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Bounds of Sombor Index for F-Sum Operation

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Abstract. Graph operations have a major impact in the aspects of theory and empirical literature of the domain. For relating the molecular topology to any real chemical attribute, the conversion of the relevant details embedded into chemical structure to some numeric value becomes so vital which ultimately paves the way for the emergence of topological indices. Topological descriptor acts as an effective graph invariant in chemical graph theory associated with certain molecular structure. Recently, the study on the sombor index is initiated by I.Gutman [16]. In the article, we utilise combinatorial inequalities, including the general sum-connectivity index, the first general zagreb index and few other indices in their formulations, for the determination of bounds for sombor index for the F-sum operation of connected graphs.

1 Introduction

Topology of a molecule is essentially a non numerical component of mathematics. Numerous measurable molecular attributes are often indicated in the form of specific numerals. For the evaluation of molecular descriptors, a chemical compound needs to amend itself to a molecular graph such that the atoms of the molecule correspond to vertices and the atomic links are depicted to be the edges. For a molecular graph $G = (V, E)$, $V(G)$ indicates the vertex set and $E(G)$ represents the edge set. p and q denote cardinality of vertex and edge set for graph, G respectively. Let $d(y)$ denote the degree of vertex y in G and $e = yz$ is the edge joining the vertex y with vertex z. The line graph of G, $L(G)$ is the graph wherein the edges of G correspond to vertices of $L(G)$ and two edges of $L(G)$ are adjacent if and only if they are incident in G. Δ_G and δ_G indicates maximum and minimum degree for the graph, G respectively.

Wiener index is a primitive and extensively explored molecular descriptor [26]. Randić index has numerous implementations in chemical and therapeutic domains [15, 25]. Since then, a huge category of molecular descriptors have come into picture and lot of activities have been achieved in this aspect of exploration of the indices of molecular networks. The First and Second Zagreb Index are the widely known topological descriptors described by Gutman to specify π-electron energy of the molecules[19, 18]. Certain generalizations were implemented by Xueliang Li and Jie Zheng [21] which ultimately led to the introduction of another topological descriptor known as the first general zagreb index, M_1^α ($\alpha \in \mathbb{R}$). Certain properties along with some edge operations have been observed with respect to the index[22].

$$M_1^\alpha(G) = \sum_{z \in V(G)} (d_G(z))^\alpha$$

Sum-connectivity index, χ was proposed by Bo Zhou and Nenad Trinajstić. Consequently several features, bounds and inequalities for respective invariant were presented in [28].

$$\chi(G) = \sum_{yz \in E(G)} (d_G(y) + d_G(z))^{-1/2}$$

Subsequently, in [29] generalizing sum-connectivity index along with first zagreb index initiated study on another descriptor called the general sum-connectivity index, χ_α ($\alpha \in \mathbb{R}$). Some derivations for the descriptor related to certain graph operations have also been performed[2].

$$\chi_\alpha(G) = \sum_{yz \in E(G)} (d_G(y) + d_G(z))^\alpha$$

Lately, the theories and applications on sombor index(SO) were postulated by Gutman[16] and further work on the graph invariant has been carried out[8, 24, 17].

$$SO(G) = \sum_{yz \in E(G)} \sqrt{d_G(y)^2 + d_G(z)^2}$$

The analysis of sombor index related to some graph operations have been resolved in [12, 6, 20]. On a connected graph G, [11, 27] defines the graphs related as:

(i) $S(G)$ or subdivision graph is generated by replacing every link of the graph with a node of degree two, keeping the original nodes unchanged.

(ii) $R(G)$ or triangle parallel graph is the graph including the links of $S(G)$ along with the links of G.

(iii) $Q(G)$ or line superposition graph is the one including links of $S(G)$ along with the links of $L(G)$.

(iv) $T(G)$ or total graph includes links of $S(G)$, $L(G)$ along with the links of G.

