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GEODESIC ORBITS OF HOMOGENEOUS FINSLER SPACE WITH (α, β)-METRICS

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Abstract: A geodesic of a Finsler space (M, F) is an orbit of a one-parameter transformation group of isometries of (M, F) and is known as homogeneous geodesic. In this paper we obtain the necessary and sufficient condition for a vector of Finsler space with (α, β)-metric to be a geodesic vector. Further we show the existence of a homogeneous geodesic through any point of the Finsler space with invariant (α, β)-metrics.

1 Introduction

Finsler geometry is a part of differential geometry, which is an intriguing field. It has many important applications in several areas like physics, magnetic fields, biology and mechanics [1]. Let (M, F) be a Finsler space, where F is positively homogeneous. A homogeneous Riemannian or Finsler manifold is called a geodesic orbit space, if any geodesic is the orbit of a one-parameter subgroup of isometries [9].

O. Kowalski and J. Szenthe [6] have proved that any homogeneous Riemannian manifold admits a homogeneous geodesic through the origin and it is the generalized work of V.V. Kajzer [4] for invariant metrics on Lie groups which has atleast one homogeneous geodesic over the identity. Later, geodesic orbits in homogeneous Finsler spaces were studied by Dariush Latifi, who described the non-zero vector to be geodesic orbit which is criticized by some simple conditions [8]. Next, in 2009, Latifi collaborated his work with Razavi [10] and studied homogeneous geodesics in a 3-dimensional connected Lie group using left invariant Randers metric and has shown that all the geodesics on spaces which are equipped with a such metrics are homogeneous. After that, in 2014, Zaili Yan and S. Deng gave a necessary and sufficient condition for a Randers space to be a Geodesic Orbit space.

In this paper, we study the condition for non-zero vector to be a geodesic vector for the Shen's square metric and a special (α, β)-metric. This article is designed as following manner: In part 2 preliminaries, we recapitulate some basic definitions related to the Finsler space, homogeneous space, reductive decomposition, isometry group, geodesic orbit space and another important definition a geodesic vector. In section 3 we discuss the invariant Finsler metric, necessary and sufficient condition for a non-zero vector to be a geodesic vector for square metric and existence of a geodesic vector. Later on, in the last part, we discuss the same results, as in section 3, for the special (α, β)-metric.

2 Preliminaries

Definition 2.1. Let M be a smooth connected manifold. A Finsler metric on M is a function $F : TM \to [0, \infty)$ such that
1. F is smooth on the slit tangent bundle $TM \setminus \{0\}$;
2. the restriction of F to any $T_xM, x \in M$, is a Minkowski norm.
The pair (M, F) is called a Finsler space.

Definition 2.2. [2] Let positively homogeneous, connected isometry group $I(M, F)$ act transitively on M, then Finsler space (M, F) is said to be homogeneous.

The connected homogeneous Finsler space M is defined as $M = G/H$, where G and H represent the Lie group of M isometries and the isotropy subgroup of a point in M, respectively. In this case, a homogeneous Finsler space is reductively decomposed if an $Ad(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ [7] exists, where $\mathfrak{h}$ and $\mathfrak{g}$ represent the Lie algebras of H and G, respectively, and $\mathfrak{m}$ is the subspace of $\mathfrak{g}$.

Definition 2.3. [2] Let (M, F) be a Finsler space. An isometry is a mapping ϕ of M onto itself if ϕ is a diffeomorphism and for any $x \in M, X \in T_x(M), F(\phi(x), d\phi_x(X)) = F(x, X)$.

Definition 2.4. [15] Let (M, F) be a Finsler space and $G = I(M, F)$ the group of isometries. The space (M, F) is called a Finsler geodesic orbit space if every geodesic of (M, F) is the orbit of a one-parameter subgroup of G. That is, if γ is a geodesic, then there exist $W \in \mathfrak{g} = Lie(G)$ and $o \in M$, such that $\gamma(t) = exp(tW) \cdot o$.

