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Integrally closed ring extensions with three non-Artinian intermediate rings

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Abstract The purpose of this paper is twofold: firstly, to partially answer a question which was left open in [1], concerning the "Artinian" property. More precisely, we provide a complete characterization of integrally closed ring extensions with exactly three non-Artinian intermediate rings. Secondly, to deepen the study of ring extensions with few non-Artinian intermediate rings, which was initiated in [19].

1 Introduction

All rings we consider are commutative and unital. All inclusions of rings, all subrings and all ring extensions are assumed to be unital. For rings $R \subseteq S$, $[R, S]$ denotes the set of intermediate rings; that is, rings T such that $R \subseteq T \subseteq S$. If $S := \text{tq}(R)$ is the total quotient ring of R, then any ring in $[R, \text{tq}(R)]$ is called an *overring* of R. Next, we will retain the same notation as in [1]. For a given ring theoretic property $\mathcal{P}$ and given rings $R \subseteq S$, we let $[R, S]_{\text{non}-\mathcal{P}} := \{T \in [R, S] \mid T \text{ does not satisfy } \mathcal{P}\}$ and $[R, S]_{\mathcal{P}} := [R, S] \setminus [R, S]_{\text{non}-\mathcal{P}}$. If $R \subseteq S$ is a ring extension, then we say that (R, S) is a $\mathcal{P}$-pair if $[R, S]_{\mathcal{P}} = [R, S]$; that is, if any ring in $[R, S]$ satisfies $\mathcal{P}$. We say that R is a maximal non-$\mathcal{P}$ subring of S if $R \subset S$ (as usual $\subset$ denotes proper inclusion) and $[R, S]_{\text{non}-\mathcal{P}} = \{R\}$; that is, if R does not satisfy $\mathcal{P}$ while every ring in $]R, S]$ satisfies $\mathcal{P}$, where $]R, S] = [R, S] \setminus \{R\}$. It is worth mentioning that many integral domains have been characterized in the last few decades by properties satisfied by their overrings. In recent years, $\mathcal{P}$-pairs of rings and maximal non-$\mathcal{P}$ subrings have received attention of many researchers. They have been studied for many properties $\mathcal{P}$ such as Noetherian [3], Jaffard [5, 6, 12], treed [4], valuation [7], Prüfer [18], universally catenarian [2], Artinian [19], zero-dimensional [15], integrally closed [9, 20], pseudo-valuation [21]. In this paper we are concerned with the following question which was raised by Al Subaiei et al., see [1]: "Let $\mathcal{P}$ be a ring-theoretic property and let n be a nonnegative integer. Provide necessary and sufficient conditions in order that a ring extension $R \subset S$ satisfies $|[R, S]_{\text{non}-\mathcal{P}}| = n$."

This question was answered in [1] in case the property $\mathcal{P} :=$ Prüfer and $0 \leq n \leq 2$. A similar study was carried out for the property $\mathcal{P} :=$ Artinian and $0 \leq n \leq 2$ (see [19] for more details). Recall that a ring R is called *Artinian* if it satisfies the descending chain condition on ideals; that is, there is no infinite descending sequence of ideals. Examples of Artinian rings are finite rings and rings that are finite-dimensional vector spaces over fields. It is worth noticing that Artinian rings, particularly local Artinian rings, play an important role in algebraic geometry, especially in deformation theory.

To complete this circle of ideas, we aim to answer the above question in case $\mathcal{P} :=$ Artinian, $n = 3$ and R is integrally closed in S. Our study is motivated both by the significance of Artinian rings for commutative algebra and several other related fields as outlined in the preceding paragraph and the recent increasing interest in the study of extensions with few non-$\mathcal{P}$ intermediate rings as explained above. A brief summary of our results is now given: In Lemma 3.1, we prove that if $R \subset S$ is an integrally closed extension such that $|[R, S]_{\text{non}-\mathcal{P}}| = n \geq 1$, then $R \subset S$ is an FIP extension (that is, $|[R, S]| < \infty$). FIP extensions were investigated and extensively studied in the literature, especially in the nice paper [10]. In Lemma 3.2, we prove that if $R \subset S$ is an integrally closed extension satisfying FIP, then $[R, S] = [R, S]_{\text{non}-\mathcal{P}} \cup \{S\}$ if S is Artinian and