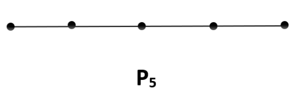

Figure 1: Base graph P_5

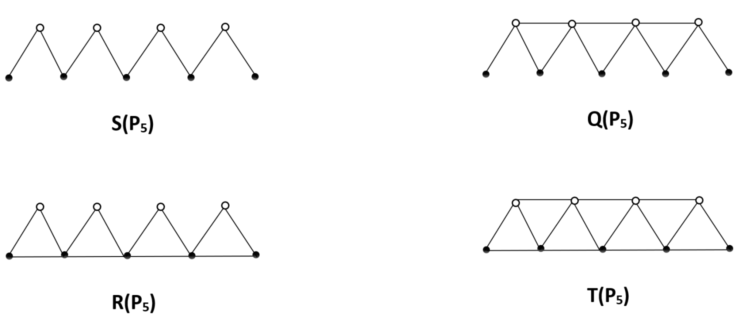

Figure 2: $S(P_5)$, $R(P_5)$, $Q(P_5)$, $T(P_5)$

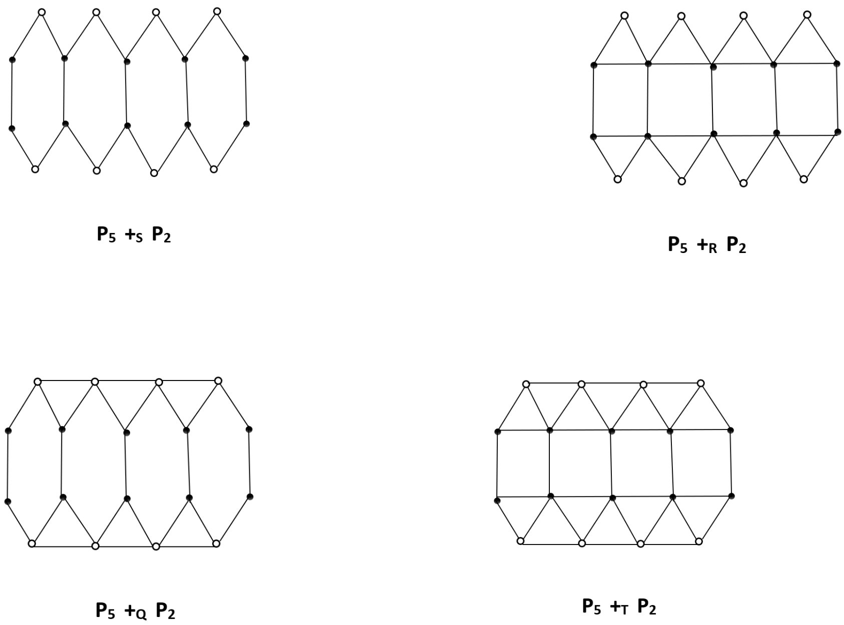

Figure 3: $P_5 +_F P_2$

The F-sum represented as $G_1 +_F G_2$ for $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$, is the graph which includes p_2 replicas of $F(G_1)$ and mark these replicas as nodes of G_2 [11]. It has $V(G_1 +_F G_2) = (V(G_1) \cup E(G_1)) \times V(G_2)$ and $(a_1, b_1)(a_2, b_2) \in E(G_1 +_F G_2)$, iff $[a_1 = b_1 \in V(G_1)$ and $a_2 b_2 \in E(G_2)]$ or $[a_2 = b_2$ and $(a_1, b_1) \in E(F(G_1))]$. The graphs related to the four F-sum operations are depicted in Figures (1), (2), (3). The formulations of the zagreb indices for F-sum graph operation have been established in [9]. Also findings related to other significant graph invariants of graph operations have also been investigated [4, 1, 10, 5, 3, 13, 7, 23, 14]. Here, some bounds related to the sombor index for F-Sum operation for two connected graphs are considered.

2 Methodology

This segment discusses and reveals our key findings on the relevant versions of graph operation. The arguments over the below stated theorems have been obtained from the descriptions.

Theorem 2.1. *Assume $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ represent connected graphs,*

$$\begin{aligned} SO(G_1 +_S G_2) \quad &\leq \quad p_1 SO(G_2) + p_2 SO(S(G_1)) + \sqrt{2}\big[M_1^{1/2}(G_1)\big(M_1^{1/2}(G_2) \\ &\quad + \chi_{1/2}(G_2)\big) + \big(2 + 2\sqrt{2}\big) q_1 q_2\big] \end{aligned}$$

Proof.

$$\begin{aligned} SO(G_1 +_S G_2) &= \sum_{(a,b)(c,d) \in E(G_1 +_S G_2)} \sqrt{d_{G_1 +_S G_2}(a,b)^2 + d_{G_1 +_S G_2}(c,d)^2} \\ &= \sum_{y \in V(G_1)} \sum_{bd \in E(G_2)} \sqrt{d_{G_1 +_S G_2}(y,b)^2 + d_{G_1 +_S G_2}(y,d)^2} \\ &+ \sum_{z \in V(G_2)} \sum_{ac \in E(S(G_1))} \sqrt{d_{G_1 +_S G_2}(a,z)^2 + d_{G_1 +_S G_2}(c,z)^2} \\ &= \sum f + \sum g \end{aligned}$$

where,

$$\sum f = \sum_{y\in V(G_1)} \sum_{bd\in E(G_2)} \sqrt{d_{G_1+_SG_2}(y,b)^2 + d_{G_1+_SG_2}(y,d)^2}$$

$$\sum g = \sum_{z\in V(G_2)} \sum_{ac\in E(S(G_1))} \sqrt{d_{G_1+_SG_2}(a,z)^2 + d_{G_1+_SG_2}(c,z)^2}$$

For the computation of $\sum f$,

$$\sqrt{d_{G_1+_SG_2}(y,b)^2 + d_{G_1+_SG_2}(y,d)^2} = \sqrt{\left(d_{G_1}(y)+d_{G_2}(b)\right)^2 + \left(d_{G_1}(y)+d_{G_2}(d)\right)^2}$$

$$= \sqrt{2d_{G_1}(y)^2 + (d_{G_2}(b)^2 + d_{G_2}(d)^2) + 2d_{G_1}(y)\left(d_{G_2}(b)+d_{G_2}(d)\right)}$$

$$\Rightarrow \sum f \quad \leq \quad p_1 SO(G_2) + \sqrt{2}\left(M_1^{1/2}(G_1)\chi_{1/2}(G_2) + 2q_1q_2\right)$$