Definition 2.5. Let $(G/H, F)$ be a homogeneous Finsler space, and $p = eH \in G/H$. A vector $X \in \mathfrak{g} \setminus \{0\}$ is called a geodesic vector if the curve $exp(tX) \cdot p$ is a geodesic.

Theorem 2.6. *[8] If (M, F) is a Finsler geodesic orbit space, then it has vanishing S-curvature.*

For further calculation, we use the following lemma: [8]

Lemma 2.7. *A vector $X \in \mathfrak{g} \setminus \{0\}$ is a geodesic vector if $g_{X_\mathfrak{m}}(X_\mathfrak{m}, [X, Z]_\mathfrak{m}) = 0, \forall Z \in \mathfrak{m}$ and conversely.*
Here, g is fundamental tensor of F on $\mathfrak{m}$ and $\mathfrak{m}$ is the corresponding projection.

3 Homogeneous geodesic for square metric

This section explores the Finsler geodesic orbit of the square metric. A function $F = \dfrac{(\alpha+\beta)^2}{\alpha}$ is a square metric constructed from a Riemannian metric α and a smooth 1-form β on M. Let $(G/H, F)$ be a homogeneous Finsler space. Here the Riemannian metric α and 1-form β are both G-invariant. Following this, the invariant metric F induces a Minkowski norm on $\mathfrak{m}$, such that

$$F(X) = \frac{\left(\sqrt{\langle X, X\rangle} + \langle u, X\rangle\right)^2}{\sqrt{\langle X, X\rangle}}, \forall X \in \mathfrak{m}, \tag{3.1}$$

where $\langle , \rangle$ is the restriction of the Riemannian metric α to $\mathfrak{m}$, $\langle u, X\rangle = \beta(X)$ and $\alpha(X) = \sqrt{\langle X, X\rangle}, \forall X \in \mathfrak{m}$. The geodesic vectors for the Finsler metric $F = \dfrac{(\alpha+\beta)^2}{\alpha}$ can be described as follows:

Theorem 3.1. *Let M be a homogeneous Finsler space with square metric $F = \dfrac{(\alpha+\beta)^2}{\alpha}$. Consequently, a vector $X(\neq 0) \in \mathfrak{g}$ is a geodesic vector if and only if*

$$\left\langle [X, Z]_\mathfrak{m}, (|X_\mathfrak{m}| - \langle u, X_\mathfrak{m}\rangle)X_\mathfrak{m} + 2|X_\mathfrak{m}|^2 u\right\rangle = 0, \tag{3.2}$$

holds for every $Z \in \mathfrak{m}$.

Proof. From the equation (3.1), the Finsler metric F can be expressed as

$$F(X) = \frac{\left(\sqrt{\langle X, X\rangle} + \langle u, X\rangle\right)^2}{\sqrt{\langle X, X\rangle}},$$

we have

$$F^2(X + sU + tV) = \frac{\left(\sqrt{\langle X + sU + tV, X + sU + tV\rangle} + \langle u, X + sU + tV\rangle\right)^4}{\langle X + sU + tV, X + sU + tV\rangle}.$$

Applying the formula, that is

$$g_X(U,V) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t} F^2(X + sU + tV)\Big|_{(t=s=0)},$$

we get

$$g_X(U,V) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[\frac{\left(\sqrt{\langle X + sU + tV, X + sU + tV\rangle} + \langle u, X + sU + tV\rangle\right)^4}{\langle X + sU + tV, X + sU + tV\rangle}\right]\Bigg|_{(t=s=0)}.$$