$[R,S] = [R,S]_{\text{non}-\mathcal{P}}$ if S is not Artinian. Our titular result is Theorem 3.3, which states that if $R \subset S$ is an integrally closed extension, then $|[R,S]_{\text{non}-\mathcal{P}}| = 3$ if and only if either $[R,S]$ ordered by the usual set inclusion is a chain of length 2 and S is not Artinian; or $[R,S]$ ordered by the usual set inclusion is a chain of length 3 and S is Artinian; or $[R,S]$ ordered by the usual set inclusion consists of two chains of length 2 and S is Artinian. As a consequence, we characterize in Corollary 3.4 integrally closed extensions $R \subset S$ of integral domains satisfying $|[R,S]_{\text{non}-\mathcal{P}}| = 3$.

As usual, $|\Omega|$ denotes the cardinal number of a set Ω. If R is an integral domain, then we let qf(R) denote its quotient field. Most of our notation is standard and can for instance be found in [13] and [22].

2 Preliminaries

In this brief section, we recall some definitions and results needed in our study. Our principal tool is the concept of minimal (ring) extensions, which was introduced by Ferrand-Olivier in [11]. We say that a ring extension $R \subset S$ is *minimal* if $|[R,S]| = 2$; that is, if $R \subset S$ and there does not exists a ring T such that $R \subset T \subset S$. A minimal ring extension is called *closed* in case R is integrally closed in S and *integral* in case S is integral over R. In [23], Knebusch and Zhang have introduced the concept of *Prüfer extensions* (which are a relativization of Prüfer rings). More precisely, a ring extension $R \subseteq S$ is called *Prüfer* if $R \subseteq T$ is a flat epimorphism for each $T \in [R,S]$. It is not difficult to check that any closed minimal extension is a Prüfer extension. One of the most important results concerning the last concept is [23, Theorem 5.2], which states that $R \subseteq S$ is a Prüfer extension if and only if each ring $T \in [R,S]$ is integrally closed in S.

We recall also from [17] that if $R \subseteq S$ is a ring extension, then an element s of S is said to be *primitive* over R if s is a root of a polynomial $f(X) \in R[X]$ with unit content. The ring extension $R \subseteq S$ is termed a *P-extension* if any element of S is primitive over R. In [8, Theorem 1], it was proved that $R \subseteq S$ is a Prüfer extension if and only if $R \subseteq S$ is an integrally closed P-extension.

3 Main results

We start our investigation with the following lemma.

Lemma 3.1. *Let $R \subset S$ be an integrally closed extension such that $|[R,S]_{\text{non}-\mathcal{P}}| = n \geq 1$. Then $R \subset S$ is an FIP extension.*

Proof. We argue by induction on n. If $n = 1$, then $R \subset S$ is a (closed) minimal extension according to [19, Theorem 1]. Thus, $R \subset S$ is an FIP extension. Assume now that $n \geq 2$ and that the result holds for any ring extension $A \subset B$ such that $|[A,B]_{\text{non}-\mathcal{P}}| < n$ and let $R \subset S$ be a ring extension satisfying $|[R,S]_{\text{non}-\mathcal{P}}| = n$. As R is not Artinian by virtue of [19, Proposition 1] and $n \geq 2$, one can consider a ring $T \in]R,S]$ which is minimal with respect to being non-Artinian. According to [19, Proposition 1], $R \subset T$ is a (closed) minimal extension. Thus, $R \subset T$ is an FIP extension. On the other hand, since $|[T,S]_{\text{non}-\mathcal{P}}| < n$, then the induction hypothesis guarantees that $T \subset S$ is an FIP extension. It follows that $R \subset T$ and $T \subseteq S$ are P-extensions. Hence, $R \subset S$ is a P-extension too (see [8, Theorem 2]). Since R is integrally closed in S, then $R \subset S$ is a Prüfer extension by virtue of [8, Theorem 1]. This implies that both $R \subset T$ and $T \subseteq S$ are integrally closed extensions. Therefore, $R \subset S$ is an FIP extension according to [10, Corollary 6.5]. This completes the proof. □

Lemma 3.2. *Let $R \subset S$ be an integrally closed FIP extension. Then the following statements hold true:*

(i) $R \subset S$ is a Prüfer extension.