Also for $\sum g$ when $a \in V(G_1)$ and $c \in V(S(G_1)) - V(G_1)$,

$$\sqrt{d_{G_1+_SG_2}(a,z)^2 + d_{G_1+_SG_2}(c,z)^2} = \sqrt{\left(d_{S(G_1)}(a)+d_{G_2}(z)\right)^2 + d_{S(G_1)}(c)^2}$$

$$= \sqrt{\left(d_{S(G_1)}(a)^2 + d_{S(G_1)}(c)^2\right) + d_{G_2}(z)^2 + 2d_{S(G_1)}(a)d_{G_2}(z)}$$

$$\Rightarrow \sum g \quad \leq \quad p_2 SO(S(G_1)) + \sqrt{2}\left(M_1^{1/2}(G_1)M_1^{1/2}(G_2) + 4q_1q_2\right)$$

Hence from the computations,

$$\begin{aligned} SO(G_1 +_S G_2) \quad &\leq \quad p_1SO(G_2) + p_2SO(S(G_1)) + \sqrt{2}\big[M_1^{1/2}(G_1)\big(M_1^{1/2}(G_2) \\ &\qquad + \chi_{1/2}(G_2)\big) + \big(2+2\sqrt{2}\big)q_1q_2\big] \end{aligned}$$

This proves the result.

□

Theorem 2.2. *Assume $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ represent connected graphs,*

$$\begin{aligned} SO(G_1 +_R G_2) \quad &\leq \quad p_1SO(G_2) + 2p_2\Big[SO(G_1) + M_1(G_1)\Big] + 2\Big[M_1^{1/2}(G_1)\chi_{1/2}(G_2) \\ &\qquad + M_1^{1/2}(G_2)\big(\chi_{1/2}(G_1) + M_1^{3/2}(G_1)\big) + \big(3\sqrt{2}+2\big)q_1q_2 + 2q_1p_2\Big] \end{aligned}$$

Proof.

$$\begin{aligned} SO(G_1 +_R G_2) \quad &\leq \sum_{(a,b)(c,d)\in E(G_1+_RG_2)} \sqrt{d_{G_1+_RG_2}(a,b)^2 + d_{G_1+_RG_2}(c,d)^2} \\ &= \sum_{a\in V(G_1)} \sum_{bd\in E(G_2)} \sqrt{d_{G_1+_RG_2}(a,b)^2 + d_{G_1+_RG_2}(a,d)^2} \\ &+ \sum_{b\in V(G_2)} \sum_{ac\in E(R(G_1))} \sqrt{d_{G_1+_RG_2}(a,b)^2 + d_{G_1+_RG_2}(c,b)^2} \\ &= \sum f + \sum g \end{aligned}$$

where,

$$\sum f = \sum_{a\in V(G_1)} \sum_{bd\in E(G_2)} \sqrt{d_{G_1+_RG_2}(a,b)^2 + d_{G_1+_RG_2}(a,d)^2}$$
$$\sum g = \sum_{b\in V(G_2)} \sum_{ac\in E(R(G_1))} \sqrt{d_{G_1+_RG_2}(a,b)^2 + d_{G_1+_RG_2}(c,b)^2}$$

For the computation of $\sum f$,

$$\begin{aligned}
&\sqrt{d_{G_1+_RG_2}(a,b)^2 + d_{G_1+_RG_2}(a,d)^2}\\
&\quad = \sqrt{\left(d_{R(G_1)}(a) + d_{G_2}(b)\right)^2 + \left(d_{R(G_1)}(a) + d_{G_2}(d)\right)^2}\\
&\quad = \sqrt{2d_{R(G_1)}(a)^2 + (d_{G_2}(b)^2 + d_{G_2}(d)^2) + 2d_{R(G_1)}(a)\,(d_{G_2}(b) + d_{G_2}(d))}
\end{aligned}$$

Note : $d_{R(G_1)}(a) = 2d_{G_1}(a)$ for $a \in V(G_1)$, and $d_{R(G_1)}(b) = 2$ for $b \in V(R(G_1)) - V(G_1)$

$$\begin{aligned}
&\sqrt{d_{G_1+_RG_2}(a,b)^2 + d_{G_1+_RG_2}(a,d)^2}\\
&\qquad = \sqrt{8d_{G_1}(a)^2 + (d_{G_2}(b)^2 + d_{G_2}(d)^2) + 4d_{G_1}(a)\,(d_{G_2}(b) + d_{G_2}(d))}
\end{aligned}$$

$$\Rightarrow \quad \sum f \quad \leq \quad p_1 SO(G_2) + 2M_1^{1/2}(G_1)\chi_{1/2}(G_2) + 4\sqrt{2}q_1q_2$$