After simplifying $g_X(U,V)$, we obtain

$$g_X(U,V) = \frac{F(X)}{\langle X, X\rangle^2} K, \tag{3.3}$$

where

$$\begin{aligned} K = & -4\sqrt{\langle X,X\rangle}\langle X,U\rangle\langle X,V\rangle + 2\langle X,X\rangle\langle X,U\rangle\langle u,V\rangle + 2\langle X,X\rangle\langle X,V\rangle\langle u,U\rangle \\ & + 6\langle X,X\rangle^{\frac{3}{2}}\langle u,U\rangle\langle u,V\rangle - 10\langle u,X\rangle\langle X,U\rangle\langle X,V\rangle - 4\sqrt{\langle X,X\rangle}\langle u,X\rangle\langle X,U\rangle\langle u,V\rangle \\ & - 4\sqrt{\langle X,X\rangle}\langle u,X\rangle\langle X,V\rangle\langle u,U\rangle + 2\langle X,X\rangle\langle u,X\rangle\langle U,V\rangle + 2\langle X,X\rangle^{\frac{3}{2}}\langle U,V\rangle \\ & - F(X)\langle X,X\rangle\langle U,V\rangle + 4F(X)\langle X,U\rangle\langle X,V\rangle. \end{aligned}$$

From the above equation, we have

$$g_X(X,V) = \frac{F(X)}{|X|^2}\left(|X| + \langle u, X\rangle\right)\langle V, (|X| - \langle u, X\rangle)X + 2|X|^2 u\rangle. \tag{3.4}$$

From equation (3.4), we obtain

$$g_{X_\mathfrak{m}}(X_\mathfrak{m}, [X,Z]_\mathfrak{m}) = \frac{F(X_\mathfrak{m})}{|X_\mathfrak{m}|^2}\left(|X_\mathfrak{m}| + \langle u, X_\mathfrak{m}\rangle\right)\left\langle [X,Z]_\mathfrak{m}, (|X_\mathfrak{m}| - \langle u, X_\mathfrak{m}\rangle)X_\mathfrak{m} + 2|X_\mathfrak{m}|^2 u\right\rangle. \tag{3.5}$$

Thus, from lemma (2.7) $g_{X_\mathfrak{m}}(X_\mathfrak{m}, [X,Z]_\mathfrak{m}) = 0$ if and only if

$$\left\langle [X,Z]_\mathfrak{m}, (|X_\mathfrak{m}| - \langle u, X_\mathfrak{m}\rangle)X_\mathfrak{m} + 2|X_\mathfrak{m}|^2 u\right\rangle = 0.$$

This completes the proof. □

Corollary 3.2. *For a homogeneous Finsler space $(G/H, F)$ with $F = \frac{(\alpha+\beta)^2}{\alpha}$ defined by an invariant Riemannian metric $\langle,\rangle$ and an invariant vector field at $\tilde{u}$ such that $\tilde{u}(H) = u$, a vector field $X \in \mathfrak{g}$ such that $\langle u, [X,Z]_\mathfrak{m}\rangle = 0$ is a geodesic vector if and only if it is a geodesic vector of $(G/H, \langle,\rangle)$.*

Proof. Considering the above theorem, a necessary and sufficient condition of a geodesic vector $X(\neq 0) \in \mathfrak{g}$

$$\left\langle [X,Z]_\mathfrak{m}, (|X_\mathfrak{m}| - \langle u, X_\mathfrak{m}\rangle)X_\mathfrak{m} + 2|X_\mathfrak{m}|^2 u\right\rangle = 0.$$

Since $\langle u, [X,Z]_\mathfrak{m}\rangle = 0$, we get $\langle X_\mathfrak{m}, [X,Z]_\mathfrak{m}\rangle = 0$. Thus, $X \in \mathfrak{g}$ is a geodesic vector of $(G/H, F)$ is also a geodesic vector of $(G/H, \langle,\rangle)$ and vice versa. □

Theorem 3.3. *On a homogeneous Finsler space (M, F), there exits at least one homogeneous geodesic, with metric $F = \frac{(\alpha+\beta)^2}{\alpha}$.*