(ii) If $A \in [R,S]_{\mathcal{P}}$, then $A = S$.

(iii) We have:

$$[R,S] = \begin{cases} [R,S]_{\text{non}-\mathcal{P}} \cup \{S\} & \text{if } S \text{ is Artinian} \\ [R,S]_{\text{non}-\mathcal{P}} & \text{if } S \text{ is not Artinian} \end{cases}$$

Proof. (i) Since $R \subset S$ is an integrally closed extension satisfying FIP, then [10, Theorem 6.3] ensures that $R \subset S$ is a Prüfer extension.

(ii) Let A be an Artinian ring in $[R,S]$ and assume by way of contradiction that $A \neq S$. First, we will show that (A,S) is an Artinian pair. To this end, let $B \in [A,S]$. Since $R \subset S$ is an FIP extension, then so is $A \subseteq B$. Therefore, there exists a (finite) chain of rings $A = A_0 \subset A_1 \subset ... \subset A_l = B$ going from A to B. As $A \subset A_1$ is a minimal extension and A is Artinian, then so is A_1 accordingly to [15, Theorem 2]. Again, as $A_1 \subset A_2$ is minimal and A_1 is Artinian, then so is A_2. Proceed along the same lines, one can easily check that B is Artinian. Hence, (A,S) is an Artinian pair. It follows from [16, Corollary 4.2] that $A \subset S$ is an integral extension. But, by (i), $A \subset S$ is integrally closed. The desired contradiction completing the proof of this assertion.

(iii) Assume first that S is Artinian. Then, assertion (ii) ensures that $[R,S]_{\mathcal{P}} = \{S\}$. Hence, $[R,S] = [R,S]_{\text{non}-\mathcal{P}} \cup \{S\}$. Suppose now that S is not Artinian. Then, using again assertion (ii), we get $[R,S] = [R,S]_{\text{non}-\mathcal{P}}$. The proof is complete.

□

The next result provides a complete characterization of integrally closed ring extensions $R \subset S$ with exactly three non-Artinian intermediate rings.

Theorem 3.3. *Let $R \subset S$ be a ring extension such that R is integrally closed in S. Then the following statements are equivalent:*

(i) $|[R,S]_{\text{non}-\mathcal{P}}| = 3$.

(ii) *(Exactly) one of the following conditions holds true:*

- *a. $[R,S]$ ordered by the usual set inclusion is a chain of length 2 and S is not Artinian.*
- *b. $[R,S]$ ordered by the usual set inclusion is a chain of length 3 and S is Artinian.*
- *c. $[R,S]$ ordered by the usual set inclusion consists of two chains of length 2 and S is Artinian.*

Proof. (i)⇒(ii) If S is not Artinian, then $|[R,S]| = 3$ by virtue of Lemmas 3.1 and 3.2. Thus, $[R,S]$ ordered by the usual set inclusion is a chain of length 2. Assume now that S is Artinian. Then, by using again Lemmas 3.1 and 3.2, we get $|[R,S]| = 4$. Thus, either $[R,S]$ ordered by the usual set inclusion is a chain of length 3 or $[R,S]$ ordered by the usual set inclusion consists of two chains of length 2.
(ii)⇒(i) If $[R,S] = \{R \subset A \subset S\}$ is a chain of length 2 and S is not Artinian, then both R and A are not Artinian according to [15, Theorem 2]. Assume now that $[R,S] = \{R \subset A \subset B \subset S\}$ is a chain of length 3 and S is Artinian. As $R \subset S$ is an integrally closed FIP extension, it follows from Lemma 3.2 that $[R,S]_{\text{non}-\mathcal{P}} = \{R, A, B\}$. Finally, assume that $[R,S]$ ordered by the usual set inclusion consists of two chains of length 2, say $R \subset A \subset S$ and $R \subset B \subset S$, with S an Artinian ring. Then, using again Lemma 3.2, we get readily $[R,S]_{\text{non}-\mathcal{P}} = \{R, A, B\}$. It follows that in all cases, we have $|[R,S]_{\text{non}-\mathcal{P}}| = 3$. The proof is complete. □

As a consequence, we characterize integrally closed ring extensions $R \subset S$ of integral domains such that $|[R,S]_{\text{non}-\mathcal{P}}| = 3$.