Now for $\sum g$,

$$\begin{aligned}
&\sum_{b\in V(G_2)} \sum_{ac\in E(R(G_1))} \sqrt{d_{G_1+_RG_2}(a,b)^2 + d_{G_1+_RG_2}(c,b)^2}\\
&\quad = \sum_{b\in V(G_2)} \sum_{\substack{ac\in E(R(G_1))\\ a,c\in V(G_1)}} \sqrt{d_{G_1+_RG_2}(a,b)^2 + d_{G_1+_RG_2}(c,b)^2}\\
&\quad + \sum_{b\in V(G_2)} \sum_{\substack{ac\in E(R(G_1))\\ a\in V(G_1)\\ c\in V(R(G_1))-V(G_1)}} \sqrt{d_{G_1+_RG_2}(a,b)^2 + d_{G_1+_RG_2}(c,b)^2}
\end{aligned}$$

$$\Rightarrow \quad \sum g \quad = \quad \sum g_1 + \sum g_2$$

where,

$$\sum g_1 \quad = \quad \sum_{b\in V(G_2)} \sum_{\substack{ac\in E(R(G_1))\\ a,c\in V(G_1)}} \sqrt{d_{G_1+_RG_2}(a,b)^2 + d_{G_1+_RG_2}(c,b)^2}$$
$$\sum g_2 \quad = \quad \sum_{b\in V(G_2)} \sum_{\substack{ac\in E(R(G_1))\\ a\in V(G_1)\\ c\in V(R(G_1))-V(G_1)}} \sqrt{d_{G_1+_RG_2}(a,b)^2 + d_{G_1+_RG_2}(c,b)^2}$$

To compute $\sum g_1$, for $b \in V(G_2)$, $ac \in E(R(G_1))$ iff $ac \in E(G_1)$ we get,

$$\begin{aligned}
&\sum_{b\in V(G_2)} \sum_{\substack{ac\in E(R(G_1))\\ a,c\in V(G_1)}} \sqrt{d_{G_1+_RG_2}(a,b)^2 + d_{G_1+_RG_2}(c,b)^2}\\
&= \sum_{b\in V(G_2)} \sum_{ac\in E(G_1)} \sqrt{d_{G_1+_RG_2}(a,b)^2 + d_{G_1+_RG_2}(c,b)^2}\\
&= \sum_{b\in V(G_2)} \sum_{ac\in E(G_1)} \sqrt{\left(d_{R(G_1)}(a) + d_{G_2}(b)\right)^2 + \left(d_{R(G_1)}(c) + d_{G_2}(b)\right)^2}\\
&= \sum_{b\in V(G_2)} \sum_{ac\in E(G_1)} \sqrt{\left(d_{R(G_1)}(a)^2 + d_{R(G_1)}(c)^2\right) + 2d_{G_2}(b)^2 + 2d_{G_2}(b)\left(d_{R(G_1)}(a) + d_{R(G_1)}(c)\right)}\\
&= \sum_{b\in V(G_2)} \sum_{ac\in E(G_1)} \sqrt{4\left(d_{G_1}(a)^2 + d_{G_1}(c)^2\right) + 2d_{G_2}(b)^2 + 4d_{G_2}(b)\left(d_{G_1}(a) + d_{G_1}(c)\right)}
\end{aligned}$$

$$\begin{aligned}
\sum_{b\in V(G_2)} \sum_{\substack{ac\in E(R(G_1))\\ a,c\in V(G_1)}} \sqrt{d_{G_1+_RG_2}(a,b)^2 + d_{G_1+_RG_2}(c,b)^2} \quad &\leq \quad 2p_2 SO(G_1)\\
&+ 2\chi_{1/2}(G_1)M_1^{1/2}(G_2) + 2\sqrt{2}q_1q_2
\end{aligned}$$

$$\Rightarrow \quad \sum g_1 \quad \leq \quad 2p_2 SO(G_1) + 2\chi_{1/2}(G_1)M_1^{1/2}(G_2) + 2\sqrt{2}q_1q_2$$

For the computation of $\sum g_2$ we get,

$$\begin{aligned}
&\sum_{b\in V(G_2)} \sum_{\substack{ac\in E(R(G_1))\\ a\in V(G_1)\\ c\in V(R(G_1))-V(G_1)}} \sqrt{d_{G_1+_RG_2}(a,b)^2 + d_{G_1+_RG_2}(c,b)^2}\\
&= \sum_{b\in V(G_2)} \sum_{\substack{ac\in E(R(G_1))\\ a\in V(G_1)\\ c\in V(R(G_1))-V(G_1)}} \sqrt{\left(d_{R(G_1)}(a) + d_{G_2}(b)\right)^2 + d_{R(G_1)}(c)^2}\\
&= \sum_{b\in V(G_2)} \sum_{\substack{ac\in E(R(G_1))\\ a\in V(G_1)\\ c\in V(R(G_1))-V(G_1)}} \sqrt{d_{R(G_1)}(a)^2 + d_{R(G_1)}(c)^2 + d_{G_2}(b)^2 + 2d_{R(G_1)}(a)d_{G_2}(b)}\\
&= \sum_{b\in V(G_2)} \sum_{\substack{ac\in E(R(G_1))\\ a\in V(G_1)\\ c\in V(R(G_1))-V(G_1)}} \sqrt{4d_{G_1}(a)^2 + 4 + d_{G_2}(b)^2 + 4d_{G_1}(a)d_{G_2}(b)}
\end{aligned}$$

$$\Rightarrow \quad \sum g_2 \quad \leq \quad 2p_2M_1(G_1) + 2M_1^{3/2}(G_1)M_1^{1/2}(G_2) + 4q_1(p_2 + q_2)$$