Proof. Let $(M = G/H, F)$ be a homogeneous Finsler space and a connected isometry group $I(M, F)$ of G acts transitively on M, at fixed point $p \in M$, an isotropy group of G is H. For G and H, let $\mathfrak{g}$ and $\mathfrak{h}$ be the respective Lie algebras. A Killing form of $\mathfrak{g}$ can be represented by K and its null space by $radK$. According to Kowalski and Szenthe's study [6], in the case of a compact H; on $\mathfrak{h}$, K is nondegenerate. For K, let $\mathfrak{h}$ and $\mathfrak{m}$ be the orthogonal to each other, which can be written as

$$\mathfrak{m} = \mathfrak{h}^{\perp} = \{W \in \mathfrak{g} : K(W; Z) = 0; \forall Z \in \mathfrak{h}\}.$$

As a result, the reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ and $radK \subset \mathfrak{m}$. From the Theorem 1.1 of [16], we get the result. □

Example 3.4. For the reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ with $\mathfrak{m} = \mathfrak{h}^{-1}$ as defined in theorem 3.3, the $\mathfrak{m}$-projection $[\mathfrak{g}, \mathfrak{g}]_{\mathfrak{m}}$ is a proper subspace of $\mathfrak{m}$. Let $Y \neq 0$ belongs to the orthogonal complement of $[\mathfrak{g}, \mathfrak{g}]_{\mathfrak{m}}$ in $\mathfrak{m}$ with respect to $\langle , \rangle$ and $\langle Y, Y \rangle = 1$. Suppose $X = \phi(Y, u)Y - u$ be a vector with $\phi(Y, u)$ satisfying,

$$4\phi^4 - 12\langle Y, u\rangle\phi^3 + (9\langle Y, u\rangle^2 + 4\langle u, u\rangle - 1)\phi^2 - 2\langle Y, u\rangle(3\langle u, u\rangle - 2)\phi + \langle u, u\rangle^2 - \langle u, u\rangle = 0, \tag{3.6}$$

then X satisfies the condition $|X| - \langle u, X\rangle = 2|X|^2$. Using this X, we get

$$\big\langle [X, Z]_{\mathfrak{m}}, (|X_{\mathfrak{m}}| - \langle u, X_{\mathfrak{m}}\rangle)X_{\mathfrak{m}} + 2|X_{\mathfrak{m}}|^2 u \big\rangle = 0.$$

Thus X forms a geodesic vector for the Finsler space $(G/H, F)$ with $F = \dfrac{(\alpha + \beta)^2}{\alpha}$.

4 Homogeneous geodesic for special metric

This section explores the Finsler geodesic orbit of the special metric. A function $F = \alpha + \sqrt{\alpha^2 + \beta^2}$ is a special metric constructed from a Riemannian metric α and a smooth 1-form β on M. Let $(G/H, F)$ be a homogeneous Finsler space. Here the Riemannian metric α and 1-form β are both G-invariant. Following this, the invariant metric F induces a Minkowski norm on $\mathfrak{m}$, such that

$$F(X) = \sqrt{\langle X, X\rangle} + \sqrt{\langle X, X\rangle + \langle u, X\rangle^2}, \ \forall X \in \mathfrak{m}, \tag{4.1}$$

where $\langle , \rangle$ is the restriction of the Riemannian metric α to $\mathfrak{m}$, $\langle u, X\rangle = \beta(X)$ and $\alpha(X) = \sqrt{\langle X, X\rangle}$, $\forall X \in \mathfrak{m}$. The geodesic vectors for the Finsler metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$ can be described as follows:

Theorem 4.1. *Let (M, F) be a homogeneous Finsler space with special metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$. Then a non-zero vector $X \in \mathfrak{g}$ is a geodesic vector if and only if*

$$\langle [X, Z]_{\mathfrak{m}}, F(X_{\mathfrak{m}})X_{\mathfrak{m}} + |X_{\mathfrak{m}}|\langle u, X_{\mathfrak{m}}\rangle u\rangle = 0, \tag{4.2}$$

holds for every $Z \in \mathfrak{m}$.