Corollary 3.4. *Let $R \subset S$ be an extension of integral domains such that R is integrally closed in S. Then the following statements are equivalent:*

(i) $|[R,S]_{\text{non}-\mathcal{P}}| = 3$.

(ii) *(Exactly) one of the following conditions holds true:*

a. $[R, S]$ *ordered by the usual set inclusion is a chain of length* 2 *and* S *is not a field.*

b. R *is a rank 3 valuation domain with quotient field* S*.*

c. R *is a one-dimensional Prüfer domain with quotient field* S *and with exactly two maximal ideals.*

Proof. (i)$\Rightarrow$(ii) If S is not a field, then it follows from Theorem 3.3. that $[R, S]$ ordered by the usual set inclusion is a chain of length 2. Suppose now that S is a field. If $[R, S]$ ordered by the usual set inclusion is a chain of length 3, say $[R, S] = \{R \subset A \subset B \subset S\}$, then R, A and B cannot be fields by virtue of Lemma 3.2. Thus, [24, p 1738] guarantees that $S = \mathrm{qf}(R)$. As R is integrally closed in S, then R would be a rank 3 valuation domain. Now, if $[R, S]$ ordered by the usual set inclusion consists of two chains of length 2. Using again Lemma 3.2, we infer that R is not a field. Hence, $S = \mathrm{qf}(R)$ accordingly to [24, p 1738]. Since R is integrally closed and $[R, \mathrm{qf}(R)]$ is finite, then [14, Theorem 1.5] ensures that R is a Prüfer domain. The ring R cannot be local since $[R, \mathrm{qf}(R)]$ is not totally ordered. Thus R has at least two maximal ideals M and N. Note that M and N are height one. Indeed, suppose for instance that $\mathrm{ht}(M) := m \geq 2$. Then there exists a maximal chain $(0) \subset P_1 \subset ... \subset P_{m-1} \subset P_m = M$ of prime ideals of R. Thus, we get the following chain: $R \subset R_M \subset R_{P_{m-1}} \subset ... \subset R_{P_1} \subset S = \mathrm{qf}(R)$ of overrings of R, which contradicts the fact that $[R, \mathrm{qf}(R)]$ consists of two chains of length 2. On the other hand, we claim that M and N are the unique maximal ideals of R. Indeed, assume the contrary and let M' be a third maximal ideal of R. Then we get $\{R, R_M, R_N, R_{M'}, \mathrm{qf}(R)\} \subseteq [R, \mathrm{qf}(R)]$, contradicting the fact that $|[R, \mathrm{qf}(R)]| = 4$. We conclude that R is a one-dimensional Prüfer domain with exactly two maximal ideals M and N and $S = \mathrm{qf}(R)$.
(ii)$\Rightarrow$(i) If $[R, S] = \{R \subset T \subset S\}$ is a chain of length 2 and S is not a field, then R and T cannot be fields by virtue of Lemma 3.2. Hence, $|[R, S]_{\text{non}-\mathcal{P}}| = 3$. Assume now that R is a rank 3 valuation domain with quotient field S. Then the spectrum of R is a chain $(0) \subset P \subset Q \subset M$ and so the overrings of R are exactly R, R_P, R_Q and S. Thus, $|[R, S]_{\text{non}-\mathcal{P}}| = 3$. Finally, suppose that R is a one-dimensional Prüfer domain with exactly two maximal ideals M and N and $S = \mathrm{qf}(R)$. Then the overrings of R are R, R_M, R_N and S. Therefore, $|[R, S]_{\text{non}-\mathcal{P}}| = 3$. This completes the proof. □

References

[1] B. Al Subaiei, N. Jarboui, *On the commutative ring extensions with at most two non-Prüfer intermediate rings*, Mediterr. J. Math. **18** (4) (2021), Paper No. 132, 19 pp.