Thus, we get

$$\begin{aligned}
\sum g \quad \leq \quad & 2\Big[p_2\big(SO(G_1) + M_1(G_1)\big) + M_1^{1/2}(G_2)\big(\chi_{1/2}(G_1) + M_1^{3/2}(G_1)\big)\\
&+ \big(\sqrt{2} + 2\big)q_1q_2 + 2q_1p_2\Big]
\end{aligned}$$

From all the computations,

$$\begin{aligned} SO(G_1 +_R G_2) \quad &\leq \quad p_1 SO(G_2) + 2p_2\Big[SO(G_1) + M_1(G_1)\Big] + 2\Big[M_1^{1/2}(G_1)\chi_{1/2}(G_2) \\ &+ M_1^{1/2}(G_2)\big(\chi_{1/2}(G_1) + M_1^{3/2}(G_1)\big) + \big(3\sqrt{2}+2\big)q_1q_2 + 2q_1p_2\Big] \end{aligned}$$

This proves the result. □

Theorem 2.3. *Assume $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ represent connected graphs,*

$$\begin{aligned} SO(G_1 +_Q G_2) \quad &\leq \quad 3p_2M_1(G_1) + p_1SO(G_2) + \sqrt{2}\Big[M_1^{1/2}(G_1)\chi_{1/2}(G_2) \\ &+ M_1^{1/2}(G_2)M_1^{3/2}(G_1) + 2\big(\sqrt{2}+1\big)q_1q_2 + 2p_2\Delta_{G_1}\Big(\frac{1}{2}M_1(G_1) - q_1\Big)\Big] \end{aligned}$$

Proof.

$$\begin{aligned} SO(G_1 +_Q G_2) \quad &= \quad \sum_{(a,b)(c,d)\in E(G_1+_QG_2)} \sqrt{d_{G_1+_QG_2}(a,b)^2 + d_{G_1+_QG_2}(c,d)^2} \\ &= \quad \sum_{a\in V(G_1)}\sum_{bd\in E(G_2)} \sqrt{d_{G_1+_QG_2}(a,b)^2 + d_{G_1+_QG_2}(a,d)^2} \\ &+ \quad \sum_{b\in V(G_2)}\sum_{ac\in E(Q(G_1))} \sqrt{d_{G_1+_QG_2}(a,b)^2 + d_{G_1+_QG_2}(c,b)^2} \\ &= \quad \sum f + \sum g \end{aligned}$$

where,

$$\begin{aligned} \sum f = &\quad \sum_{a\in V(G_1)}\sum_{bd\in E(G_2)} \sqrt{d_{G_1+_QG_2}(a,b)^2 + d_{G_1+_QG_2}(a,d)^2} \\ \sum g = &\quad \sum_{b\in V(G_2)}\sum_{ac\in E(Q(G_1))} \sqrt{d_{G_1+_QG_2}(a,b)^2 + d_{G_1+_QG_2}(c,b)^2} \end{aligned}$$

For the computation of $\sum f$, we have $a \in V(G_1)$ and $bd \in E(G_2)$,

$$\begin{aligned} &\sum_{a\in V(G_1)}\sum_{bd\in E(G_2)} \sqrt{d_{G_1+_QG_2}(a,b)^2 + d_{G_1+_QG_2}(a,d)^2} \\ &= \sum_{a\in V(G_1)}\sum_{bd\in E(G_2)} \sqrt{\big(d_{Q(G_1)}(a) + d_{G_2}(b)\big)^2 + \big(d_{Q(G_1)}(a) + d_{G_2}(d)\big)^2} \\ &= \sum_{a\in V(G_1)}\sum_{bd\in E(G_2)} \sqrt{2d_{Q(G_1)}(a)^2 + (d_{G_2}(b)^2 + d_{G_2}(d)^2) + 2d_{Q(G_1)}(a)\,(d_{G_2}(b) + d_{G_2}(d))} \\ &= \sum_{a\in V(G_1)}\sum_{bd\in E(G_2)} \sqrt{2d_{G_1}(a)^2 + (d_{G_2}(b)^2 + d_{G_2}(d)^2) + 2d_{G_1}(a)\,(d_{G_2}(b) + d_{G_2}(d))} \end{aligned}$$

$$\Rightarrow \quad \sum f \quad \leq \quad p_1SO(G_2) + \sqrt{2}\Big[M_1^{1/2}(G_1)\chi_{1/2}(G_2) + 2q_1q_2\Big]$$

For the computation of $\sum g$, we have $ac \in E(Q(G_1))$ and $b \in V(G_2)$,

$$\sum_{b\in V(G_2)} \sum_{ac\in E(Q(G_1))} \sqrt{d_{G_1+_QG_2}(a,b)^2 + d_{G_1+_QG_2}(c,b)^2}$$

$$= \sum_{b\in V(G_2)} \sum_{\substack{ac\in E(Q(G_1))\\ a\in V(G_1)\\ c\in V(Q(G_1))-V(G_1)}} \sqrt{d_{G_1+_QG_2}(a,b)^2 + d_{G_1+_QG_2}(c,b)^2}$$

$$+ \sum_{b\in V(G_2)} \sum_{\substack{ac\in E(Q(G_1))\\ a,c\in V(Q(G_1))-V(G_1)}} \sqrt{d_{G_1+_QG_2}(a,b)^2 + d_{G_1+_QG_2}(c,b)^2}$$

$$\Rightarrow \quad \sum g \quad = \quad \sum g_1 + \sum g_2$$

where

$$\sum g_1 = \sum_{b\in V(G_2)} \sum_{\substack{ac\in E(Q(G_1))\\ a\in V(G_1)\\ c\in V(Q(G_1))-V(G_1)}} \sqrt{d_{G_1+_QG_2}(a,b)^2 + d_{G_1+_QG_2}(c,b)^2}$$

$$\sum g_2 = \sum_{b\in V(G_2)} \sum_{\substack{ac\in E(Q(G_1))\\ a,c\in V(Q(G_1))-V(G_1)}} \sqrt{d_{G_1+_QG_2}(a,b)^2 + d_{G_1+_QG_2}(c,b)^2}$$