Proof. From equation (4.1) the Finsler metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$ can be written as

$$F^2(X + sU + tV) = 2\langle X + sU + tV, X + sU + tV\rangle + \langle u, X + sU + tV\rangle^2 + 2L,$$

where

$$L = \sqrt{\langle X + sU + tV, X + sU + tV\rangle^2 + \langle X + sU + tV, X + sU + tV\rangle\langle u, X + sU + tV\rangle^2}.$$

Applying the formula,

$$g_X(U, V) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[2\langle X + sU + tV, X + sU + tV\rangle + \langle u, X + sU + tV\rangle^2 + 2L\right]\Bigg|_{(t=s=0)},$$

we obtain

$$g_X(U,V) = 2\langle U,V\rangle + \langle u,U\rangle\langle u,V\rangle + \frac{L_1}{\sqrt{\langle X,X\rangle^2 + \langle u,X\rangle^2\langle X,X\rangle}} - \frac{L_2}{(\langle X,X\rangle^2 + \langle u,X\rangle^2\langle X,X\rangle)^{\frac{3}{2}}}, \tag{4.3}$$

where,

$$\begin{aligned} L_1 =& 4\langle X,U\rangle\langle X,V\rangle + 2\langle X,X\rangle\langle U,V\rangle + 2\langle u,X\rangle\left(\langle u,U\rangle\langle X,V\rangle + \langle u,V\rangle\langle X,U\rangle\right) \\ &+ \langle u,U\rangle\langle u,V\rangle\langle X,X\rangle + \langle u,X\rangle^2\langle U,V\rangle, \\ L_2 =& 4\langle X,X\rangle^2\langle X,U\rangle\langle X,V\rangle + 2\langle u,X\rangle\langle X,X\rangle^2\left(\langle u,U\rangle\langle X,V\rangle + \langle u,V\rangle\langle X,U\rangle\right) \\ &+ 4\langle u,X\rangle^2\langle X,X\rangle\langle X,U\rangle\langle X,V\rangle + \langle u,X\rangle^2\langle X,X\rangle^2\langle u,U\rangle\langle u,V\rangle \\ &+ \langle u,X\rangle^3\langle X,X\rangle\left(\langle u,U\rangle\langle X,V\rangle + \langle u,V\rangle\langle X,U\rangle\right) + \langle u,X\rangle^4\langle X,U\rangle\langle X,V\rangle. \end{aligned}$$

Using the above equation, we have

$$g_X(X,V) = \frac{F(X)}{|X|\sqrt{|X|^2 + \langle u,X\rangle^2}}\left\langle V, F(X)X + |X|\langle u,X\rangle u\right\rangle. \tag{4.4}$$

From equation (4.4), we obtain

$$g_{X_{\mathfrak{m}}}(X_{\mathfrak{m}},[X,Z]_{\mathfrak{m}}) = \frac{F(X_{\mathfrak{m}})}{|X_{\mathfrak{m}}|\sqrt{|X_{\mathfrak{m}}|^2 + \langle u,X_{\mathfrak{m}}\rangle^2}}\left\langle V, F(X_{\mathfrak{m}})X_{\mathfrak{m}} + |X_{\mathfrak{m}}|\langle u,X_{\mathfrak{m}}\rangle u\right\rangle. \tag{4.5}$$

Thus, from lemma (2.7) $g_{X_{\mathfrak{m}}}(X_{\mathfrak{m}},[X,Z]_{\mathfrak{m}}) = 0$ if and only if

$$\langle [X,Z]_{\mathfrak{m}}, F(X_{\mathfrak{m}})X_{\mathfrak{m}} + |X_{\mathfrak{m}}|\langle u,X_{\mathfrak{m}}\rangle u\rangle = 0.$$