[2] A. Ayache, M. Ben Nasr, O. Echi, N. Jarboui, *Universally catenarian and going-down pairs of rings*, Math. Z. **238** (4) (2001), 695-731.

[3] A. Ayache, N. Jarboui, *Maximal non-Noetherian subring of a domain*, J. Algebra **248** (2) (2002), 806–823.

[4] A. Ayache, N. Jarboui, E. Massaoud, *Pairs of domains where all intermediate domains are treed*, Arab. J. Sci. Eng. **36** (6) (2011), 933–946.

[5] M. Ben Nasr, O. Echi, L. Izelgue, N. Jarboui, *Pairs of domains where all intermediate domains are Jaffard*, J. Pure Appl. Algebra **145** (1) (2000), 1-18.

[6] M. Ben Nasr, N. Jarboui, *Maximal non-Jaffard subrings of a field*, Publ. Math. **44** (1) (2000), 157-175.

[7] M. Ben Nasr, N. Jarboui, *On maximal non-valuation subrings*, Houston J. Math. **37** (1) (2011), 47-59.

[8] M. Ben Nasr, N. Jarboui, *New results about normal pairs of rings with zero divisors*, Ric. Mat. **63** (1) (2014), 149–155.

[9] E. D. Davis, *Overrings of commutative rings, III: Normal pairs*, Trans. Amer. Math. Soc. **182** (1973), 175-185.

[10] D. E. Dobbs, G. Picavet, M. Picavet-L'Hermitte, *Characterizing the ring extensions that satisfy FIP or FCP*, J. Algebra **371** (2012), 391-429.

[11] D. Ferrand, J.-P. Olivier, *Homomorphismes minimaux d'anneaux*, J. Algebra **16** (1970), 461-471.

[12] B. Gasmi, N. Jarboui, *A note on intermediate rings between* $D + I$ *and* $K[y_1]] \cdots [y_t]]$, Palest. J. Math. **7** (2) (2018), 414–416.

[13] R. Gilmer, *Multiplicative Ideal Theory*. Dekker, New York, (1972).

[14] R. Gilmer, *Some finiteness conditions on the set of overrings of an integral domain*, Proc. Amer. Math. Soc. **131** (8) (2003), 2337-2346.

[15] R. Gilmer, W. Heinzer, *An application of Jonsson modules to some questions concerning proper subrings*, Math. Scand. **70** (1) (1992), 34–42.

[16] R. Gilmer, W. Heinzer, *Products of commutative rings and zero-dimensionality* , Trans. Amer. Math. Soc. **331** (2) (1992), 663–680.

[17] R. Gilmer, J. Hoffmann, *A characterization of Prüfer domains in terms of polynomials*, Pacific J. Math. **60** (1) (1975), 81–85.

[18] N. Jarboui, *Pairs of domains where most of the intermediate domains are Prüfer*, J. Algebra Appl. **20** (6) (2021), 2150101, 13 pp.

[19] N. Jarboui, N. Al-Kuleab, O. Almallah, *Ring extensions with finitely many non-Artinian intermediate rings*, J. Math. (2020), Art. ID 7416893, 6 pp.

[20] N. Jarboui, M. Toumi, S. Trabelsi, *Some questions concerning proper subrings*, Ric. Mat. **64** (1) (2015), 51–55.

[21] N. Jarboui, S. Trabelsi, *Some results about proper overrings of pseudo-valuation domains*, J. Algebra Appl. **15** (5) (2016), 1650099, 16 pp.

[22] I. Kaplansky, *Commutative Rings*, rev. ed., Univ. Chicago Press, Chicago, (1974).

[23] M. Knebusch, D. Zhang, *Manis Valuations and Prüfer Extensions I*, Lecture Notes Math. 1791, Springer-Verlag, Berlin-Heidelberg, (2002).

[24] J. Sato, T. Sugatani, K. I. Yoshida, *On minimal overrings of a Noetherian domain*, Comm. Algebra **20** (6) (1992), 1746–1753.

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