To compute $\sum g_1$, for $b \in V(G_2)$, $ac \in E(Q(G_1))$ iff $a \in V(G_1)$ and $c \in V(Q(G_1)) - V(G_1)$ we get,

$$\sum_{b\in V(G_2)} \sum_{\substack{ac\in E(Q(G_1))\\ a\in V(G_1)\\ c\in V(Q(G_1))-V(G_1)}} \sqrt{d_{G_1+_QG_2}(a,b)^2 + d_{G_1+_QG_2}(c,b)^2}$$

$$= \sum_{b\in V(G_2)} \sum_{\substack{ac\in E(Q(G_1))\\ a\in V(G_1)\\ c\in V(Q(G_1))-V(G_1)}} \sqrt{\left(d_{Q(G_1)}(a) + d_{G_2}(b)\right)^2 + d_{Q(G_1)}(c)^2}$$

$$= \sum_{b\in V(G_2)} \sum_{\substack{ac\in E(Q(G_1))\\ a\in V(G_1)\\ c\in V(Q(G_1))-V(G_1)}} \sqrt{d_{Q(G_1)}(a)^2 + 2d_{Q(G_1)}(a)d_{G_2}(b) + d_{G_2}(b)^2 + d_{Q(G_1)}(c)^2}$$

Since for $a \in V(G_1)$, $d_{Q(G_1)}(a) = d_{G_1}(a)$

$$\sum_{b\in V(G_2)} \sum_{\substack{ac\in E(Q(G_1))\\ a\in V(G_1)\\ c\in V(Q(G_1))-V(G_1)}} \sqrt{d_{G_1+_QG_2}(a,b)^2 + d_{G_1+_QG_2}(c,b)^2}$$

$$= \sum_{b\in V(G_2)} \sum_{\substack{ac\in E(Q(G_1))\\ a\in V(G_1)\\ c\in V(Q(G_1))-V(G_1)}} \sqrt{d_{G_1}(a)^2 + 2d_{G_1}(a)d_{G_2}(b) + d_{G_2}(b)^2 + d_{Q(G_1)}(c)^2}$$

$$\leq p_2 M_1(G_1) + \sqrt{2} M_1^{3/2}(G_1) M_1^{1/2}(G_2) + 4q_1q_2 + \sum_{b\in V(G_2)} \sum_{\substack{ac\in E(Q(G_1))\\ a\in V(G_1)\\ c\in V(Q(G_1))-V(G_1)}} d_{Q(G_1)}(c)$$

Now for $c \in V(Q(G_1)) - V(G_1)$, let vertex c be inserted into edge u_iu_j of G_1,

$$d_{Q(G_1)}(c) = d_{G_1}(u_i) + d_{G_1}(u_j)$$

$$\begin{aligned}\sum_{b\in V(G_2)} \sum_{\substack{ac\in E(Q(G_1))\\ a\in V(G_1)\\ c\in V(Q(G_1))-V(G_1)}} \sqrt{d_{G_1+_QG_2}(a,b)^2 + d_{G_1+_QG_2}(c,b)^2} &\le p_2M_1(G_1) + \sqrt{2}M_1^{3/2}(G_1)M_1^{1/2}(G_2) + 4q_1q_2\\ &\quad + 2\sum_{b\in V(G_2)}\sum_{u_iu_j\in E(G_1)} (d_{G_1}(u_i) + d_{G_1}(u_j))\end{aligned}$$

$$\Rightarrow \quad \sum g_1 \le 3p_2M_1(G_1) + \sqrt{2}M_1^{3/2}(G_1)M_1^{1/2}(G_2) + 4q_1q_2$$

To compute $\sum g_2$, for $b \in V(G_2)$, $ac \in E(Q(G_1))$ when $a, c \in V(Q(G_1)) - V(G_1)$ we get,

$$\begin{aligned}\sum_{b\in V(G_2)} \sum_{\substack{ac\in E(Q(G_1))\\ a,c\in V(Q(G_1))-V(G_1)}} \sqrt{d_{G_1+_QG_2}(a,b)^2 + d_{G_1+_QG_2}(c,b)^2} &= \sum_{b\in V(G_2)} \sum_{\substack{ac\in E(Q(G_1))\\ a,c\in V(Q(G_1))-V(G_1)}} \sqrt{d_{Q(G_1)}(a)^2 + d_{Q(G_1)}(c)^2}\end{aligned}$$

