb{Z}$, $a$ and $c$ are even, we deduce that $b$ is also even. We suppose that $a = 2s$, $b = 2u$, $c = 2v$ and we replace in $t$, we obtain: $t = \frac{2(2a^2 + 4s^2 + 8su^2 + v^2 + 4su - 8s^2 - 8 - 4u^2 - 4v^2 - 2a^2 - 2uv)}{2}$

What is a contradiction because the numerator of $t$ is odd. Then $d_3 \neq 8$.

4. Case $d_3 = 4$

Let $\alpha = \omega^3 + a\omega^2 + b\omega + c$ (Using the calculus of paragraph 3-8) with $p = 2$ we have the following expressions:

\[
\begin{cases}
  x = c \\
y = \frac{3c - 4b - 2a^2}{2} \\
z = \frac{3c - 4b - 2a^2}{2} \\
t = \frac{3c - 4b - 2a^2}{2}
\end{cases}
\]

with $a, b, c \in \{0, 1, 2, 3\}$, If $c$ is even, $y \in \mathbb{Z}$, we deduce that $a$ is even, and from $t \in \mathbb{Z}$ we obtain $b$ is even. We suppose that

$\alpha = 2s$, $b = 2u$, $c = 2v$ and replace this in $t$, we obtain: $t = \frac{2(2a^2 + 4s^2 + 8su^2 + v^2 + 4su - 8s^2 - 8 - 4u^2 - 4v^2 - 2a^2 - 2uv)}{2}$

This is a contradiction because the numerator of $t$ is odd. If $c$ is odd then $y = \frac{3c - 4b - 2a^2}{2} \notin \mathbb{Z}$ because the numerator of $y$ is odd. We deduce that: $d_3 \neq 4$.

5. Case $d_3 = 2$

Let $\alpha = \omega^3 + a\omega^2 + b\omega + c$ (Using the calculus of paragraph 3-9) (with $p = 2$) we have the following solutions:

\[
\begin{cases}
  x = -2c \\
y = \frac{3c - 4b - 2a^2}{2} \\
z = \frac{3c - 4b - 2a^2}{2} \\
t = \frac{3c - 4b - 2a^2}{2}
\end{cases}
\]

with $a, b, c \in \{0, 1\}$. If $c = 0$ then $t = \frac{4b^2 + 2a^2 - 8b - 4b - 2a^2}{2}$ Z this implies that $b$ is even. So $b = 0$, we deduce that $t = \frac{6a - 1}{2}$. So it’s a contradiction. If $c = 1$ then $y = \frac{3 - 4b - 2a^2}{2} \notin \mathbb{Z}$ because $3 - 4b - 2a^2$ is odd. Then $d_3 \neq 2$. In conclusion we have $d_3 = 1$.

**Corollary 4.1.** Let $p$ be a prime number, $\omega = \sqrt[p]{q}$ and $K = \mathbb{Q}(\omega)$ then:

1) If $p = 2$ or $p \equiv 3 \mod 4$ we have: $\text{disc}(K) = -2^8p^3$.

2) If $p \equiv 5 \mod 8$ we have: $\text{disc}(K) = -2^4p^3$.

3) If $p \equiv 1 \mod 8$ we have: $\text{disc}(K) = -2^2p^3$.

**Remark 4.2.** Let $m$ be an integer number without quartic factors, $\omega = \sqrt[p]{m}$ and $K = \mathbb{Q}(\omega)$. By theorem 1, $K$ admits an integral basis of the form: $(1; f_1(\omega), f_2(\omega), f_3(\omega))$

Let’s put $m = fg^2h^2$ where $f, g$ and $h$ are integers mutually coprimes $f = a_1a_2...a_r$, $g = b_1b_2...b_s$ and $h = c_1c_2...c_t$ where $a_1, ..., a_r, b_1, ..., b_s, c_1, ..., c_t$ are prime numbers We have $\text{disc}(\omega) = -2^8m^3 = -2^8a_1^3...a_r^3b_1^3...b_s^3c_1^3...c_t^3$. Under the proposal we have $d_2^t$ divides $\text{disc}(\omega)$. It follows that $d_2 \in\{1, 2, 4, u_1, ..., a_r, b_1, ..., b_s, c_1, ..., c_t, ...\}$ Which proves the complexity of the determination of $d_2$. This is how limited ourselves to the case $m = p$ where $p$ is a prime integer.
References


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