This completes the proof. □

Corollary 4.2. *For a homogeneous Finsler space $(G/H,F)$ with $F = \alpha + \sqrt{\alpha^2+\beta^2}$ defined by an invariant Riemannian metric $\langle,\rangle$ and an invariant vector field at $\tilde{u}$ such that $\tilde{u}(H) = u$, a vector field $X \in \mathfrak{g}$ such that $\langle u,[X,Z]_{\mathfrak{m}}\rangle = 0$ is a geodesic vector if and only if it is a geodesic vector of $(G/H,\langle,\rangle)$.*

Proof. From the above theorem, $X(\neq 0) \in \mathfrak{g}$ is a geodesic vector if and only if

$$\langle [X,Z]_{\mathfrak{m}}, F(X_{\mathfrak{m}})X_{\mathfrak{m}} + |X_{\mathfrak{m}}|\langle u,X_{\mathfrak{m}}\rangle u\rangle = 0.$$

Since $\langle u,[X,Z]_{\mathfrak{m}}\rangle = 0$, we get $\langle X_{\mathfrak{m}},[X,Z]_{\mathfrak{m}}\rangle = 0$. Thus, $X \in \mathfrak{g}$ is geodesic vector of $(G/H,F)$ if and only if it is a geodesic vector of $(G/H,\langle,\rangle)$. □

Theorem 4.3. *On a homogeneous Finsler space (M,F), there exits at least one homogeneous geodesic, with metric $F = \alpha + \sqrt{\alpha^2+\beta^2}$.*

Proof. Let $(M = G/H,F)$ be a homogeneous Finsler space and a connected isometry group $I(M,F)$ of G acts transitively on M, at fixed point $p \in M$, an isotropy group of G is H. For G and H, let $\mathfrak{g}$ and $\mathfrak{h}$ be the respective Lie algebras. A Killing form of $\mathfrak{g}$ can be represented by K and its null space by $radK$. According to Kowalski and Szenthe's study [6], in the case of a compact H; on $\mathfrak{h}$, K is nondegenerate. For K, let $\mathfrak{h}$ and $\mathfrak{m}$ be the orthogonal to each other, which can be written as

$$\mathfrak{m} = \mathfrak{h}^{\perp} = \{W \in \mathfrak{g} : K(W;Z) = 0; \forall Z \in \mathfrak{h}\}$$

As a result, the reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ and $radK \subset \mathfrak{m}$. From the Theorem 1.1 [16], we get the result. □

Example 4.4. For the reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ with $\mathfrak{m} = \mathfrak{h}^{-1}$ as defined in theorem 4.3, the $\mathfrak{m}$-projection $[\mathfrak{g}, \mathfrak{g}]_{\mathfrak{m}}$ is a proper subspace of $\mathfrak{m}$. Let $Y \neq 0$ belongs to the orthogonal complement of $[\mathfrak{g}, \mathfrak{g}]_{\mathfrak{m}}$ in $\mathfrak{m}$ with respect to $\langle , \rangle$ and $\langle Y, Y \rangle = 1$. Suppose $X = \phi(Y, u)Y - u$ be a vector with $\phi(Y, u)$ satisfying,

$$\langle Y, u\rangle \phi^3 - (2\langle Y, u\rangle^2 + \langle u, u\rangle + 2)\phi^2 + (\langle Y, u\rangle\langle u, u\rangle - \langle Y, u\rangle + 2\langle u, u\rangle + 4)\phi \\ -\langle u, u\rangle^2 - \langle u, u\rangle = 0, \tag{4.6}$$

then X satisfies the condition $F(X) = |X|\langle u, X\rangle$. Using this X, we get

$$\langle [X, Z]_{\mathfrak{m}}, F(X_{\mathfrak{m}})X_{\mathfrak{m}} + |X_{\mathfrak{m}}|\langle u, X_{\mathfrak{m}}\rangle u\rangle = 0.$$

Thus X forms a geodesic vector for the Finsler space $(G/H, F)$ with $F = \alpha + \sqrt{\alpha^2 + \beta^2}$.

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