While $a, c \in V(Q(G_1)) - V(G_1)$ we consider, let the vertices a and c be inserted into the edges u_iu_j and u_ju_k of G_1 respectively then,

$$\begin{aligned}\sum_{b\in V(G_2)} \sum_{\substack{ac\in E(Q(G_1))\\ a,c\in V(Q(G_1))-V(G_1)}} \sqrt{d_{G_1+_QG_2}(a,b)^2 + d_{G_1+_QG_2}(c,b)^2} &= p_2 \sum_{\substack{u_iu_j\in E(G_1)\\ u_ju_k\in E(G_1)}} \sqrt{\left(d_{G_1}(u_i) + d_{G_1}(u_j)\right)^2 + \left(d_{G_1}(u_j) + d_{G_1}(u_k)\right)^2}\end{aligned}$$

$$\Rightarrow \quad \sum g_2 \le 2\sqrt{2}p_2\Delta_{G_1}\left(\frac{1}{2}M_1(G_1) - q_1\right)$$

Thus we get,

$$\sum g \le 3p_2M_1(G_1) + \sqrt{2}M_1^{3/2}(G_1)M_1^{1/2}(G_2) + 2\sqrt{2}p_2\Delta_{G_1}\left(\frac{1}{2}M_1(G_1) - q_1\right) + 4q_1q_2$$

From all the computations,

$$\begin{aligned}SO(G_1 +_Q G_2) &\le 3p_2M_1(G_1) + p_1SO(G_2) + \sqrt{2}\Big[M_1^{1/2}(G_1)\chi_{1/2}(G_2)\\ &\quad + M_1^{1/2}(G_2)M_1^{3/2}(G_1) + 2(\sqrt{2}+1)q_1q_2 + 2p_2\Delta_{G_1}\left(\frac{1}{2}M_1(G_1) - q_1\right)\Big]\end{aligned}$$

This proves the result. □

Theorem 2.4. *Assume $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ represent connected graphs*

$$\begin{aligned}SO(G_1 +_T G_2) &\le p_1SO(G_2) + 2p_2SO(G_1) + 2\Big[M_1^{1/2}(G_1)\chi_{1/2}(G_2)\\ &\quad + M_1^{1/2}(G_2)\big(M_1^{3/2}(G_1) + \chi_{1/2}(G_1)\big)\Big] + 2(3\sqrt{2}+2)q_1q_2\\ &\quad + p_2M_1(G_1)\left(4 + \sqrt{2}\Delta_{G_1}\right) - 2\sqrt{2}\Delta_{G_1}p_2q_1\end{aligned}$$

Proof. We have for every vertex $a \in V(G_1)$ and $b \in V(G_2)$,

$$d_{G_1+_T G_2}(a,b) = d_{G_1+_R G_2}(a,b) \tag{2.1}$$

Also for every vertex $a \in V(T(G_1)) - V(G_1)$ and $b \in V(G_2)$,

$$d_{G_1+_T G_2}(a,b) = d_{G_1+_Q G_2}(a,b) \tag{2.2}$$

$$\begin{aligned} SO(G_1 +_T G_2) &= \sum_{(a,b)(c,d)\in E(G_1+_T G_2))} \sqrt{d_{G_1+_T G_2}(a,b)^2 + d_{G_1+_T G_2}(c,d)^2} \\ &= \sum_{a\in V(G_1)} \sum_{bd\in E(G_2)} \sqrt{d_{G_1+_T G_2}(a,b)^2 + d_{G_1+_T G_2}(a,d)^2} \\ &+ \sum_{b\in V(G_2)} \sum_{ac\in E(T(G_1))} \sqrt{d_{G_1+_T G_2}(a,b)^2 + d_{G_1+_T G_2}(c,b)^2} \\ &= \sum f + \sum g \end{aligned}$$

where

$$\begin{aligned} \sum f &= \sum_{a\in V(G_1)} \sum_{bd\in E(G_2)} \sqrt{d_{G_1+_T G_2}(a,b)^2 + d_{G_1+_T G_2}(a,d)^2} \\ \sum g &= \sum_{b\in V(G_2)} \sum_{ac\in E(T(G_1))} \sqrt{d_{G_1+_T G_2}(a,b)^2 + d_{G_1+_T G_2}(c,b)^2} \end{aligned}$$

For the computation of $\sum f$, we proceed by equation (2.1) and applying theorem (2.2)

$$\sum f \quad \leq \quad p_1 SO(G_2) + 2\left(M_1^{1/2}(G_1)\chi_{1/2}(G_2) + 2\sqrt{2}q_1q_2\right)$$

Now for $\sum g$,

$$\begin{aligned} &\sum_{b\in V(G_2)} \sum_{ac\in E(T(G_1))} \sqrt{d_{G_1+_T G_2}(a,b)^2 + d_{G_1+_T G_2}(c,b)^2} \\ &= \sum_{b\in V(G_2)} \sum_{\substack{ac\in E(T(G_1)) \\ a,c\in V(G_1)}} \sqrt{d_{G_1+_T G_2}(a,b)^2 + d_{G_1+_T G_2}(c,b)^2} \\ &+ \sum_{b\in V(G_2)} \sum_{\substack{ac\in E(T(G_1)) \\ a\in V(G_1) \\ c\in V(T(G_1))-V(G_1)}} \sqrt{d_{G_1+_T G_2}(a,b)^2 + d_{G_1+_T G_2}(c,b)^2} \\ &+ \sum_{b\in V(G_2)} \sum_{\substack{ac\in E(T(G_1)) \\ a,c\in V(T(G_1))-V(G_1)}} \sqrt{d_{G_1+_T G_2}(a,b)^2 + d_{G_1+_T G_2}(c,b)^2} \end{aligned}$$

$$\Rightarrow \quad \sum g \quad = \quad \sum g_1 + \sum g_2 + \sum g_3$$

where

$$\sum g_1 = \sum_{b\in V(G_2)} \sum_{\substack{ac\in E(T(G_1))\\ a,c\in V(G_1)}} \sqrt{d_{G_1+_TG_2}(a,b)^2 + d_{G_1+_TG_2}(c,b)^2}$$

$$\sum g_2 = \sum_{b\in V(G_2)} \sum_{\substack{ac\in E(T(G_1))\\ a\in V(G_1)\\ c\in V(T(G_1))-V(G_1)}} \sqrt{d_{G_1+_TG_2}(a,b)^2 + d_{G_1+_TG_2}(c,b)^2}$$

$$\sum g_3 = \sum_{b\in V(G_2)} \sum_{\substack{ac\in E(T(G_1))\\ a,c\in V(T(G_1))-V(G_1)}} \sqrt{d_{G_1+_TG_2}(a,b)^2 + d_{G_1+_TG_2}(c,b)^2}$$

For $\sum g_1$ proceeding by equation (2.1) and applying theorem (2.2),

$$\sum g_1 \quad \leq \quad 2\big(p_2 SO(G_1) + \chi_{1/2}(G_1)M_1^{1/2}(G_2) + \sqrt{2}q_1q_2\big)$$

For the computation of $\sum g_2$,

$$\sum_{b\in V(G_2)} \sum_{\substack{ac\in E(T(G_1))\\ a\in V(G_1)\\ c\in V(T(G_1))-V(G_1)}} \sqrt{d_{G_1+_TG_2}(a,b)^2 + d_{G_1+_TG_2}(c,b)^2}$$

$$= \sum_{b\in V(G_2)} \sum_{\substack{ac\in E(T(G_1))\\ a\in V(G_1)\\ c\in V(T(G_1))-V(G_1)}} \sqrt{\big(d_{T(G_1)}(a) + d_{G_2}(b)\big)^2 + d_{T(G_1)}(c)^2}$$

By applying equation (2.1) and (2.2),

$$\sum_{b\in V(G_2)} \sum_{\substack{ac\in E(T(G_1))\\ a\in V(G_1)\\ c\in V(T(G_1))-V(G_1)}} \sqrt{d_{G_1+_TG_2}(a,b)^2 + d_{G_1+_TG_2}(c,b)^2}$$

$$= \sum_{b\in V(G_2)} \sum_{\substack{ac\in E(T(G_1))\\ a\in V(G_1)\\ c\in V(T(G_1))-V(G_1)}} \sqrt{4d_{G_1}(a)^2 + d_{G_2}(b)^2 + 4d_{G_1}(a)d_{G_2}(b) + d_{Q(G_1)}(c)^2}$$

$$\Rightarrow \quad \sum g_2 \quad \leq \quad 2\big(2p_2M_1(G_1) + M_1^{3/2}(G_1)M_1^{1/2}(G_2) + 2q_1q_2\big)$$

Now for $\sum g_3$, we proceed by equation (2.2) and applying theorem (2.3),

$$\sum g_3 \quad \leq \quad 2\sqrt{2}p_2\Delta_{G_1}\Big(\frac{1}{2}M_1(G_1) - q_1\Big)$$

Thus we get

$$\begin{aligned}\sum g \quad \leq \quad & 2\Big[p_2\left(SO(G_1) + 2M_1(G_1)\right) + \Big(\chi_{1/2}(G_1)M_1^{1/2}(G_2) \\ & + M_1^{3/2}(G_1)M_1^{1/2}(G_2)\Big) + \sqrt{2}p_2\Delta_{G_1}\Big(\frac{1}{2}M_1(G_1) - q_1\Big) + \sqrt{2}(\sqrt{2}+1)q_1q_2\Big]\end{aligned}$$

From all the computations,

$$\begin{aligned}SO(G_1 +_T G_2) \quad \leq \quad & p_1SO(G_2) + 2p_2SO(G_1) + 2\Big[M_1^{1/2}(G_1)\chi_{1/2}(G_2) \\ & + M_1^{1/2}(G_2)\big(M_1^{3/2}(G_1) + \chi_{1/2}(G_1)\big)\Big] + 2\big(3\sqrt{2}+2\big)q_1q_2 \\ & + p_2M_1(G_1)\left(4 + \sqrt{2}\Delta_{G_1}\right) - 2\sqrt{2}\Delta_{G_1}p_2q_1\end{aligned}$$

This proves the result. □

3 Conclusion

In molecular graph theory, the proper exploration of topological descriptors by analysing every feature of a chemical graph, aids in the process for acquiring advancements in the domain. Accordingly, this paper investigates the four graph operations termed as F-Sum by analysing the bounds and inequalities of sombor index of these operations on connected graphs. The outcomes procured have favourable angles towards further research for the analysis of degree and distance dependent descriptors and series of graph operation.

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